Gravitational waves from an effective field theory

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In collaboration with

Johannes Blümlein Peter Marquard Gerhard Schäfer

Gravitational wave observatories

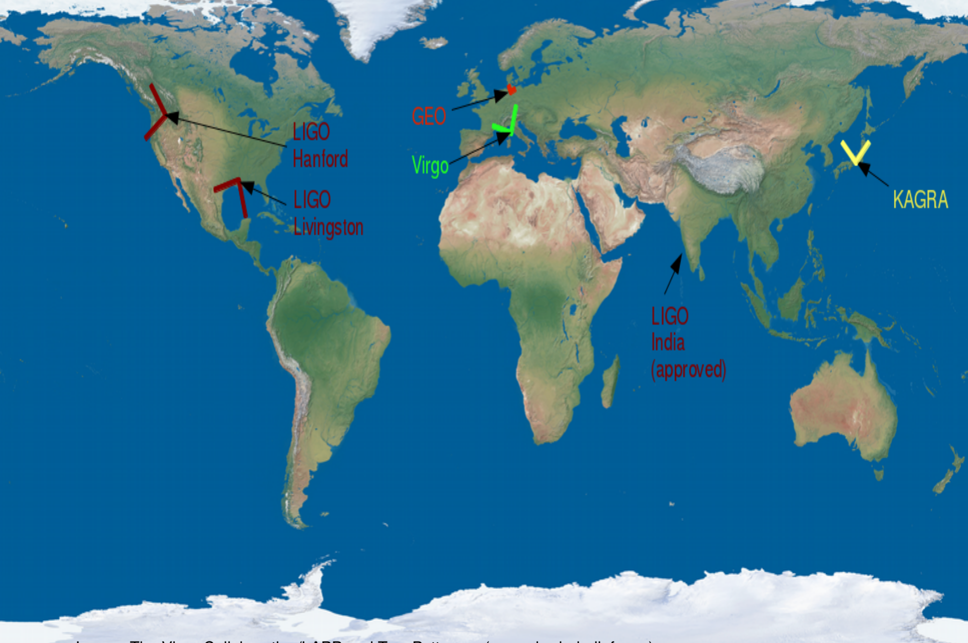
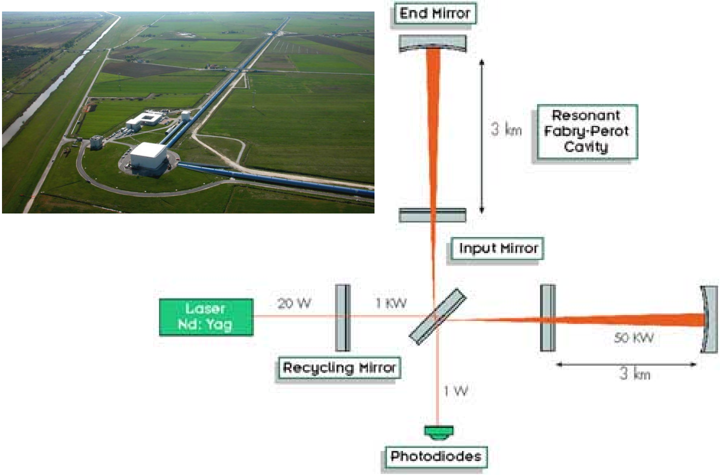


Image: The Virgo Collaboration/LAPP and Tom Patterson (www.shadedrelief.com)

Basic observatory layout

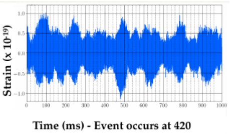
Example: Virgo

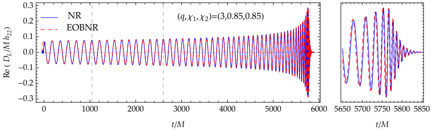


Photograph courtesy Caltech/MIT/LIGO Laboratory

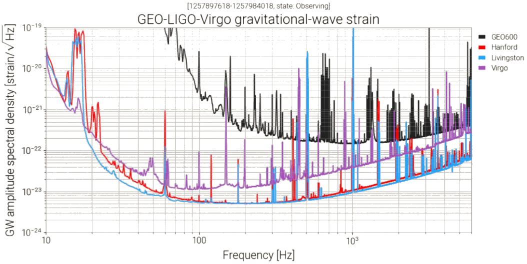
Signal extraction

Matched filtering

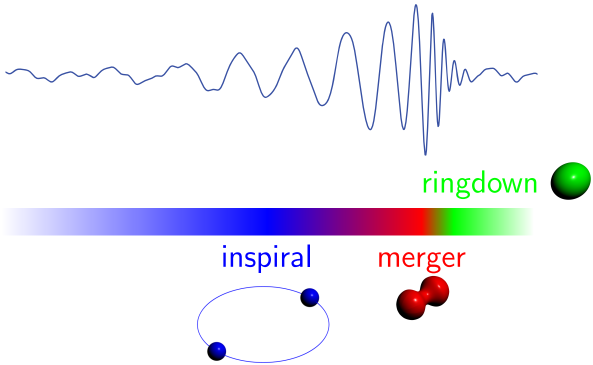




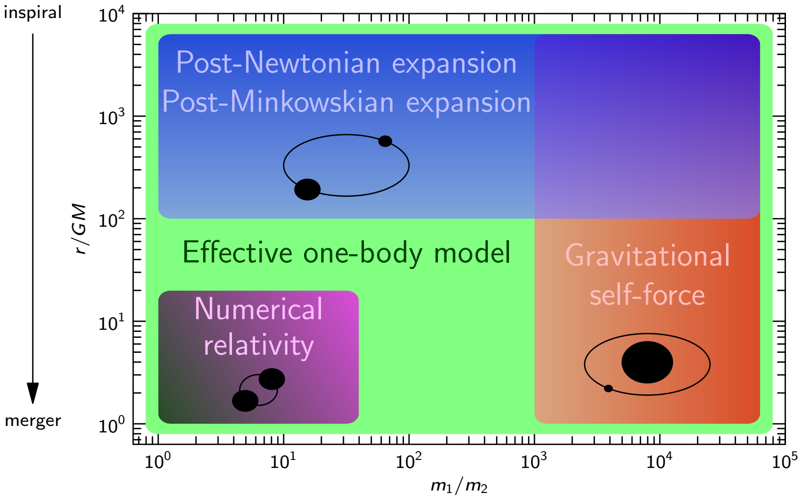
$$(s|h) = 4\Re \int_0^\infty df \, \frac{\tilde{s}(f)\tilde{h}^\star(f)}{S_h(f)}$$



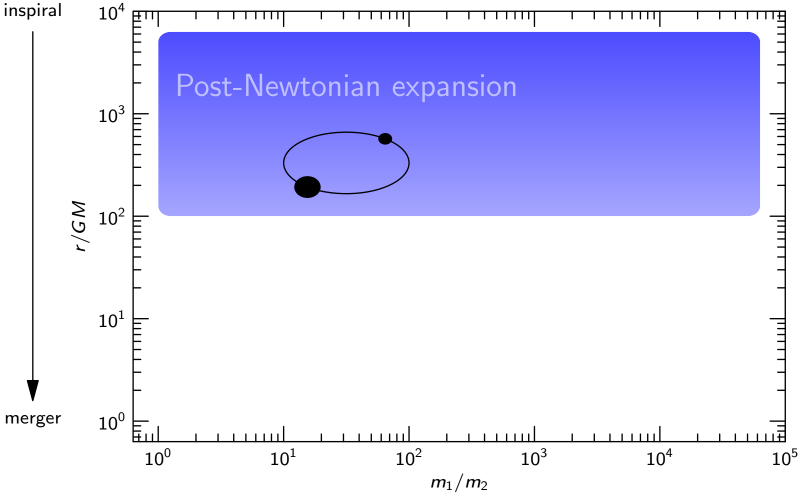
Gravitational waves from binary mergers



Theory approaches to binary systems

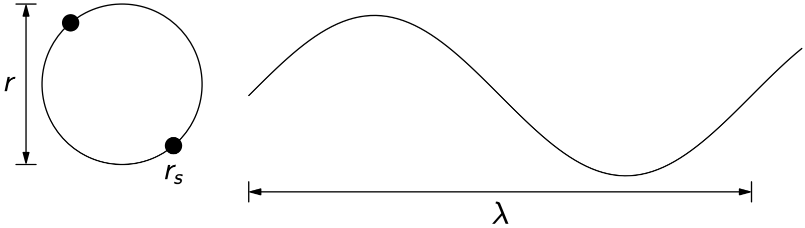


adapted from [Buonanno, Sathyaprakash arXiv:1410.7832]


inspiral
merger
Post-Newtonian expansion
r/GM
m_1/m_2
10^0
10^1
10^2
10^3
10^4
10^5

Compact binary systems

Power counting

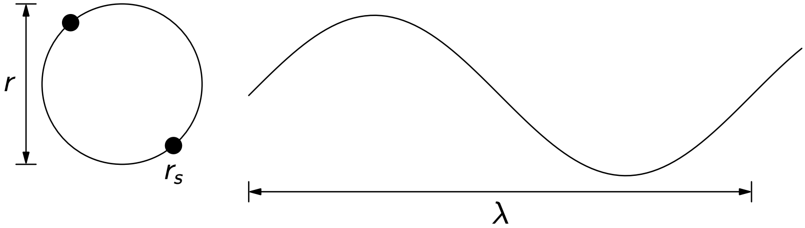


- Masses comparable: $m \equiv m_1 \sim m_2$
  Generalisation to different masses straightforward
- Nonrelativistic system: $v \ll 1$
- Virial theorem: $mv^2 \sim \frac{Gm^2}{r}$

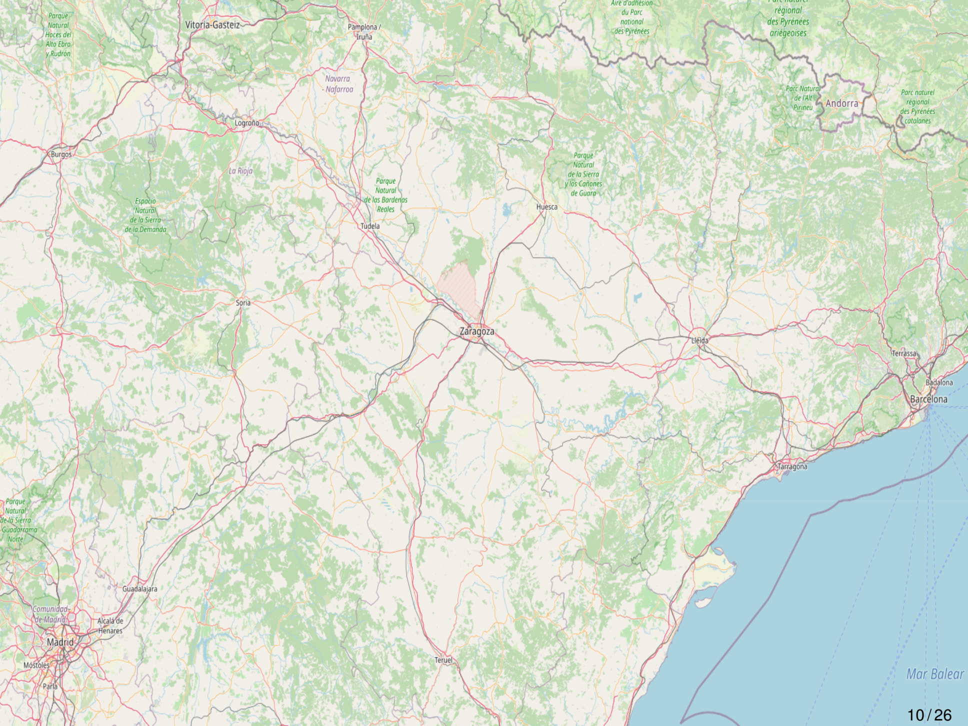
Post-Newtonian (PN) expansion:
Combined expansion in $v \sim \sqrt{Gm/r} \ll 1$

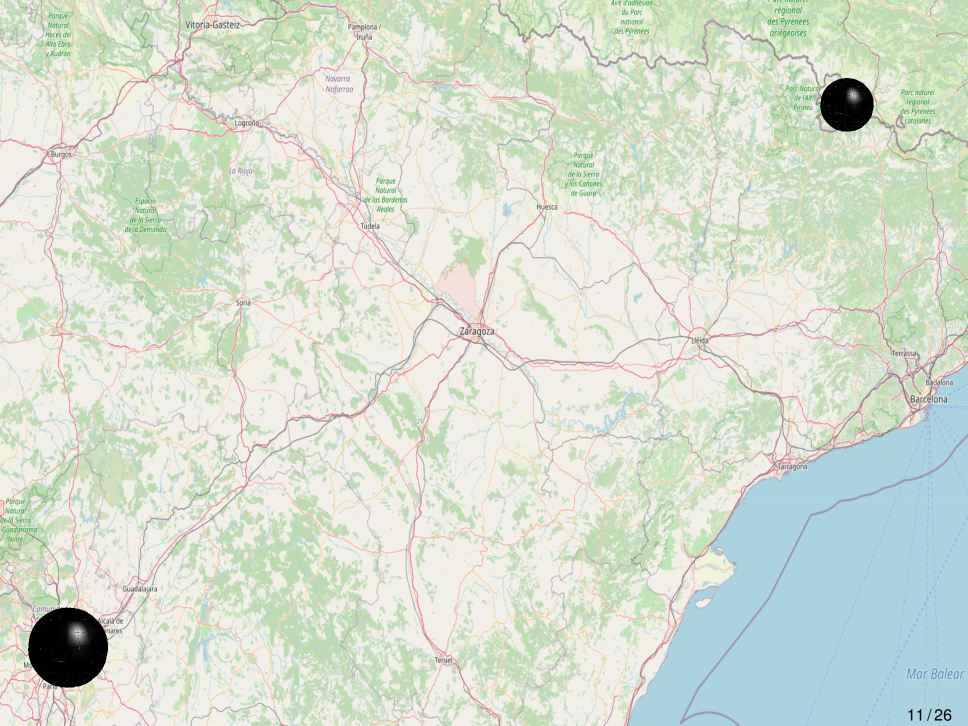
Post-Newtonian expansion

Scales



- $\omega_{\rm GW} = \frac{2v}{r} \Rightarrow$ $\boxed{\lambda \sim \frac{r}{v}}$
- $r_s = 2Gm \Rightarrow$ $\boxed{r_s \sim r v^2}$


Vitoria-Gasteiz
Pamplona / Iruña
Parque Natural Hoces del Alto Ebro y Rudrón
Navarra - Nafarroa
Aire d'adhésion du Parc national des Pyrénées
régional des Pyrénées ariégeoises
Parc Natural de l'Alt Pirineu
Andorra
Parc naturel régional des Pyrénées catalanes
Logroño
Burgos
La Rioja
Parque Natural de las Bardenas Reales
Parque Natural de la Sierra y los Cañones de Guara
Huesca
Espacio Natural de la Sierra de la Demanda
Tudela
Soria
Zaragoza
Lleida
Terrassa
Badalona
Barcelona
Tarragona
Parque Natural de la Sierra Guadarrama Norte
Guadalajara
Comunidad de Madrid
Alcalá de Henares
Madrid
Móstoles
Parla
Teruel
Mar Balear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Teruel
Mar Balear

Central quantities

Energy and flux

- Emission of quadrupole waves: $\omega_{\mathrm{GW}}(t) = 2\omega(t)$
- Orbital frequency ω from continuity equation:

$$\frac{dE(\omega)}{dt} = -F(\omega) \qquad \Rightarrow \qquad \dot{\omega}(t) = -\frac{F(\omega)}{dE(\omega)/d\omega}$$

E: Centre-of-mass energy
F: Gravitational wave luminosity

$$\dot{\omega}_{\mathrm{GW}}(t) = -2\frac{F(\omega)}{dE(\omega)/d\omega}$$

Determine $F(\omega)$, $E(\omega)$ from a Post-Newtonian Lagrangian

General relativity

General relativity action:

$$S_{\text{GR}}[g^{\mu\nu}] = S_{\text{EH}} + S_{\text{GF}} + S_{\text{matter}}$$

With $\eta^{\mu\nu} = \text{diag}(-1,1,1,1)$, $g = \det(g^{\mu\nu})$:

- Einstein-Hilbert action:

$$S_{\text{EH}} = \frac{1}{16G\pi}\int d^d x\sqrt{-g}R$$

- *Harmonic* gauge $\partial_\mu \sqrt{-g} g^{\mu\nu} = 0$:

$$S_{\text{GF}} = -\frac{1}{32G\pi}\int d^d x\sqrt{-g}\Gamma_\mu\Gamma^\mu\,, \qquad \Gamma^\mu = g^{\alpha\beta}\Gamma^\mu{}_{\alpha\beta}$$

- Assume point-like matter:

$$S_{\text{matter}} = -\sum_{a=1}^{2} m_a \int d\tau_a$$

General relativity

Post-Newtonian expansion

Expand S_{GR} in $v \sim \sqrt{Gm/r} \ll 1$, e.g.

$$-m_a \int d\tau_a = -m_a \int dt \sqrt{-g_{\mu\nu} \frac{\partial x_a^\mu}{\partial t} \frac{\partial x_a^\nu}{\partial t}} = -m_a \int dt \sqrt{-g_{00}} + \mathcal{O}(v_a)$$

Coupling to spatial components of metric suppressed

Temporal Kaluza-Klein decomposition [Kol, Smolkin 2010]

$$g^{\mu\nu} = e^{2\phi} \begin{pmatrix} -1 & A_j \\ A_i & e^{-c_d \phi}(\delta_{ij} + \sigma_{ij}) - A_i A_j \end{pmatrix}, \qquad c_d = 2\frac{d-2}{d-3}$$

Flat spacetime for $\sqrt{Gm/r} \to 0$:

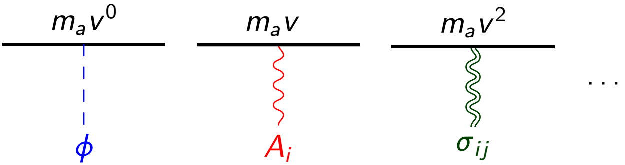
Expand in $\phi, A_i, \sigma_{ij} \sim \sqrt{Gm/r} \sim v$

General relativity

Expanded action

$$
\begin{aligned}
S_{\mathrm{GR}}[\phi, A_i, \sigma_{ij}] = \\
&\sum_{a=1}^{2} \int dt \left(m_a + \frac{1}{2} m_a v_a^2 + \mathcal{O}(v^4) \right) \\
&+ \sum_{a=1}^{2} m_a \int dt \left(-\phi + v_{ai} A_i + v_{ai} v_{aj} \sigma_{ij} - \frac{1}{2}\phi^2 + \ldots \right) \\
&+ \int \frac{d^d x}{32\pi G} \left[-c_d (\partial_\mu \phi)^2 + (\partial_\mu A_i)^2 + \frac{1}{4}(\partial_\mu \sigma_{ii})^2 - \frac{1}{2}(\partial_\mu \sigma_{ij})^2 + \ldots \right]
\end{aligned}
$$

Consider as quantum field theory:



Constructing an Effective Field Theory

Starting from a full theory (General Relativity)

1. Identify relevant scales and small scaling parameter
2. Identify corresponding field modes, characterised by scaling of momentum (components)
3. Formulate ansatz for effective theory action with lower number of field modes, guided by symmetries
4. Determine free coefficients in effective theory action by matching to full theory

Nonrelativistic effective theory

1. Scaling parameter $v \sim \sqrt{Gm/r} \ll 1$

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   - Potential: $k_0 \sim \frac{v}{r}, \vec{k} \sim \frac{1}{r}$
   - Radiation/ultrasoft: $k_0 \sim \frac{v}{r}, \vec{k} \sim \frac{v}{r}$
   - hard, soft modes: negligible quantum corrections

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   - hard, soft modes: negligible quantum corrections
3. Effective theory (NRGR) [Goldberger, Rothstein 2004]
   Analogue to pNRQCD/vNRQCD

   [Pineda, Soto 1997; Luke, Manohar, Rothstein 2000; . . .]
   - Absorb potential modes into near-zone potential
   - Keep radiation/ultrasoft modes

   Ansatz:

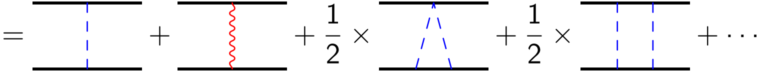
$$S_{\text{NRGR}} = S_{\text{matter}} + S_{\text{mixed}} + S_{\text{radiation}}$$

- $S_{\text{matter}} = \int dt\, (T - V_{\text{NZ}})$
- S_{mixed}: coupling of radiation modes to matter
- $S_{\text{radiation}}$: pure radiation modes

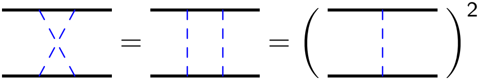
Nonrelativistic effective theory

Potential matching

4. Matching between effective and full theory
   e.g. potential interaction:

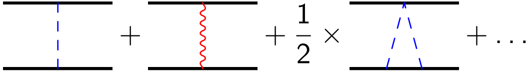


No matter propagators:



Factorising diagrams vanish in potential: cf. [Fischler 1977]

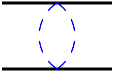
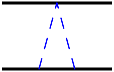
$-iV_{\mathrm{NZ}} =$



Potential matching

Scaling

- Coupling to matter proportional to black hole mass:

 $\sim m_1 m_2$  $\sim m_1 m_2^2$

$\Rightarrow$ Graviton loops suppressed by $\frac{E}{m_a} \ll \frac{m_{\text{Pl}}}{m_a} \approx 10^{-40}$

- Newton potential $\sim \frac{Gm}{r} \sim v^2$: 0PN

  L loop diagram $\sim \left(\frac{Gm}{r}\right)^L v^X$: $\geq L$PN

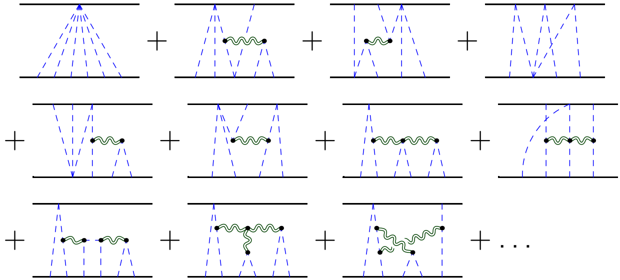
State of the art: 5PN

Potential matching

5PN calculation

1. Generate diagrams with up to 5 loops with `QGRAF` [Nogueira 1991]
2. Discard unwanted diagrams, e.g. graviton loops

$$-iV_{\text{5PN}} =$$



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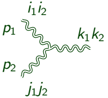
3. Compute and insert Feynman rules with FORM [Vermaseren et al.]



$$= \frac{i}{32 m_{\mathrm{Pl}}} \left(\tilde{V}_{\sigma\sigma\sigma}^{i_1 i_2, j_1 j_2, k_1 k_2} + \tilde{V}_{\sigma\sigma\sigma}^{i_2 i_1, j_1 j_2, k_1 k_2}\right)$$

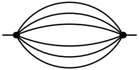
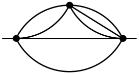
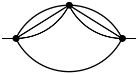
$$\tilde{V}_{\sigma\sigma\sigma}^{i_1 i_2, j_1 j_2, k_1 k_2} = V_{\sigma\sigma\sigma}^{i_1 i_2, j_1 j_2, k_1 k_2} + V_{\sigma\sigma\sigma}^{i_1 i_2, j_2 j_1, k_1 k_2} + V_{\sigma\sigma\sigma}^{i_1 i_2, j_1 j_2, k_2 k_1} + V_{\sigma\sigma\sigma}^{i_1 i_2, j_2 j_1, k_2 k_1}$$

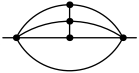
$$\begin{aligned}
V_{\sigma\sigma\sigma}^{i_1 i_2, j_1 j_2, k_1 k_2} \overset{v=0}{\equiv} & (\vec{p}_1^{\,2} + \vec{p}_1 \cdot \vec{p}_2 + \vec{p}_2^{\,2})\Big(-\delta^{j_1 j_2}(2\delta^{i_1 k_1}\delta^{i_2 k_2} - \delta^{i_1 i_2}\delta^{k_1 k_2}) \\
& \quad + 2\big[\delta^{i_1 j_1}(4\delta^{i_2 k_1}\delta^{j_2 k_2} - \delta^{i_2 j_2}\delta^{k_1 k_2}) - \delta^{i_1 i_2}\delta^{j_1 k_1}\delta^{j_2 k_2}\big]\Big) \\
& + 2\Big\{ 4(p_1^{k_2} p_2^{i_2} - p_1^{i_2} p_2^{k_2})\delta^{i_1 j_1}\delta^{j_2 k_1} \\
& \quad + 2\big[(p_1^{i_1} + p_2^{i_1}) p_2^{i_2}\delta^{j_1 k_1}\delta^{j_2 k_2} - p_1^{k_1} p_2^{k_2}\delta^{i_1 j_1}\delta^{i_2 j_2}\big] \\
& \quad + \delta^{j_1 j_2}\big[p_1^{k_1} p_2^{k_2}\delta^{i_1 i_2} + 2(p_1^{k_2} p_2^{i_2} - p_1^{i_2} p_2^{k_2})\delta^{i_1 k_1} - (p_1^{i_1} + p_2^{i_1}) p_2^{i_2}\delta^{k_1 k_2}\big] \\
& \quad + p_2^{j_2}\Big(4 p_1^{i_2}\delta^{i_1 k_1}\delta^{j_1 k_2} + p_1^{j_1}(2\delta^{i_1 k_1}\delta^{i_2 k_2} - \delta^{i_1 i_2}\delta^{k_1 k_2}) \\
& \qquad + 2\big[\delta^{i_1 j_1}(p_1^{i_2}\delta^{k_1 k_2} - 2 p_1^{k_2}\delta^{i_2 k_1}) - p_1^{k_2}\delta^{i_1 i_2}\delta^{j_1 k_1}\big]\Big) \\
& \quad + p_1^{j_2}\Big(p_1^{j_1}(2\delta^{i_1 k_1}\delta^{i_2 k_2} - \delta^{i_1 i_2}\delta^{k_1 k_2}) - 4 p_2^{i_2}\delta^{i_1 k_1}\delta^{j_1 k_2} \\
& \qquad + 2\big[p_2^{k_2}\delta^{i_1 i_2}\delta^{j_1 k_1} + \delta^{i_1 j_1}(2 p_2^{k_2}\delta^{i_2 k_1} - p_2^{i_2}\delta^{k_1 k_2})\big]\Big)\Big\}
\end{aligned}$$

Potential matching

5PN calculation

1. Generate diagrams with up to 5 loops with `QGRAF` [Nogueira 1991]
2. Discard unwanted diagrams, e.g. graviton loops
3. Compute and insert Feynman rules with `FORM` [Vermaseren et al.]
4. Reduce massless propagators to master integrals using Laporta's algorithm [Chetyrkin, Tkachov 1981, Laporta 2000] implemented in `crusher`

$V_{5\text{PN}} \overset{v=0}{=} c_0$  $+\, c_1$  $+\, c_2$ 

$+\, c_3$  $+\, \mathcal{O}(\epsilon)$

c_j: Laurent series in $\epsilon = \frac{3-d}{2}$,
polynomials in m_1, m_2, r^{-1}, G^{-1}

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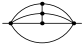
5. Insert known (factorising) master integrals

[Lee, Mingulov 2015; Damour, Jaranowski 2017]



$$= 6\pi^{7/2}\left[\frac{2}{\epsilon} - 4 - 4\ln(2) + \mathcal{O}(\epsilon^1)\right]$$

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$$V_{5PN} \overset{v=0}{=} \frac{G^6}{r^6} m_1 m_2 \left[\frac{5}{16}(m_1^5 + m_2^5) + \frac{91}{6} m_1 m_2 (m_1^3 + m_2^3) + \frac{653}{6} m_1^2 m_2^2 (m_1 + m_2) \right]$$

Potential matching

Known results

Confirmation of previous results, $x\text{PN} \mathrel{\hat{=}} \left(G, v^2\right)^{x+1}$:

- 1PN: [Goldberger, Rothstein 2004]
- 2PN: [Gilmore, Ross 2008]
- 3PN: [Foffa, Sturani 2011]
- 4PN:
  - “static” contribution $v = 0$:
    [Foffa, Mastrolia, Sturani, Sturm 2016; Damour, Jaranowski 2017]
  - $v \neq 0$: [Foffa, Sturani 2019; Foffa, Porto, Rothstein, Sturani 2019]
  - [Blümlein, Maier, Marquard, Schäfer 2020]

confirming [Damour, Jaranowski, Schäfer 2014; Bernard, Blanchet, Bohé, Faye, Marchant, Marsat 2017]

New:

- 5PN static contribution:
  [Foffa, Mastrolia, Sturani, Sturm, Torres Bobadilla 2019; Blümlein, Maier, Marquard 2019]
- 5PN $v \neq 0$ [Blümlein, Maier, Marquard, Schäfer, 2020]
- Partial 6PN $v \neq 0$ [Blümlein, Maier, Marquard, Schäfer, 2020 + 2021]

Classical theory

V_{NZ} is not physical:

- Gauge dependent
- Infrared divergence at $\geq$4PN

$\Rightarrow$ combine with contribution from radiation/ultrasoft modes

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Construct Post-Newtonian Lagrangian *without* any field:

$$S_{\mathrm{PN}}[x_a, v_a] = \int dt\, (T - V - \Gamma)$$

Absorb radiation modes radiation/ultrasoft modes into

- Far-zone potential V_{FZ} ("tail")

$$V = V_{\mathrm{NZ}} + V_{\mathrm{FZ}}$$

- Radiation loss Γ

Radiation interaction

- Matter-radiation interaction in electrodynamics:

$$S \supset \int d^d x \, J^\mu A_\mu$$

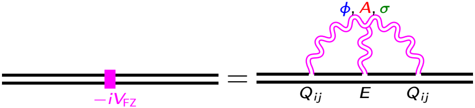
- Matter-radiation interaction in NRGR:

$$S_{\text{mixed}} = \frac{1}{2} \int d^d x \, T^{\mu\nu} \delta g_{\mu\nu} + \mathcal{O}(\delta g_{\mu\nu}^2), \qquad \delta g_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$$

Wavelength of radiation modes: $\lambda \sim \frac{r}{v} \gg r$
$\Rightarrow$ multipole expansion
$\Rightarrow$ ϕ, A_i, σ_{ij} coupling to multipole moments $E, P_i, L_i, Q_{ij}, \ldots$

Far-zone potential

Matching at 4PN:



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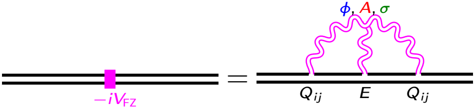


At 5PN: [Foffa, Sturani 2019–2021]

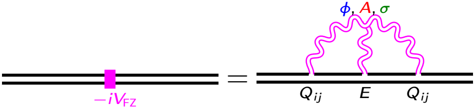
- additional multipole moments J_{ij}, O_{ijk}
- More 2-loop diagrams
- 1PN corrections to E, Q_{ij} in d dimensions

  [Marchand, Henry, Larrouturou, Marsat, Faye, Blanchet 2020]

Combine with V_{NZ} and compute 5PN energy:

Far-zone potential

Matching at 4PN:



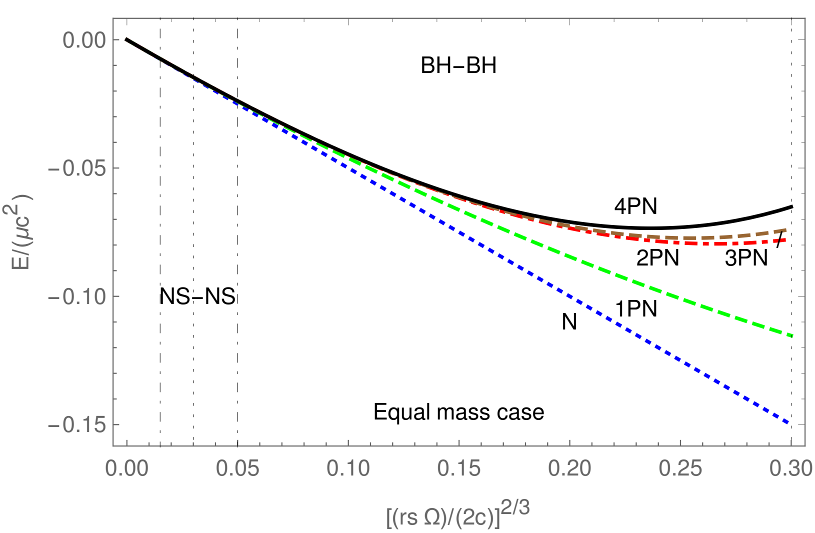
At 5PN: [Foffa, Sturani 2019–2021]

- additional multipole moments J_{ij}, O_{ijk}
- More 2-loop diagrams
- 1PN corrections to E, Q_{ij} in d dimensions

[Marchand, Henry, Larrouturou, Marsat, Faye, Blanchet 2020]

Combine with V_{NZ} and compute 5PN energy:

Post-Newtonian Energy



Conclusion

- Measurements require accurate, fast, complete gravitational wave form computations
- Post-Newtonian (PN) expansion for black-hole binaries using particle physics techniques
  - Non-relativistic effective field theory
  - Multiloop Feynman integrals
- Latest result: 5PN near-zone potential

  [Blümlein, Maier, Marquard, Schäfer, 2020]

  partial results: [Foffa, Mastrolia, Torres Bobadilla 2020]

  $v = 0$: [Foffa, Mastrolia, Sturani, Sturm, Torres Bobadilla 2019; Blümlein, Maier, Marquard 2019]
- 5PN conservative dynamics complete, but inconsistent with self-force results