New $\tau$-based evaluation of the hadronic contribution to the vacuum polarization piece of the muon anomalous magnetic moment

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IFAE Seminar

## Outline

(1) Motivation
(2) Hadronic vacuum polarization
(3) $\tau^{-} \rightarrow \pi^{-} \pi^{0} \gamma \nu_{\tau}$ decays
(4) Isospin-breaking corrections to $a_{\mu}^{\mathrm{HVP}, \mathrm{LO} \pi \pi}$
(5) Conclusions

## Motivation

- The fermionic anomalous magnetic moment is given by

$$
\begin{equation*}
\vec{\mu}=g \frac{Q e}{2 m} \vec{S}, \text { where } Q= \pm 1 \text { and } e>0 \tag{1}
\end{equation*}
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One of the biggest successes of the Dirac theory was the prediction of $g \equiv 2$.

- For a few years, this situation was kept. The electron had $\mathrm{g}=2$ and the Dirac eq. seems to describe nature.
- Motivated by an excess in the measurements of the hyperfine structure of hydrogen atom, in 1937 J. Schwinger showed that these discrepancies can be explained by an additional contribution from QED,

$$
\begin{equation*}
\frac{\delta \mu}{\mu}=\frac{\alpha}{2 \pi} \simeq 0.001162 \tag{2}
\end{equation*}
$$

- This prediction was confirmed by the Kush and Foley experiments. Since the electron anomaly has been measured up to a few ppb $\left(10^{-9}\right)$, the QED calculation has to be extended to tenth-order (5 loops).
- It is useful to split the magnetic moment into two terms:

$$
\begin{equation*}
\mu_{\ell}=\left(1+a_{\ell}\right) \frac{e \hbar}{2 m}, \text { where } a_{\ell} \equiv \frac{g_{\ell}-2}{2} . \tag{3}
\end{equation*}
$$

- The anomalous magnetic moment of electron ( $a_{e}$ ) and muon $\left(a_{\mu}\right)$ have been measured with a precision of a few $\mathrm{ppb}\left(10^{-9}\right)$ and $\mathrm{ppm}\left(10^{-6}\right)$, respectively.
- A deviation of $a_{\ell}^{\text {exp }}$ concerning the SM theoretical value would be a signal of NP.
- Since the contributions from heavier NP to $a_{\ell}$ are proportional to $\Delta a_{\ell} \sim \frac{m_{\ell}^{2}}{M_{N P}^{2}}$, the NP effects in $a_{\mu}$ are magnified by a factor $\left(m_{\mu} / m_{e}\right)^{2} \sim 4 \times 10^{4}$. For the $a_{\tau}$ these effects would be better, however the short lifetime of the $\tau$ makes it harder to measure.

$$
\begin{equation*}
\tau_{\mu}=(2.1969811 \pm 0.0000022) \times 10^{-6} s, \quad \tau_{\tau}=(290.3 \pm 0.5) \times 10^{-15} s \tag{4}
\end{equation*}
$$

- Nowadays, there is a discrepancy between the theoretical prediction and the experimental value for both, $\ell=e, \mu$,

$$
\begin{aligned}
\Delta a_{\mu} & \equiv a_{\mu}^{\exp }-a_{\mu}^{S M}=(2.79 \pm 0.76) \times 10^{-9}, \quad \Rightarrow+3.7 \sigma \\
\Delta a_{e} & \equiv a_{e}^{e x p}-a_{e}^{S M}=(4.8 \pm 3.0) \times 10^{-13}, \quad \Rightarrow+1.6 \sigma
\end{aligned}
$$

## The Anomalous Magnetic Moment of the Muon



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- In the SM, the Anomalous Magnetic Moment of the Muon receives three contributions:

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\begin{equation*}
a_{\mu}^{\text {th }}=a_{\mu}^{Q E D}+a_{\mu}^{\text {weak }}+a_{\mu}^{\text {had }} \tag{5}
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- The QED contribution is the most important and has been evaluated up to tenth order in the perturbative expansion, i.e., $\mathcal{O}\left(\alpha^{5}\right)$.
- Electroweak contributions are suppressed by a factor $\left(m_{\mu} / M_{W}\right)^{2}$ and have been calculated up to two loops.


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\end{equation*}
$$

- The hadronic vacuum polarization (HVP) contributions arise at $\mathcal{O}\left(\alpha^{2}\right)$ and the hadronic light-by-light scattering (HLbL) contributions at $\mathcal{O}\left(\alpha^{3}\right)$.





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An impressive effort was made by the Muon g-2 Theory Initiative to publish a consensus value for the muon's magnetic moment.

| Contribution | Value $\left(\times 10^{11}\right)$ |
| :---: | :---: |
| QED | $116584718.931 \pm 0.104$ |
| Electroweak | $153.6 \pm 1.0$ |
| HVP LO $\left(e^{+} e^{-}\right)$ | $6931 \pm 40$ |
| HVP NLO $\left(e^{+} e^{-}\right)$ | $-98.3 \pm 0.7$ |
| HVP NNLO $\left(e^{+} e^{-}\right)$ | $12.4 \pm 0.1$ |
| HLBL (phenomenology+lattice + NLO $)$ | $92 \pm 18$ |
| Total | $116591810 \pm 43$ |

For more details
Phys.Rept. 887 (2020) 1-166

- The most precise measurement was obtained by the muon $g-2$ experiment E821 at BNL using a storage ring. They measured $a_{\mu}$ for both $\mu^{+}$and $\mu^{-}$:

$$
\begin{align*}
& a_{\mu^{+}}^{\exp }=11659204(6)_{\text {stat }}(5)_{\text {syst }} \times 10^{-10} \\
& a_{\mu^{-}}^{\exp }=11659215(8)_{\text {stat }}(3)_{\text {syst }} \times 10^{-10} \tag{6}
\end{align*}
$$

- Supposing CPT-invariance and correlation between systematics, it is found

$$
\begin{equation*}
a_{\mu}^{e x p}=11659208.9(5.4)_{\text {stat }}(3.3)_{\text {syst }}(6.3)_{\mathrm{tot}} \times 10^{-10}( \pm 0.54 \mathrm{ppm}) \tag{7}
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- With these values,

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- Soon there will be two new measurements of $a_{\mu}$, one from the Muon $g-2$ experiment at Fermilab/USA and the other from the E34 experiment at J-PARC/Japan, that aim to reduce the errors by a factor of 4 .


## HVP

- Based on analyticity and unitarity, loop integrals containing HVP insertions in photon propagators can be expressed as dispersive integrals over the cross-section of a virtual photon decaying into hadrons:

$$
\begin{equation*}
a_{\mu}^{\mathrm{HVP}, \mathrm{LO}}=\frac{\alpha^{2}}{3 \pi^{2}} \int_{m_{\pi}^{2}}^{\infty} d s \frac{K(s)}{s} R(s), \tag{9}
\end{equation*}
$$

where $K(s)$ is a Kernel function $\Rightarrow K(s) \sim 1 / s$,

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\begin{equation*}
R(s)=\frac{\sigma^{0}\left(e^{+} e^{-} \rightarrow \text { hadrons }(+\gamma)\right)}{\sigma_{p t}}, \quad \sigma_{p t}=\frac{4 \pi \alpha^{2}}{3 s} \tag{10}
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- An evaluation of the $a_{\mu}^{\mathrm{HVP}, \mathrm{LO}}$ can be obtained from the measurements of $\sigma\left(e^{+} e^{-} \rightarrow\right.$ hadrons $)$ or the $\tau \rightarrow \nu_{\tau}+$ hadrons decays which can be related to the isovector component of the $e^{+} e^{-} \rightarrow$ hadrons cross-section through isospin-symmetry.


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- Since both are subject to theoretical uncertainties, it is a good strategy to keep using $e^{+} e^{-}$and $\tau$ data.
- About $73 \%$ of the contributions to the $a_{\mu}^{\text {HVP }}$ and $58 \%$ of the total uncertainty correspond to the $\pi^{+} \pi^{-}(\gamma)$ final state at low-energies $\left(4 m_{\pi}^{2} \leq s \leq 0.8 \mathrm{GeV}^{2}\right)$.
- For the two-pion final state,

$$
\begin{equation*}
\sigma_{\pi^{+} \pi^{-}}(s)=\frac{\pi \alpha^{2} \beta_{\pi^{-} \pi^{+}}^{3}(s)}{3 s}\left|F_{V}(s)\right|^{2} \tag{11}
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- Including isospin-breaking corrections at LO, we have

$$
\begin{equation*}
\sigma_{\pi^{+} \pi^{-}}(s)=\left[\frac{K_{\sigma}(s)}{K_{\Gamma}(s)} \frac{d \Gamma_{\pi \pi[\gamma]}}{d s}\right] \frac{R_{l B}(s)}{S_{E W}}, \quad s \equiv\left(p_{\pi^{-}}+p_{\pi^{0}}\right)^{2} \tag{12}
\end{equation*}
$$

where $R_{I B}(s)=\frac{F S R(s)}{G_{E M}(s)} \frac{\beta^{3}}{\beta_{\pi^{+} \pi^{-}}}\left|\frac{F_{V}(s)}{f_{+}(s)}\right|^{2}$,

$$
\begin{align*}
& K_{\Gamma}(s)=\frac{G_{F}^{2}\left|V_{u d}\right|^{2} m_{\tau}^{3}}{384 \pi^{3}}\left(1-\frac{s}{m_{\tau}^{2}}\right)^{2}\left(1+\frac{2 s}{m_{\tau}^{2}}\right),  \tag{13}\\
& K_{\sigma}(s)=\frac{\pi \alpha^{2}}{3 s},
\end{align*}
$$

- $G_{E M}(s)$ receives contributions from real and virtual photons ${ }^{1}$.

[^0]- The IR divergencies that appear at NLO for $\tau^{-} \rightarrow \pi^{-} \pi^{0} \nu_{\tau}$ decays are canceled out by the IR divergencies of the radiative decay.


Figure: Contributions at NLO for $\tau^{-} \rightarrow \pi^{-} \pi^{0} \nu_{\tau}$ decays. ${ }^{2}$

[^1]- There is a discrepancy between the values of $a_{\mu}^{H V P, L O}[\pi \pi]$ obtained through $e^{+} e^{-}$and $\tau$ decays. According to Cirigliano et al. [Phys.Rev.Lett. 122 (2019)] this could be a NP effect,

$$
\begin{equation*}
\frac{a_{\mu}^{\tau}-a_{\mu}^{e e}}{2 a_{\mu}^{e e}}=\epsilon_{L}^{\tau}-\epsilon_{L}^{e}+\epsilon_{R}^{\tau}-\epsilon_{R}^{e}+1.7 \epsilon_{T}^{\tau} \tag{14}
\end{equation*}
$$

- A global fit using hadronic tau decays to set bounds on NP effective couplings at the low-energy limit of SMEFT was studied in Gonzàlez-Solís et al. [Phys.Lett.B 804 (2020)].
- NP effects in $\tau^{-} \rightarrow \pi^{-} \pi^{0} \nu_{\tau}$ decays have been studied using an effective field theory setup for some observables in Miranda \& Roig, JHEP 1811 (2018).
- Although the determinations of $a_{\mu}^{\mathrm{HVP}}$, LO using lattice QCD are still not competitive with respect to the $e^{+} e^{-}$based evaluation, there is one lattice calculation by the BMW collaboration with an error of $\pm 53 \cdot 10^{-11}$ in which the difference concerning the experimental value is reduced to $\sim 1 \sigma$.
- There is a solution given by Jegerlehner \& Szafron that induces an additional correction due to the $\rho-\gamma$ mixing in which $\rho^{0}$ is regarded as a gauge boson.

$$
\tau^{-}(P) \rightarrow \pi^{-}\left(p_{-}\right) \pi^{0}\left(p_{0}\right) \gamma(k) \nu_{\tau}(q)
$$

- The matrix element for these decays has the following structure:

$$
\begin{gather*}
T=e G_{F} V_{u d}^{*} \epsilon^{\mu}(k)^{*}\left\{F_{\nu} \bar{u}(q) \gamma^{\nu}\left(1-\gamma_{5}\right)\left(m_{\tau}+\not P-k\right) \gamma_{\mu} u(P)\right. \\
\left.+\left(V_{\mu \nu}-A_{\mu \nu}\right) \bar{u}(q) \gamma^{\nu}\left(1-\gamma_{5}\right) u(P)\right\} \tag{15}
\end{gather*}
$$

where $F_{\nu}=\left(p_{0}-p_{-}\right)_{\nu} f_{+}(s) / 2 p \cdot k$ with $s \equiv\left(p_{-}+p_{0}\right)^{2}$.

- The $V_{\mu \nu}-A_{\mu \nu}$ term can be split into two parts, structure-independent (SI) and structure-dependent (SD), according to the Low and Burnett-Kroll theorems.

$$
\begin{aligned}
& V^{\mu \nu}=V_{S I}^{\mu \nu}+V_{S D}^{\mu \nu} \\
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$$

- At low-energies, the SM of electroweak and strong interactions is described by an effective field theory known as Chiral Perturbation Theory ( $\chi P T$ ).
- There is a difference for the effects of the SD part when they are evaluated using $\chi P T$ with resonances ${ }^{3}$ and VMD ${ }^{4}$.

[^4]
## Contributions at $\mathcal{O}\left(p^{4}\right)$

- At $\mathcal{O}\left(p^{4}\right)$ in $\chi P T$ with resonances $(R \chi T)$, the diagrams that contribute to these decays are ${ }^{5}$ :


[^5]
## Contributions at $\mathcal{O}\left(p^{6}\right)$

- Using the basis given by Cirigliano et al. Nucl. Phys. B753 (2006) and Kampf \& Novotný. Phys. Rev. D84 (2011), we get the following contributions at $\mathcal{O}\left(p^{6}\right)$ :

for $V_{\mu \nu}$, and

for $A_{\mu \nu}$.


## Short-distance constraints

- Including the contributions up to $\mathcal{O}\left(p^{6}\right)$, we have now too many parameters allowed by the discrete symmetries of QCD and chiral symmetry that prevent making phenomenological predictions.
- It is possible to reduce the number of couplings using the SD properties of QCD and its OPE.
- To estimate the remaining parameters, we rely on chiral counting and estimation of the LECs $C_{i}^{R}$ of the $\mathcal{O}\left(p^{6}\right) \chi P T$ Lagrangian ${ }^{6}$.
- Since the $\kappa_{i}^{V}$ couplings, which are related to the $\omega$-exchange, contribute significantly to the radiative decays, we perform a global fit using the relations for the resonance saturation of the anomalous sector LECs at $\mathcal{O}\left(p^{6}\right)^{78}$.

[^6]
## Decay spectrum

- In this work, we use the dispersive representation of the vector form factor ${ }^{9}$ instead of the exponential parameterization ${ }^{10}$ used by Cirigliano, Ecker \& Neufeld. JHEP 0208 (2002).


Figure: The $\pi^{-} \pi^{0}$ hadronic invariant mass distributions for $E_{\gamma}^{\text {cut }}=300 \mathrm{MeV}$.

[^7]
## Photon energy distribution



## Branching ratio



| $E_{\gamma}^{\text {cut }}$ | $\operatorname{BR}\left[\mathcal{O}\left(p^{4}\right)\right]$ | $\operatorname{BR}\left[\mathcal{O}\left(p^{6}\right)\right]$ |
| :---: | :---: | :---: |
| 100 MeV | $\left(9.5_{-0.5}^{+3.5}\right) \cdot 10^{-4}$ | $(1.9 \pm 0.3) \cdot 10^{-3}$ |
| 300 MeV | $\left(2.3_{-0.4}^{+2.8}\right) \cdot 10^{-4}$ | $(1.1 \pm 0.3) \cdot 10^{-3}$ |
| 500 MeV | $\left(0.5_{-0.2}^{+1.9}\right) \cdot 10^{-4}$ | $(0.6 \pm 0.2) \cdot 10^{-3}$ |

## $G_{E M}(s)$

- Adding the contributions due to virtual and real photons and integrating over $u \equiv\left(P-p_{1}\right)^{2}$, we get the $G_{E M}(s)$ function,

$$
\begin{equation*}
\left.\frac{d \Gamma}{d s}\right|_{\pi \pi[\gamma]}=\frac{G_{F}^{2}\left|V_{u d}\right|^{2} m_{\tau}^{3} S_{E W}}{384 \pi^{3}}\left|f_{+}(s)\right|^{2}\left(1-\frac{s}{m_{\tau}^{2}}\right)^{2}\left(1-\frac{4 m_{\pi}^{2}}{s}\right)^{3 / 2}\left(1+\frac{2 s}{m_{\tau}^{2}}\right) G_{E M}(s) \tag{16}
\end{equation*}
$$




## $\Delta a_{\mu}^{\mathrm{HVP}, \mathrm{LO}}$

We can estimate the effect of each IB correction through $\Delta a_{\mu}^{\mathrm{HVP}}$, LO

$$
\begin{equation*}
\Delta a_{\mu}^{\mathrm{HVP}, \mathrm{LO}}=\frac{1}{4 \pi^{3}} \int_{s_{1}}^{s_{2}} d s K(s)\left[\frac{K_{\sigma}(s)}{K_{\Gamma}(s)} \frac{d \Gamma_{\pi \pi[\gamma]}}{d s}\right]\left(\frac{R_{I B}(s)}{S_{E W}}-1\right), \tag{17}
\end{equation*}
$$

Contributions to $\triangle a_{\mu}^{H V P, L O}$ in units of $10^{-11}$ using the dispersive representation of the form factor.

| $\left[s_{1}, s_{2}\right]$ | $\Delta a_{\mu, \mathrm{G}_{\mathrm{EM}}^{(0)}}^{\mathrm{HVP}, \mathrm{LO}}$ | $\Delta a_{\mu, \mathrm{SI}}^{\mathrm{HVP}, \mathrm{LO}}$ | $\Delta a_{\mu,\left[\mathcal{O}\left(p^{4}\right)\right]}^{\mathrm{HVP}, \mathrm{LO}}$ | $\Delta a_{\mu,\left[\mathcal{O}\left(p^{4}\right)\right]}^{\mathrm{HVP}, \mathrm{LO}}$ | $\Delta a_{\mu,[S D]}^{\mathrm{HVP}, \mathrm{LO}}$ | $\Delta a_{\mu,\left[\mathcal{O}\left(p^{6}\right)\right]}^{\mathrm{HVP}, \mathrm{LO}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $4 m_{\pi}^{2}, 1 \mathrm{GeV}^{2}$ | +17.8 | -11.0 | -11.3 | -17.0 | -32.4 | $-74.8 \pm 44.0$ |
| $4 m_{\pi}^{2}, 2 \mathrm{GeV}^{2}{ }^{2}$ | +18.3 | -10.1 | -10.3 | -16.0 | -31.9 | $-75.9 \pm 45.5$ |
| $\left[4 m_{\pi}^{2}, 3 \mathrm{GeV}^{2}\right]$ | +18.4 | -10.0 | -10.2 | -15.9 | -31.9 | $-75.9 \pm 45.6$ |
| $\left[4 m_{\pi}^{2}, m_{\tau}^{2}\right]$ | +18.4 | -10.0 | -10.2 | -15.9 | -31.9 | $-75.9 \pm 45.6$ |

## $\Delta a_{\mu}^{\mathrm{HVP}, \mathrm{LO}}$

IB contributions to $a_{\mu}^{\mathrm{HVP}, \mathrm{LO}}[\pi \pi, \tau]$ at $\mathcal{O}\left(p^{4}\right)$.

- For FF1 we use the following numerical inputs: $\theta_{\rho \omega}=(-3.5 \pm 0.7) \times 10^{-3} \mathrm{GeV}^{2}$ [JHEP 08 (2002) 002], $\Gamma_{\rho^{0}}-\Gamma_{\rho^{+}}=0.3 \pm 1.3 \mathrm{MeV}, m_{\rho^{ \pm}}-m_{\rho^{0}}=0.7 \pm 0.8 \mathrm{MeV}$ and $m_{\rho^{0}}=775.26 \pm 0.25 \mathrm{MeV}$ from PDG.
- For FF2 we use the same inputs as FF1 except by $\Gamma_{\rho^{0} \rightarrow \pi^{+} \pi^{-} \gamma}-\Gamma_{\rho^{ \pm} \rightarrow \pi^{ \pm} \pi^{0} \gamma}=0.45 \pm 0.45 \mathrm{MeV}$ [JHEP 08 (2002) 002].

|  | $\Delta a_{\mu}^{\mathrm{HVPP}, \mathrm{LO}}[\pi \pi, \tau]\left(\times 10^{11}\right)$ |  |
| :---: | :---: | :---: |
| Source | FF1 |  |
| SFW $^{2}$ | FF2 |  |
| PS | -703.1 |  |
| FSR | -74.5 |  |
| FF | $+40.9 \pm 48.9 \quad+75.5 \pm 4.6$ |  |
| EM | $-15.9_{-16.0}^{+5.7}$ |  |
| Total | $-107.1_{-51.7}^{+49.4}-70.4_{-29.2}^{+25.1}$ |  |

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- For FF1 we use the following numerical inputs: $\theta_{\rho \omega}=(-3.5 \pm 0.7) \times 10^{-3} \mathrm{GeV}^{2}$ [JHEP 08 (2002) 002], $\Gamma_{\rho^{0}}-\Gamma_{\rho^{+}}=0.3 \pm 1.3 \mathrm{MeV}, m_{\rho^{ \pm}}-m_{\rho^{0}}=0.7 \pm 0.8 \mathrm{MeV}$ and $m_{\rho^{0}}=775.26 \pm 0.25 \mathrm{MeV}$ from PDG.
- For FF2 we use the same inputs as FF1 except by $\Gamma_{\rho^{0} \rightarrow \pi^{+} \pi^{-} \gamma}-\Gamma_{\rho^{ \pm} \rightarrow \pi^{ \pm} \pi^{0} \gamma}=0.45 \pm 0.45 \mathrm{MeV}$ [JHEP 08 (2002) 002].

|  | $\Delta a_{\mu}^{\mathrm{HVP}, \mathrm{LO}}[\pi \pi, \tau]\left(\times 10^{11}\right)$ |  |
| :---: | :---: | :---: |
| Source | FF1 | FF2 |
| S $_{\text {EW }}$ | -103.1 |  |
| PS | -74.5 |  |
| FSR | $+45.5 \pm 4.6$ |  |
| FF | $+40.9 \pm 48.9 \quad+77.6 \pm 24.0$ |  |
| EM | $-75.9_{-45.6}^{+65.7}$ |  |
| Total | $-167.1_{-67.0}^{+82.0}-130.4_{-51.7}^{+70.1}$ |  |

- It is possible to estimate the branching ratio $B_{\pi \pi^{0}}=\Gamma\left(\tau \rightarrow \pi \pi^{0} \nu_{\tau}\right) / \Gamma_{\tau}$ using $e^{+} e^{-}$data

$$
\begin{equation*}
B_{\pi \pi^{0}}^{C V C}=B_{e} \int_{4 m_{\pi}^{2}}^{m_{\tau}^{2}} d s \sigma_{\pi^{+} \pi^{-}(\gamma)}(s) \mathcal{N}(s) \frac{S_{E W}}{R_{I B}(s)} \tag{18}
\end{equation*}
$$

where $\mathcal{N}(s)=\frac{3\left|V_{u d}\right|^{2}}{2 \pi \alpha_{0}^{2} m_{\tau}^{2}} s\left(1-\frac{s}{m_{\tau}^{2}}\right)^{2}\left(1+\frac{2 s}{m_{\tau}^{2}}\right)$.
Using the most recent data obtained from BaBar for the $e^{+} e^{-} \rightarrow \pi^{+} \pi^{-}(\gamma)$ cross section, we get

$$
\begin{equation*}
B_{\pi \pi^{0}}^{C V C}=\left(24.68 \pm 0.11 \pm 0.10 \pm 0.01 \pm 0.01 \pm 0.02_{-0.00}^{+0.03}\right) \%, \text { at } \mathcal{O}\left(p^{4}\right) \tag{19}
\end{equation*}
$$

and

$$
\begin{equation*}
B_{\pi \pi^{0}}^{C V C}=\left(24.70 \pm 0.11 \pm 0.10 \pm 0.01 \pm 0.01 \pm 0.02_{-0.01}^{+0.21}\right) \%, \text { at } \mathcal{O}\left(p^{6}\right) \tag{20}
\end{equation*}
$$

- These results are in good agreement with the value reported by the Belle collaboration, $B_{\pi \pi^{0}}^{\tau}=(25.24 \pm 0.01 \pm 0.39) \%$ at $1.3 \sigma(1.2 \sigma)$,
and ALEPH, $B_{\pi \pi^{0}}^{\tau}=(25.471 \pm 0.097 \pm 0.085) \%$ at $4.0 \sigma(1.2 \sigma)$.

Comparison between the different data sets from BaBar (above) and KLOE (below) with $\Delta \Gamma_{\pi \pi \gamma}=1.5 \mathrm{MeV}$ (left-hand) and $\Delta \Gamma_{\pi \pi \gamma}=0.45 \mathrm{MeV}$ (right-hand) using the Belle spectrum.


Taking into account all di-pion tau decay data from ALEPH, Belle, CLEO and OPAL Colls., we get

$$
\begin{equation*}
10^{10} \cdot a_{\mu}^{H V P,\left.L O\right|_{\pi \pi, \tau \text { data }}=519.6 \pm 2.8_{\text {spectra }+B R s_{-2.1 I B}^{+1.9}}, \text { at } \mathcal{O}\left(p^{4}\right), ~ ., ~} \tag{21}
\end{equation*}
$$

and

When these results are supplemented with the four-pion tau decays measurements and with $e^{+} e^{-}$data ${ }^{11}$, we find the overall HVP LO contribution

$$
\begin{equation*}
\left.10^{10} \cdot a_{\mu}^{H V P, L O}\right|_{\tau \text { data }}=705.7_{-4.1}^{+4.0}, \text { at } \mathcal{O}\left(p^{4}\right) \tag{23}
\end{equation*}
$$

and

$$
\begin{equation*}
10^{10} \cdot a_{\mu}^{H V P,\left.L O\right|_{\tau \text { data }}=700.7_{-5.2}^{+6.1}, \text { at } \mathcal{O}\left(p^{6}\right) . . . . . .} \tag{24}
\end{equation*}
$$

${ }^{11}$ Eur. Phys. J., C80(3):241, 2020, Eur. Phys. J. C, 74(3):2803, 2014.

When all other (QED, EW and subleading hadronic) contributions are added, the $3.7 \sigma$ deficit of the SM prediction with respect to the BNL measurement is reduced to

$$
\begin{equation*}
\Delta a_{\mu} \equiv a_{\mu}^{e x p}-a_{\mu}^{S M}=(15.3 \pm 7.7) \cdot 10^{-10} \tag{25}
\end{equation*}
$$

at $\mathcal{O}\left(p^{4}\right)$, and

$$
\begin{equation*}
\Delta a_{\mu} \equiv a_{\mu}^{\exp }-a_{\mu}^{S M}=\left(20.3_{-8.9}^{+8.3}\right) \cdot 10^{-10} \tag{26}
\end{equation*}
$$

at $\mathcal{O}\left(p^{6}\right)$, which are 2.0 and $2.3 \sigma$, respectively.

## Conclusions

- There is a global effort in improving the hadronic contributions to $a_{\mu}$. Specifically, dedicated studies to improve the HVP part from lattice, dispersion relations and improved $e^{+} e^{-}$data and Monte Carlos are being undertaken.
- The observables for the $\tau \rightarrow \pi \pi \gamma \nu_{\tau}$ decays have the potential to reduce drastically the errors in our estimation.
- Our IB corrections improve the agreement between $e^{+} e^{-}$and tau data, on the spectrum and the branching ratio.
- Evaluating the HVP, LO contributions from tau data, we get
$a_{\mu}^{H V P,\left.L O\right|_{\tau \text { data }}}=\left(705.7_{-4.1}^{+4.0}\right) \cdot 10^{-10}$ at $\mathcal{O}\left(p^{4}\right)$, and $a_{\mu}^{H V P,\left.L O\right|_{\tau} \text { data }}=\left(700.7_{-5.2}^{+6.1}\right) \cdot 10^{-10}$ at $\mathcal{O}\left(p^{6}\right)$. This reduces the anomaly $\Delta a_{\mu} \equiv a_{\mu}^{\text {exp }}-a_{\mu}^{S M}$ to 2.0 and $2.3 \sigma$, respectively.


## References

- V. Cirigliano, G. Ecker and H. Neufeld, "Radiative tau decay and the magnetic moment of the muon," JHEP 0208, 002 (2002)
- V. Cirigliano, G. Ecker and H. Neufeld, "Isospin violation and the magnetic moment of the muon," Phys. Lett. B 513, 361 (2001)
- J. Miranda and P. Roig. "New $\tau$-based evaluation of the hadronic contribution to the vacuum polarization piece of the muon anomalous magnetic moment", Phys.Rev.D 102 (2020) 114017

The $\pi^{+} \pi^{-}$cross section from the KLOE combination compared to the BABAR, CMD-2, SND, and BESIII data points in the $0.6-0.9 \mathrm{GeV}$ range ${ }^{12}$.


## $a_{\mu}^{\mathrm{HVP}, \mathrm{LO}}[\pi \pi]$ from $e^{+} e^{-}$data

Comparison of results for $a_{\mu}^{\mathrm{HVP}, \mathrm{LO}}[\pi \pi]$, evaluated between 0.6 GeV and 0.9 GeV for the various experiments ${ }^{13}$.


## $e^{+} e^{-}$vs $\tau$ data

Relative comparison between the combined $\tau$ (after the IB corrections) and $e^{+} e^{-} \rightarrow \pi^{+} \pi^{-}$ spectral function contributions ${ }^{14}$.


The measured branching fractions for $\tau^{-} \rightarrow \pi^{-} \pi^{0} \nu_{\tau}$ compared to the predictions from the $e^{+} e^{-} \rightarrow \pi^{+} \pi^{-}$spectral functions, applying the IB corrections ${ }^{15}$.


## The anomalous magnetic moment of the muon

## No consensus in theory number:



## $\rho-\omega$ mixing

$\rho-\gamma$ mixing corrections proposed in Eur.Phys.J.C71:1632,2011.



## Form Factor




## Short-distance constraints

Using the relations for 2-point Green functions at $\mathcal{O}\left(p^{4}\right)$, we have:

$$
\begin{equation*}
F_{V}=\sqrt{2} F \quad G_{V}=\frac{F}{\sqrt{2}} \quad F_{A}=F \tag{27}
\end{equation*}
$$

Using the relations for 2 and 3 -point Green functions at $\mathcal{O}\left(p^{6}\right)$, we have

$$
\begin{equation*}
F_{V}=\sqrt{3} F \quad G_{V}=\frac{F}{\sqrt{3}} \quad F_{A}=\sqrt{2} F . \tag{28}
\end{equation*}
$$

## Short-distance constraints

For the parameters contributing to the leading-order chiral LECs:

$$
\begin{array}{cl}
F_{V} G_{V}=F^{2}, & F_{V}^{2}-F_{A}^{2}=F^{2}, \\
F_{V}^{2} M_{V}^{2}=F_{A}^{2} M_{A}^{2}, & 4 c_{d} c_{m}=F^{2},  \tag{29}\\
8\left(c_{m}^{2}-d_{m}^{2}\right)=F^{2}, & c_{m}=c_{d}=\sqrt{2} d_{m}=F / 2
\end{array}
$$

For the even-intrinsic parity sector:

$$
\begin{array}{ll}
\lambda_{13}^{P}=0, & \lambda_{17}^{S}=\lambda_{18}^{S}=0,  \tag{30}\\
\lambda_{17}^{A}=0, & \lambda_{21}^{V}=\lambda_{22}^{V}=0 .
\end{array}
$$

The analysis of the $\langle V A S\rangle$ Green function yields:

$$
\begin{gather*}
\kappa_{2}^{S}=\kappa_{14}^{A}=0, \quad \kappa_{4}^{V}=2 \kappa_{15}^{V}, \quad \kappa_{6}^{V A}=\frac{F^{2}}{32 F_{A} F_{V}}  \tag{31}\\
F_{V}\left(2 \kappa_{1}^{S V}+\kappa_{2}^{S V}\right)=2 F_{A} \kappa_{1}^{S A}=\frac{F^{2}}{16 \sqrt{2} c_{m}}
\end{gather*}
$$

## Short-distance constraints

The study of the $\langle V A P\rangle$ and $\langle S P P\rangle$ Green functions yield the following restrictions on the resonance couplings:

$$
\begin{gather*}
\sqrt{2} \lambda_{0}=-4 \lambda_{1}^{V A}-\lambda_{2}^{V A}-\frac{\lambda_{4}^{V A}}{2}-\lambda_{5}^{V A}=\frac{1}{2 \sqrt{2}}\left(\lambda^{\prime}+\lambda^{\prime \prime}\right), \\
\sqrt{2} \lambda^{\prime}=\lambda_{2}^{V A}-\lambda_{3}^{V A}+\frac{\lambda_{4}^{V A}}{2}+\lambda_{5}^{V A}=\frac{M_{A}}{2 M_{V}}, \\
\sqrt{2} \lambda^{\prime \prime}=\lambda_{2}^{V A}-\frac{\lambda_{4}^{V A}}{2}-\lambda_{5}^{V A}=\frac{M_{A}^{2}-2 M_{V}^{2}}{2 M_{V} M_{A}},  \tag{32}\\
\lambda_{1}^{P V}=-4 \lambda_{2}^{P V}=-\frac{F \sqrt{M_{A}^{2}-M_{V}^{2}}}{4 \sqrt{2} d_{m} M_{A}}, \quad \lambda_{1}^{P A}=\frac{F \sqrt{M_{A}^{2}-M_{V}^{2}}}{16 \sqrt{2} d_{m} M_{V}} .
\end{gather*}
$$

For the odd-intrinsic parity sector:

$$
\begin{align*}
\kappa_{14}^{V} & =\frac{N_{C}}{256 \sqrt{2} \pi^{2} F_{V}}, \quad 2 \kappa_{12}^{V}+\kappa_{16}^{V}=-\frac{N_{C}}{32 \sqrt{2} \pi^{2} F_{V}}, \quad \kappa_{17}^{V}=-\frac{N_{C}}{64 \sqrt{2} \pi^{2} F_{V}}, \quad \kappa_{5}^{P}=0, \\
\kappa_{2}^{V V} & =\frac{F^{2}+16 \sqrt{2} d_{m} F_{V} \kappa_{3}^{P V}}{32 F_{V}^{2}}-\frac{N_{C} M_{V}^{2}}{512 \pi^{2} F_{V}^{2}}, \quad 8 \kappa_{2}^{V V}-\kappa_{3}^{V V}=\frac{F^{2}}{8 F_{V}^{2}} . \tag{33}
\end{align*}
$$

## Fit results

Since the $\kappa_{i}^{V}$ couplings are related with the $\omega$ exchange which is known to give an important contribution to the $\tau \rightarrow \pi \pi \gamma \nu_{\tau}$ decays, we perform a global fit using the relations for the resonance saturation of the anomalous sector LECs ${ }^{16}$

$$
\begin{align*}
& \kappa_{1}^{V}=(-2.1 \pm 0.7) \cdot 10^{-2} \mathrm{GeV}^{-1},  \tag{34a}\\
& \kappa_{2}^{V}=(-8.8 \pm 9.1) \cdot 10^{-3} \mathrm{GeV}^{-1},  \tag{34b}\\
& \kappa_{3}^{V}=(2.2 \pm 5.8) \cdot 10^{-3} \mathrm{GeV}^{-1},  \tag{34c}\\
& \kappa_{6}^{V}=(-2.1 \pm 0.3) \cdot 10^{-2} \mathrm{GeV}^{-1},  \tag{34d}\\
& \kappa_{7}^{V}=(1.2 \pm 0.5) \cdot 10^{-2} \mathrm{GeV}^{-1},  \tag{34e}\\
& \kappa_{8}^{V}=(3.1 \pm 0.9) \cdot 10^{-2} \mathrm{GeV}^{-1},  \tag{34f}\\
& \kappa_{9}^{V}=(-0.1 \pm 5.9) \cdot 10^{-3} \mathrm{GeV}^{-1},  \tag{34g}\\
& \kappa_{10}^{V}=(-5.9 \pm 9.6) \cdot 10^{-3} \mathrm{GeV}^{-1},  \tag{34h}\\
& \kappa_{11}^{V}=(-3.0 \pm 0.6) \cdot 10^{-2} \mathrm{GeV}^{-1},  \tag{34i}\\
& \kappa_{12}^{V}=(1.0 \pm 0.8) \cdot 10^{-2} \mathrm{GeV}^{-1},  \tag{34j}\\
& \kappa_{13}^{V}=(-5.3 \pm 1.1) \cdot 10^{-3} \mathrm{GeV}^{-1},  \tag{34k}\\
& \kappa_{18}^{V}=(4.7 \pm 0.8) \cdot 10^{-3} \mathrm{GeV}^{-1} . \tag{341}
\end{align*}
$$

These values are in good agreement with our earlier estimation $\left|\kappa_{i}^{V}\right| \lesssim 0.025 \mathrm{GeV}^{-1}$.
${ }^{16}$ Phys.Rev.D 92 (2015) 025014
$\left|F_{0} / F_{-}\right|$


## Total corrections




| $a_{\mu}^{H V P, L O}[\pi \pi, \tau]$ at $\mathcal{O}\left(p^{4}\right)$ |  |  |  |
| :---: | :---: | :---: | :---: |
| Experiment | $2 m_{\pi} \pm-0.36 \mathrm{GeV}$ | $0.36-1.8 \mathrm{GeV}$ | TOTAL |
| Belle | $8.81 \pm 0.00 \pm 0.14_{-0.34}^{+0.16}$ | $511.14 \pm 1.94 \pm 7.99_{-2.09}^{+1.91}$ | $519.95 \pm 1.94 \pm 7.99_{-2}^{+1.91}$ |
| ALEPH | $8.89 \pm 0.00 \pm 0.05_{-0.34}^{+0.16}$ | $508.26 \pm 4.48 \pm 2.82_{-2.01}^{+1.09}$ | $517.15 \pm 4.48 \pm 2.82_{-2.12}^{+1.91}$ |
| CLEO | $8.85 \pm 0.00 \pm 0.15_{-0.34}^{+0.16}$ | $510.63 \pm 3.40 \pm 8.93_{-2.08}^{+1.90}$ | $519.48 \pm 3.40 \pm 8.93_{-2.11}^{+1.90}$ |
| OPAL | $8.89 \pm 0.00 \pm 0.12_{-0.34}^{+0.15}$ | $522.81 \pm 10.04 \pm 7.00_{-2.12}^{+1.87}$ | $531.70 \pm 10.04 \pm 7.00_{-2.15}^{+1.87}$ |


| $a_{\mu}^{H V P, L O}[\pi \pi, \tau]$ at $\mathcal{O}\left(p^{6}\right)$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| Experiment | $2 m_{\pi} \pm-0.36 \mathrm{GeV}$ | $0.36-1.8 \mathrm{GeV}$ | TOTAL |  |
| Belle | $7.77 \pm 0.00 \pm 0.12_{-0.59}^{+1.20}$ | $507.18 \pm 1.91 \pm 7.88_{-3.76}^{+4.72}$ | $514.95 \pm 1.91 \pm 7.88_{-3.81}^{+4.87}$ |  |
| ALEPH | $7.84 \pm 0.00 \pm 0.04_{-0.60}^{+1.21}$ | $504.37 \pm 4.35 \pm 2.79_{-3.73}^{+4.70}$ | $512.21 \pm 4.35 \pm 2.79_{-3.78}^{+4.75}$ |  |
| CLEO | $7.80 \pm 0.00 \pm 0.14_{-0.59}^{+1.21}$ | $506.74 \pm 3.28 \pm 8.84_{-3.71}^{+4.63}$ | $514.54 \pm 3.28 \pm 8.84_{-3.78}^{+4.76}$ |  |
| OPAL | $7.84 \pm 0.00 \pm 0.10_{-0.60}^{+1.20}$ | $518.32 \pm 9.69 \pm 6.92_{-4.12}^{+5.25}$ | $526.16 \pm 9.69 \pm 6.92_{-4.16}^{+5.39}$ |  |


[^0]:    ${ }^{1}$ Cirigliano et al. Phys.Lett. B513 (2001). JHEP 0208 (2002)

[^1]:    ${ }^{2}$ Cirigliano, Ecker \& Neufeld. Phys.Lett.B 513 (2001)

[^2]:    ${ }^{3}$ Cirigliano et al. Phys.Lett. B513 (2001). JHEP 0208 (2002)
    ${ }^{4}$ Flores-Tlalpa et al. '06, '07

[^3]:    ${ }^{3}$ Cirigliano et al. Phys.Lett. B513 (2001). JHEP 0208 (2002)
    ${ }^{4}$ Flores-Tlalpa et al. '06, '07

[^4]:    ${ }^{3}$ Cirigliano et al. Phys.Lett. B513 (2001). JHEP 0208 (2002)
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[^7]:    ${ }^{9}$ Gómez Dumm \& Roig, Eur.Phys.J. C73 (2013)
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