Strong coupling determination from relativistic quarkonium sum rules

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DB, Vicent Mateu, arXiv:1912:06237 PLB (2020),
DB, Vicent Mateu, arXiv:2001:11041 JHEP (2020)
DB, Vicent Mateu and Marcus V Rodrigues, in preparation

Institut de Física d'Altes Energies

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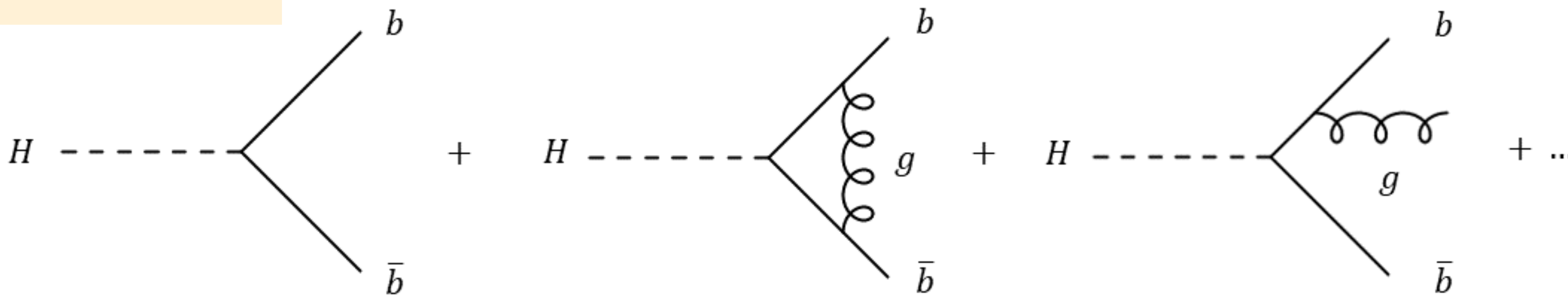
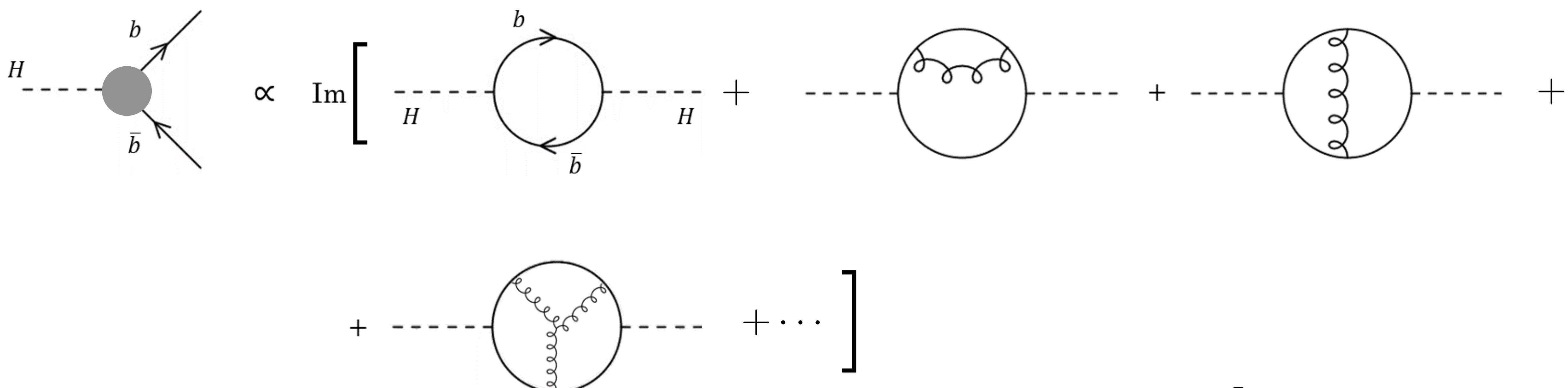
α_s the whys and the hows

Theoretical uncertainty in a Higgs decay

Decay $H \to b\bar{b}$



Optical theorem



Scalar qq correlator

$$\Pi(p^2) \equiv i \int \mathrm{d}x \;\; \mathrm{e}^{ipx} \, \langle\Omega|T\{j(x)j^\dagger(0)\}|\Omega\rangle$$

$$j(x) = m_q \; :\bar{q}_f(x)\, q_f(x):$$

Optical theorem

$$\Gamma(H \to b\bar{b}) = \mathrm{Im}\,\Pi/m_H$$

(massless limit)

Theoretical uncertainty in a Higgs decay

Decay $H \to b\bar{b}$

(massless case)

$$\mathrm{Im}\,\Pi(s) = \frac{N_c}{8\pi}\, m_b^2\, s \left[1 + \sum_{n=0}^{\infty} c_n a_s^n\right]$$

$$a_s = \frac{\alpha_s}{\pi}$$

$c_1 = \frac{17}{3}$	$c_2 = 29.1467$	$c_3 = 41.7576$	$c_4 = -825.747$
1980	1990	1997	2006
raaten, Leveille akai	Gorishny et al	Chetyrkin	Baikov, Chetyrkin, Kühn
2-loop	3-loop	4-loop	5-loop
NLO	N2LO	N3LO	N4LO

With this information we can estimate *even higher* orders using Borel-Padé approximants DB, P Masjuan, C London, in preparation

$$c_5 = -8200 \pm 308$$

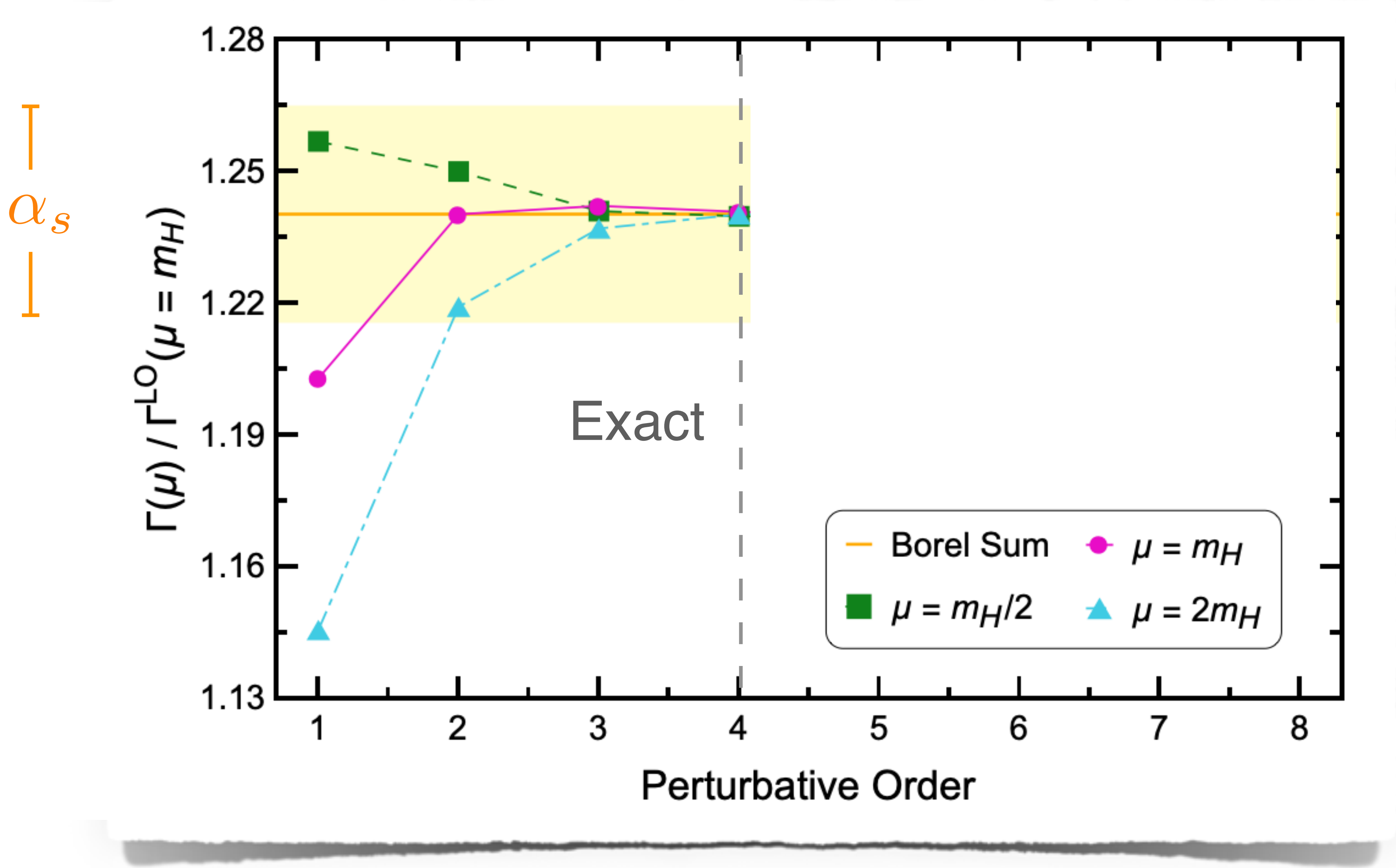
Estimated 6-loop (N5LO)

What about the theory error?

Theoretical uncertainty in a Higgs decay

Decay $H \to b\bar{b}$

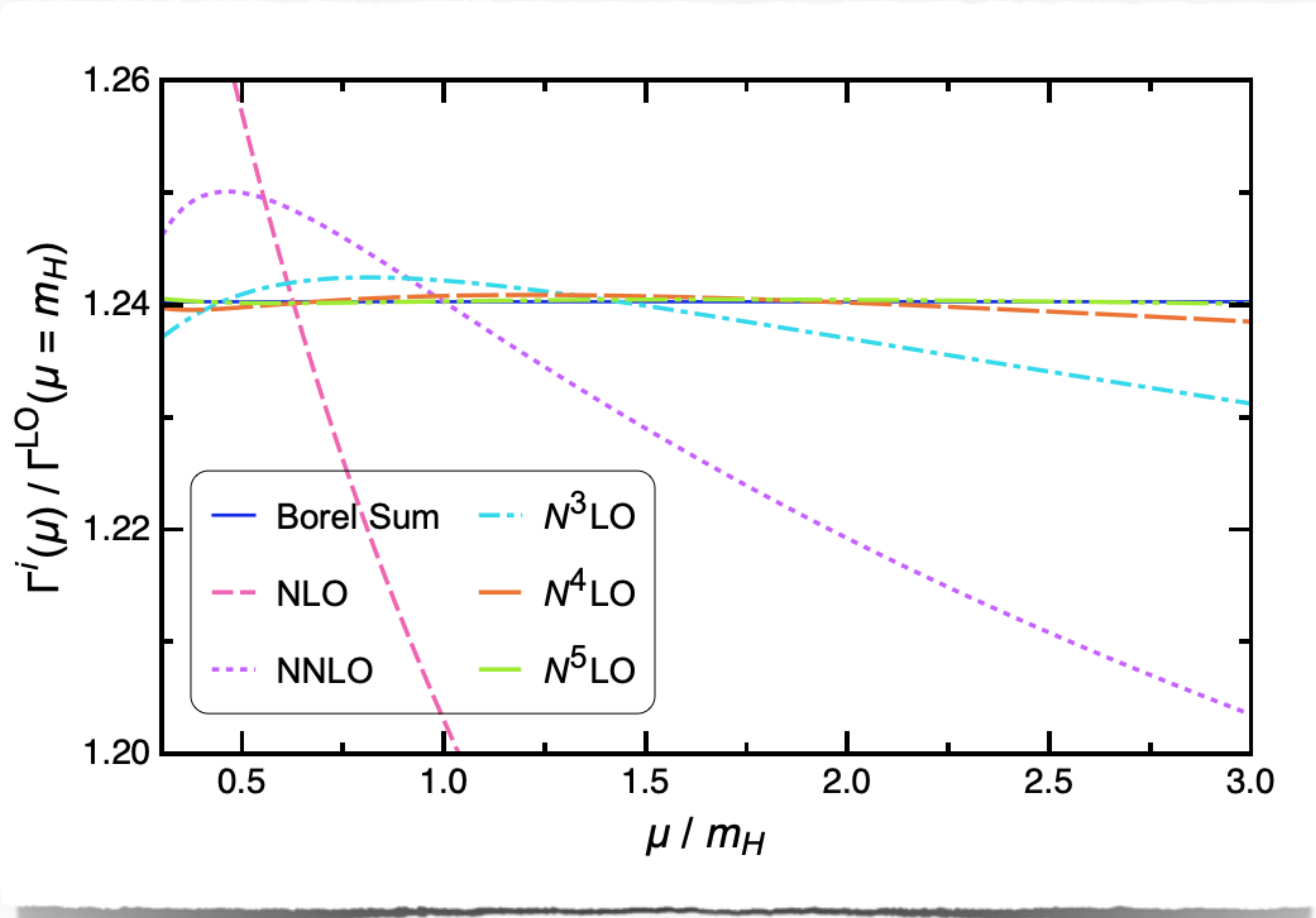
Truncation error vs. strong coupling error



Theoretical uncertainty in a Higgs decay

Decay $H \to b\bar{b}$

Renormalization scale variation



At N4LO we already have a very stable perturbative series

Theoretical uncertainty in a Higgs decay

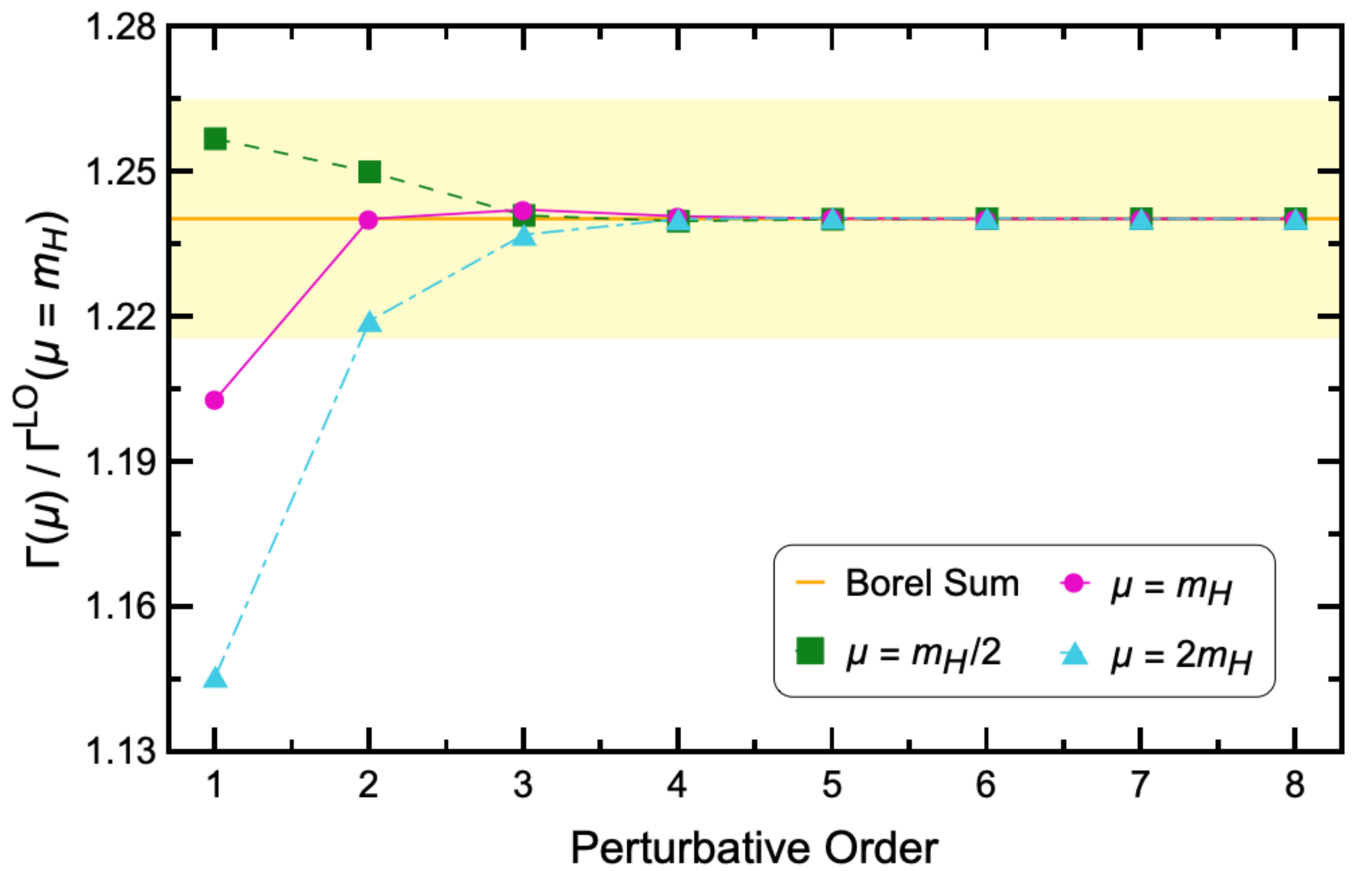
Decay $H \to b\bar{b}$

Uncertainty is dominated by the masses and couplings

$$\sigma_{\alpha_s} \sim 0.9\%$$

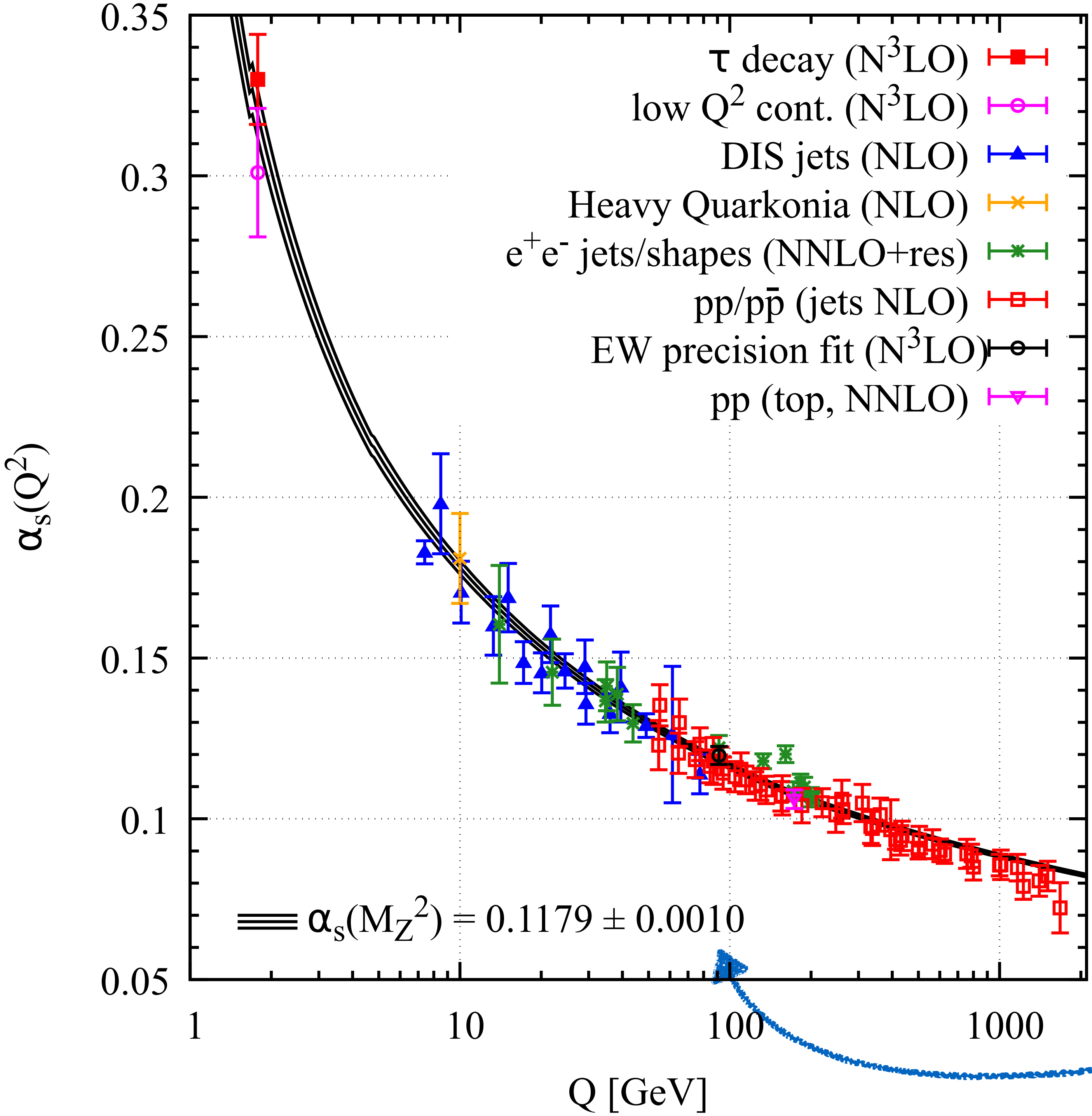
$$\sigma_{m_b} \sim 0.7\%$$

$$\sigma_{m_H} \sim 0.1\%$$



The strong coupling in 2021

PDG 2019



Overall picture is very consistent.

Discrepancies persist: uncertainty has been *enlarged.*

The PDG uncertainty was +/- 0.0007 in 2014

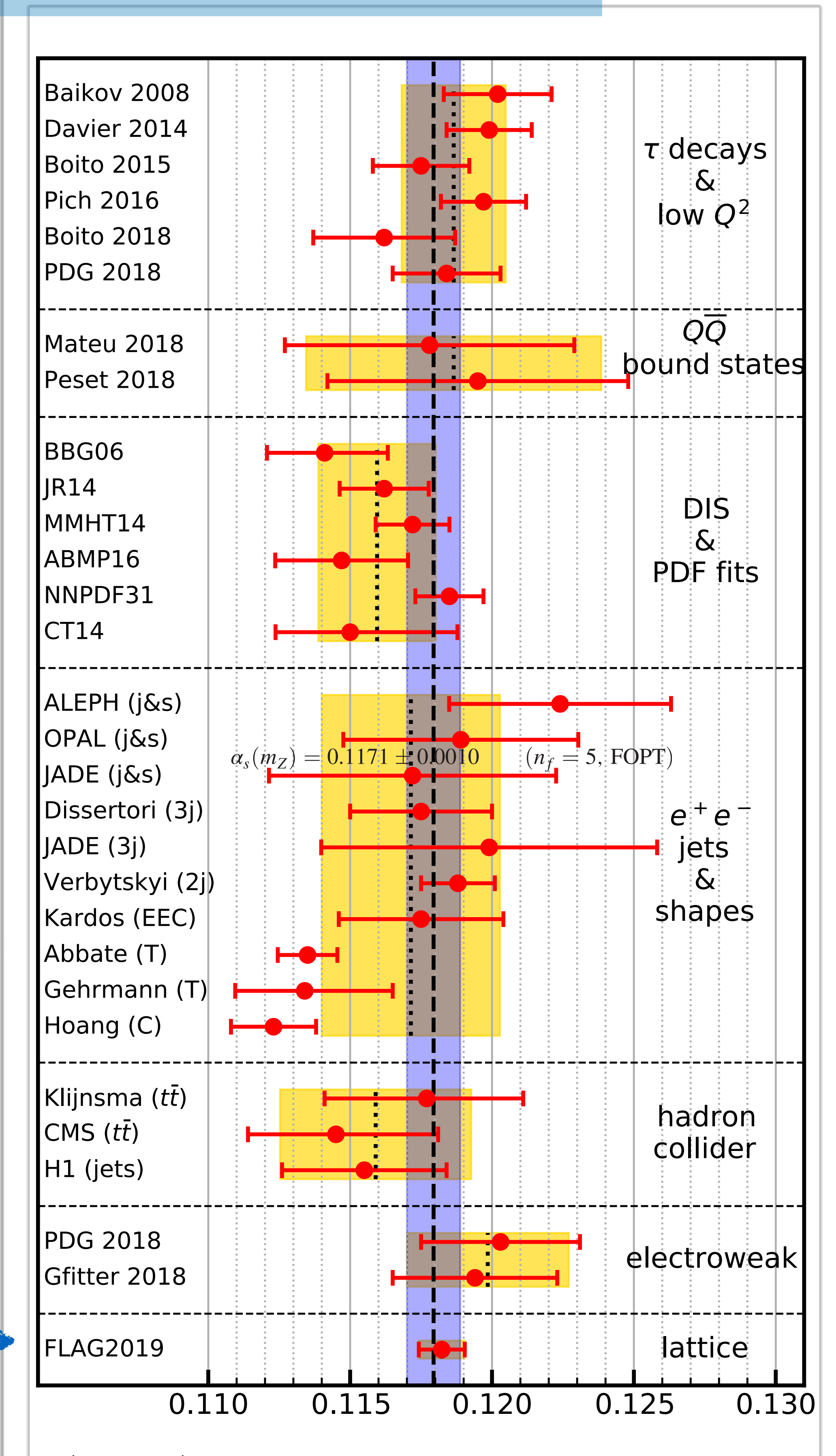
The strong coupling in 2021

PDG 2019

Tensions in determinations from same data

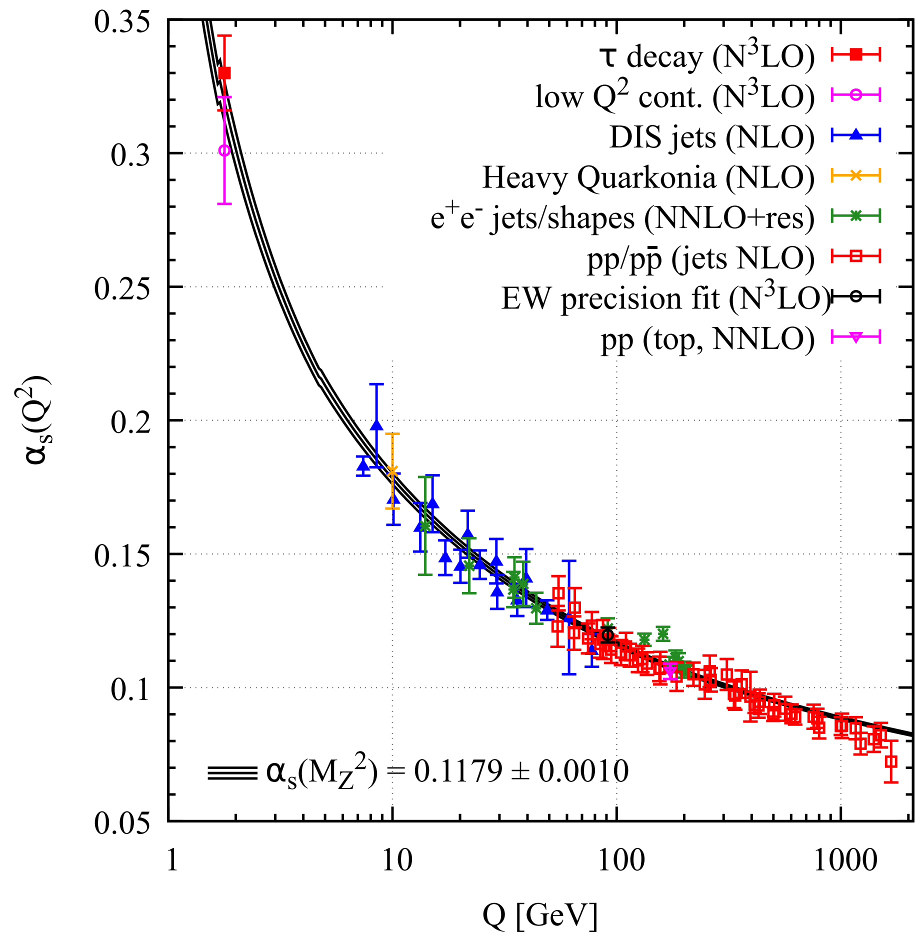
Event shapes give systematically lower results

Starting to be dominated by lattice



$$\alpha_s^{(\mathrm{PDG})}(m_Z) = 0.1179 \pm 0.0010$$

The strong coupling in 2021



Lower energies

Higher energies

Larger coupling, more sensitivity to QCD corrections.

Larger non-perturbative physics (OPE, DVs),

Problems with pt. theory (renormalons).

Smaller coupling, less sensitive to QCD corrections, more precision required from exp.

Small contamination from non-perturbative physics,

pt. series is almost convergent

Strong coupling from quarkonium sum rules

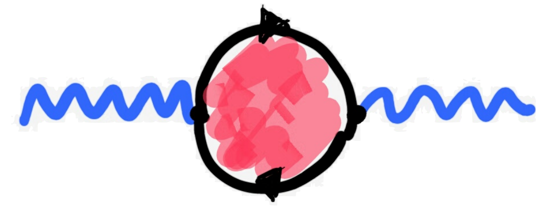
- DB, V. Mateu, arXiv:1912:06237 PLB (2020),
- DB, V. Mateu, arXiv:2001:11041 JHEP (2020)

Sum rules

Vector correlator with massive quarks

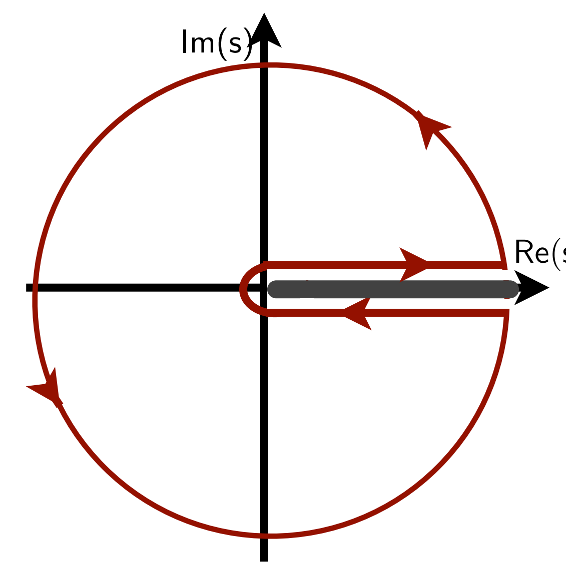
$$j^\mu(x) = \bar{q}(x)\gamma^\mu q(x)$$

$$\boxed{\Pi_{\mu\nu}(q) = i\int d^4x\, e^{iqx}\langle 0|T\{J_\mu(x)J_\nu(0)^\dagger\}|0\rangle}$$



(once subtracted) dispersion relation

$$\Pi(q^2) = \frac{q^2}{12\pi^2}\int_{s_{th}}^{\infty}\frac{R_{q\bar{q}}(s)}{s(s-q^2+i\epsilon)}$$



$$R_{q\bar{q}}(s) = \frac{\sigma_{e^+e^-\to q\bar{q}+X}(s)}{\sigma_{e^+e^-\to\mu^+\mu^-}(s)}$$

$$R_{q\bar{q}} = 12\pi\,\mathrm{Im}\Pi(q^2)$$

Many experiments devoted to R(s) mainly because of muon g-2

Sum rules

(once subtracted) dispersion relation

$$\Pi(q^2) = \frac{q^2}{12\pi^2}\int_{s_{th}}^{\infty} \frac{R_{q\bar{q}}(s)}{s(s-q^2+i\epsilon)}$$

Using analyticity and unitarity (dispersion relation): sum rules

Experiment Theory

$$M_q^{V,n} = \int \frac{\mathrm{d}s}{s^{n+1}} R_{q\bar{q}}(s) = \frac{12\pi^2 Q_q^2}{n!} \frac{\mathrm{d}^n}{\mathrm{d}s^n}\Pi_q^V(s)\Big|_{s=0}$$

Shifman, Vainshtein, Zakharov '79

We restrict the sum rules to $n \leq 4$. Typical scale m_q/n.

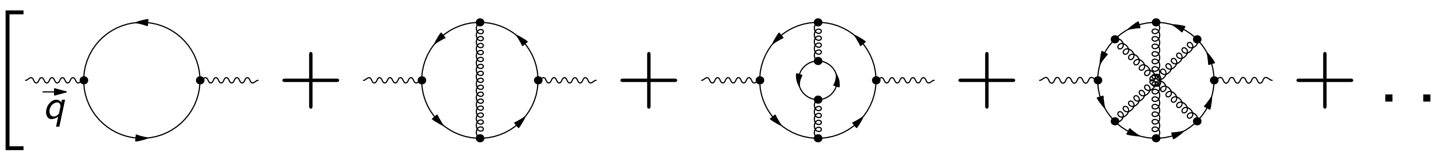
Relativistic sum rules

Theory: QCD

Small momentum expansion of the correlator

$$\hat{\Pi}_q^X(s) = \frac{1}{12\pi^2 Q_q^2} \sum_{n=0}^{\infty} s^n \hat{M}_q^{X,n}$$

$$M_q^{V,n} = \frac{12\pi^2 Q_q^2}{n!} \left(\frac{d}{dq^2}\right)^n \Big[\; \cdots \; \Big]_{q^2=0}$$



A Maier

Perturbative expansion

$$\hat{M}_q^{X,n} = \frac{1}{(2\,\overline{m}_q)^{2n}} \sum_{i=0} \left[\frac{\alpha_s(\overline{m}_q)}{\pi}\right]^i c_i^{X,n}$$

summing logs with $\mu = \bar{m}_q(\bar{m}_q)$

Known up to $\mathcal{O}(\alpha_s^3)$ for $n \leq 4$

Also for scalar, pseudoscalar and axial correlators

Chetyrkin, Kühn, Sturm '06; Boughezal, Czakon, Schutzmeier '06
Maier, Maierhöfer, Smirnov '08/'09; Maier and Marquard '17

Perturbative expansion

$$\hat{M}_q^{X,n} = \frac{1}{(2\,\overline{m}_q)^{2n}} \sum_{i=0} \left[\frac{\alpha_s(\overline{m}_q)}{\pi}\right]^i c_i^{X,n}$$

summing logs with $\mu = \bar{m}_q(\bar{m}_q)$

Known up to $\mathcal{O}(\alpha_s^3)$ for $n \leq 4$

Chetyrkin, Kühn, Sturm '06; Boughezal, Czakon, Schutzmeier '06
Maier, Maierhöfer, Smirnov '08/'09; Maier and Marquard '17

General expansion in terms of the two scales (using RG)

$$M_q^{(n)} = \frac{1}{[2\,\overline{m}_b(\mu_m)]^{2n}} \sum_{i=0} \left[\frac{\alpha_s^{(n_f)}(\mu_\alpha)}{\pi}\right]^i \sum_{a=0}^{i} \sum_{b=0}^{[i-1]} c_{i,a,b}^{(n)}(n_f)\, \ln^a\left(\frac{\mu_m}{\overline{m}_b(\mu_m)}\right) \ln^b\left(\frac{\mu_\alpha}{\overline{m}_b(\mu_m)}\right)$$

Highly sensitive to the mass, ideal for quark-mass determinations

Kühn, Steinhauser '01, Kühn, Steinhauser Sturm '07, Chetyrkin '09,
Chetyrkin Kühn, Maier, Maierhofer, Marquard, Steinhauser, '12, '17
Erler, Masjuan, Spiesberger '16
Dehnadi, Hoang, Mateu, Zebarjad '11, Dehnadi, Hoang, Mateu '15

Ratios of moments: strong coupling extraction

$$M_q^{(n)} = \frac{1}{[2\,\overline{m}_b(\mu_m)]^{2n}} \sum_{i=0} \left[\frac{\alpha_s^{(n_f)}(\mu_\alpha)}{\pi}\right]^i \sum_{a=0}^{i} \sum_{b=0}^{[i-1]} c_{i,a,b}^{(n)}(n_f)\, \ln^a\!\left(\frac{\mu_m}{\overline{m}_b(\mu_m)}\right) \ln^b\!\left(\frac{\mu_\alpha}{\overline{m}_b(\mu_m)}\right)$$

Strong mass dependence is eliminated

We consider dimensionless ratios of moments

DB,V Mateu '19

$$R_q^{X,n} \equiv \frac{\left(M_q^{X,n}\right)^{\frac{1}{n}}}{\left(M_q^{X,n+1}\right)^{\frac{1}{n+1}}}$$

Central object of this part
Of the talk

…similar to the ones used in lattice studies of the PS correlators

Maezawa, Petreczky '16

Perturbative expansion

$$R_b^{V,n} = \sum_{i=0} \left[\frac{\alpha_s(\mu_\alpha)}{\pi}\right]^i \sum_{k=0}^{[i-1]} \sum_{j=0}^{[i-2]} r_{i,j,k}^{(n)}\, \ln^j\!\left(\frac{\mu_m}{\overline{m}_b(\mu_m)}\right) \ln^k\!\left(\frac{\mu_\alpha}{\overline{m}_b(\mu_m)}\right)$$

Residual (suppressed) mass dependence

Ratios of moments: strong coupling extraction

$$R_q^{X,n} \equiv \frac{\left(M_q^{X,n}\right)^{\frac{1}{n}}}{\left(M_q^{X,n+1}\right)^{\frac{1}{n+1}}}$$

Perturbative expansion

$$R_b^{V,n} = \sum_{i=0} \left[\frac{\alpha_s(\mu_\alpha)}{\pi}\right]^i \sum_{k=0}^{[i-1]} \sum_{j=0}^{[i-2]} r_{i,j,k}^{(n)} \ln^j\left(\frac{\mu_m}{\overline{m}_b(\mu_m)}\right) \ln^k\left(\frac{\mu_\alpha}{\overline{m}_b(\mu_m)}\right)$$

Residual (suppressed) mass dependence

Example

$$R_c^{V,2} = 1.0449\big[1 + 0.57448\, a_s + (0.32576 + 2.3937\, L_\alpha)\, a_s^2 - \left(2.1093 + 4.7873 L_m - 6.4009 L_\alpha - 9.9736 L_\alpha^2\right) a_s^3 + \mathcal{O}\left(\alpha_s^4\right)\big]$$

DB, V Mateu '19
DB, V Mateu '20

Almost insensitive to the quark mass (only through logs at $\mathcal{O}(\alpha_s^2)$)
Sensitive to the coupling.
Available at N^{3}LO up to $R_q^{V,3}$
Can be accurately determined from data.

Ratios of moments: strong coupling extraction

Perturbative expansion

$$R_c^{V,2} = 1.0449\big[1 + 0.57448\, a_s + (0.32576 + 2.3937\, L_\alpha)\, a_s^2 - \left(2.1093 + 4.7873 L_m - 6.4009 L_\alpha - 9.9736 L_\alpha^2\right) a_s^3 + \mathcal{O}\left(\alpha_s^4\right)\big]$$

Typical size of pt. corrections: 13%, 7%, and 5% (for charm with *n*=1,2,3)

Non-perturbative contributions: gluon-condensate known to NLO.

$$\Delta M_n^{X,\,\langle G^2\rangle} = \frac{1}{(4M_q^2)^{n+2}} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle_{\rm RGI} \left[[a_X(n_f)]_n^0 + \frac{\alpha_s^{(n_f)}(\mu_\alpha)}{\pi} [a_X(n_f)]_n^1 \right]$$

Added as an estimate of non-perturbative uncertainties.

Completely irrelevant for the bottom-quark case.

Theory errors: scale variation

$$R_b^{V,n} = \sum_{i=0} \left[\frac{\alpha_s(\mu_\alpha)}{\pi}\right]^i \sum_{k=0}^{[i-1]} \sum_{j=0}^{[i-2]} r_{i,j,k}^{(n)} \ln^j\left(\frac{\mu_m}{\overline{m}_b(\mu_m)}\right) \ln^k\left(\frac{\mu_\alpha}{\overline{m}_b(\mu_m)}\right)$$

Independent scale variation important for conservative error estimate

$\overline{m}_q \leq \mu_\alpha, \mu_m \leq \mu_{\max}$ With $\mu_{\max} = 4\,(15)\,\mathrm{GeV}$ for charm (bottom)

Dehnadi, Hoang, Mateu '15

With the following constraint

$\boxed{1/\xi \leq (\mu_m/\mu_\alpha) \leq \xi}$ With $\xi = 2$ our (canonical) choice

Aways checking order-by-order convergence.

Experimental ratios of moments: charm

Resonance data

Pt. continuum (theory)

$$M_q^{V,n} = \int \frac{\mathrm{d}s}{s^{n+1}} R_{q\bar{q}}(s) \; = \text{(resonan.)} + \int_{s_{\mathrm{th}}}^{s_{\mathrm{max}}} \frac{ds}{s^{n+1}} R_{q\bar{q}}(s) + \int_{s_{\mathrm{max}}}^{\infty} \frac{ds}{s^{n+1}} R_{q\bar{q}}(s)$$

Combined R data

$$- \int_{s_{\mathrm{th}}}^{\infty} \frac{ds}{s^{n+1}} R_{uds}$$

α_s

Non-charm background (theory) (+secondary charm production)

(singlet contributions are very small and can be neglected)

(no light-quark background for the bottom moments)

Parametrize the continuum contribution (highly linear dependence on the coupling) (including mass corrections)

Experimental ratios of moments: charm

Resonance data

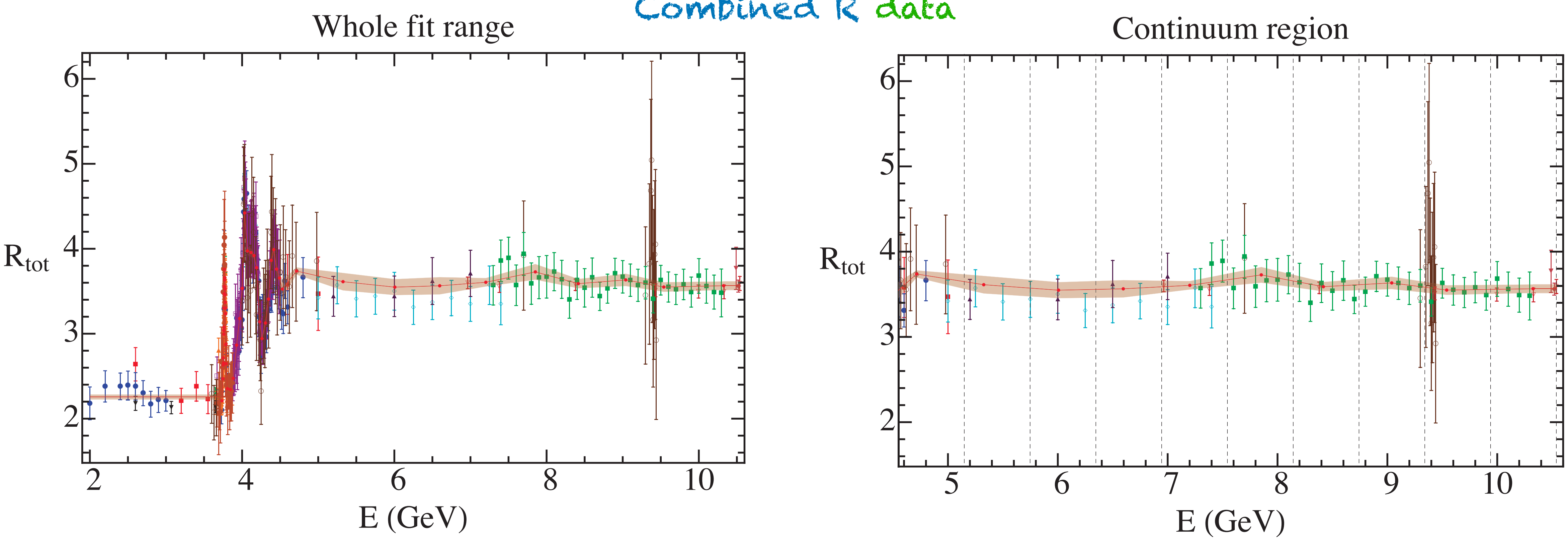
Pt. continuum (theory)

$$M_q^{V,n} = \boxed{(\text{resonan.})} + \boxed{\int_{s_{\rm th}}^{s_{\rm max}} \frac{ds}{s^{n+1}} R_{q\bar{q}}(s)} + \boxed{\int_{s_{\rm max}}^{\infty} \frac{ds}{s^{n+1}} R_{q\bar{q}}(s)} \quad \boxed{- \int_{s_{\rm th}}^{\infty} \frac{ds}{s^{n+1}} R_{uds}}$$

Combined R data

Non-charm background (theory)

Combined R data



Dehnadi, Hoang, Mateu, Zebarjad '11, Dehnadi, Hoang, Mateu '15

Exp moments determined from resonances and combined *R* data.

Correlations must be taken into account in the procedure.

Parametrize the continuum contribution (highly linear dependence on the coupling) (including mass corrections)

Experimental ratios of moments: charm

Resonance data

Pt. continuum (theory)

$$M_q^{V,n} = (\text{resonan.}) + \int_{s_{\rm th}}^{s_{\max}} \frac{ds}{s^{n+1}} R_{q\bar q}(s) + \int_{s_{\max}}^{\infty} \frac{ds}{s^{n+1}} R_{q\bar q}(s) \quad - \int_{s_{\rm th}}^{\infty} \frac{ds}{s^{n+1}} R_{uds}$$

Combined R data

Non-charm background (theory)

Slightly update as compared with the original works. Dehnadi, Hoang, Mateu, Zebarjad '11, Dehnadi, Hoang, Mateu '15
Cross checked with other R-data combinations Keshavarzi, Nomura, Teubner '18

$$R_q^{X,n} \equiv \frac{\left(M_q^{X,n}\right)^{\frac{1}{n}}}{\left(M_q^{X,n+1}\right)^{\frac{1}{n+1}}}$$

For the **charm** quark ratios we have

$$\Delta_\alpha = 0.1181 - \alpha_s$$

$$R_c^{V,1} = (1.770 - 0.705\,\Delta_\alpha) \pm 0.017, \qquad [\sigma_{\rm rel} = 0.98\%]$$
$$R_c^{V,2} = (1.1173 - 0.1330\,\Delta_\alpha) \pm 0.0022, \qquad [\sigma_{\rm rel} = 0.22\%]$$
$$R_c^{V,3} = (1.03535 - 0.04376\,\Delta_\alpha) \pm 0.00084. \qquad [\sigma_{\rm rel} = 0.104\%]$$

Continuum contribution smaller for higher n

Small uncertainties partially due to positive correlation among the moments Mn.

Experimental ratios of moments: bottom

Resonance data   Pt. continuum (theory)

$$M_q^{V,n} = \boxed{(\text{resonan.})} + \boxed{\int_{s_{\rm th}}^{s_{\rm max}} \frac{ds}{s^{n+1}} R_{q\bar{q}}(s)} + \boxed{\int_{s_{\rm max}}^{\infty} \frac{ds}{s^{n+1}} R_{q\bar{q}}(s)}$$

Combined R data

For the **bottom** quark ratios we have

	$R_q^{V,1}$	$R_q^{V,2}$	$R_q^{V,3}$
bottom	$0.8020(14) + 0.4083\,\Delta_\alpha$	$0.8465(20) + 0.14955\,\Delta_\alpha$	$0.8962(11) + 0.06905\,\Delta_\alpha$

$\sigma_{\rm rel} = 0.55\%$ $\sigma_{\rm rel} = 0.23\%$ $\sigma_{\rm rel} = 0.12\%$

Smaller errors partially due to cancellations arising from the positive correlations between moments

$$R_q^{X,n} \equiv \frac{\left(M_q^{X,n}\right)^{\frac{1}{n}}}{\left(M_q^{X,n+1}\right)^{\frac{1}{n+1}}}$$

Results for charmonium sum rules

α_s with $n_f = 4$ and $R_c^{V,n}$ with $n = 1,\, 2,$ and 3

α_s from charm moment ratios

$$R_q^{V,\mathrm{exp}} = R_q^{V,\mathrm{th}}$$

$$R_c^{V,2}$$

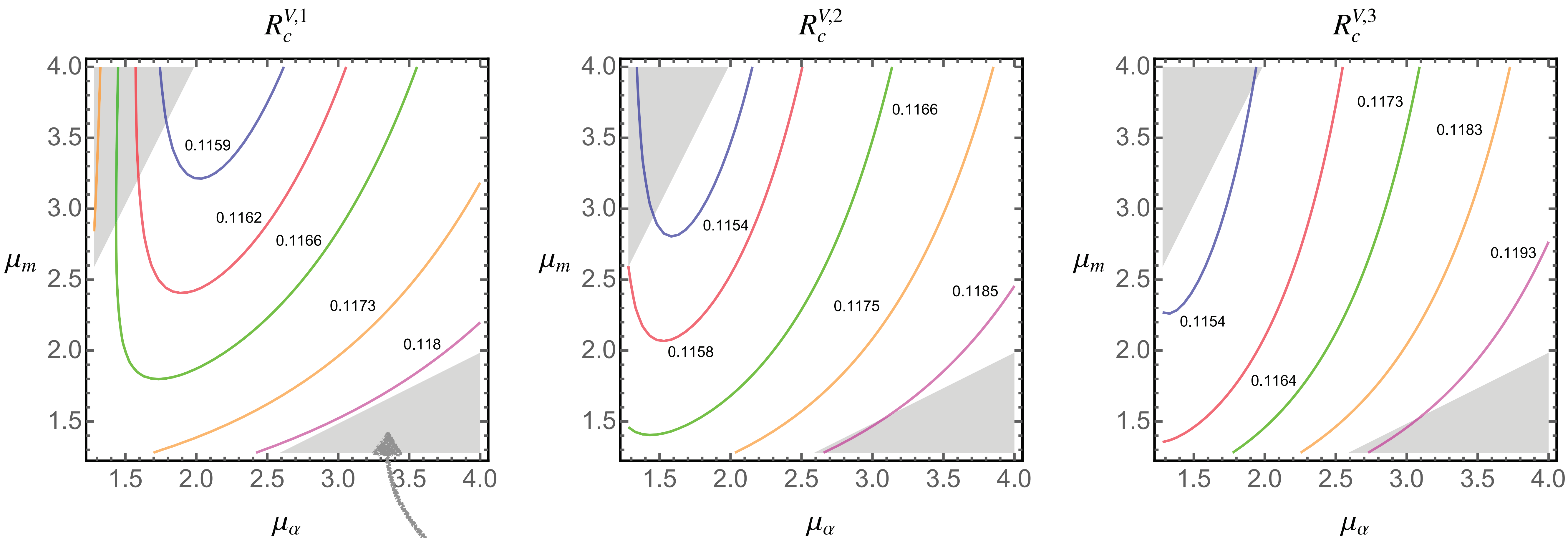
$$(1.1173 - 0.1330\,\Delta_\alpha) \pm 0.0022$$

$$= 1.0449\big[1 + 0.57448\,a_s + (0.32576 + 2.3937\,L_\alpha)\,a_s^2$$

$$- \left(2.1093 + 4.7873 L_m - 6.4009 L_\alpha - 9.9736 L_\alpha^2\right) a_s^3 + \mathcal{O}\left(\alpha_s^4\right)\big]$$

- Scan for different values of the renormalization scale
- Include (and remove) the gluon condensate
- Vary the quark mass

Perturbative error analysis



$$1/\xi \leq (\mu_m/\mu_\alpha) \leq \xi$$

The gray areas are not included in the analysis

$$\xi = 2$$

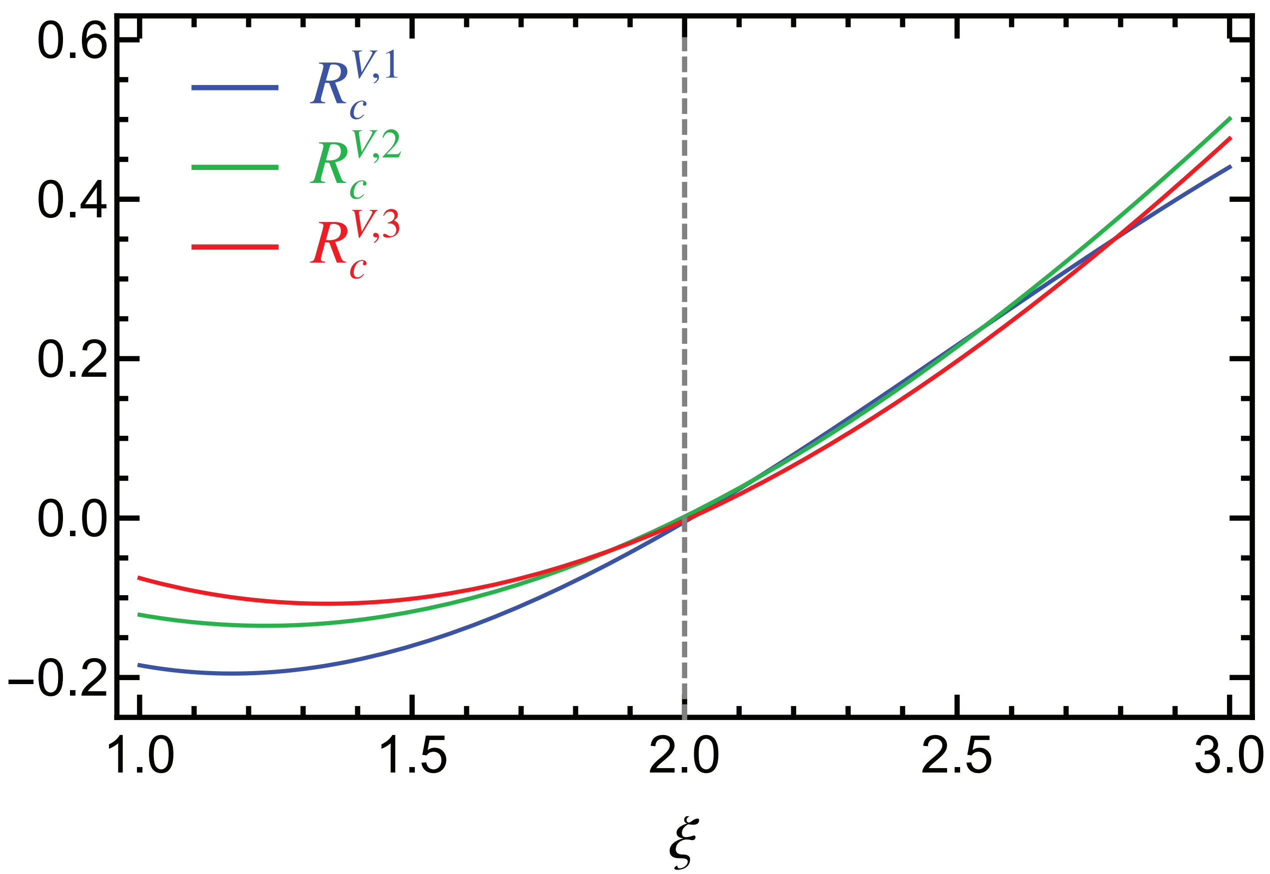
Perturbative error analysis

$$1/\xi \leq (\mu_m/\mu_\alpha) \leq \xi$$

$$\xi = 1 \rightarrow \mu_m = \mu_\alpha$$

$\xi = 2$ our (canonical) choice

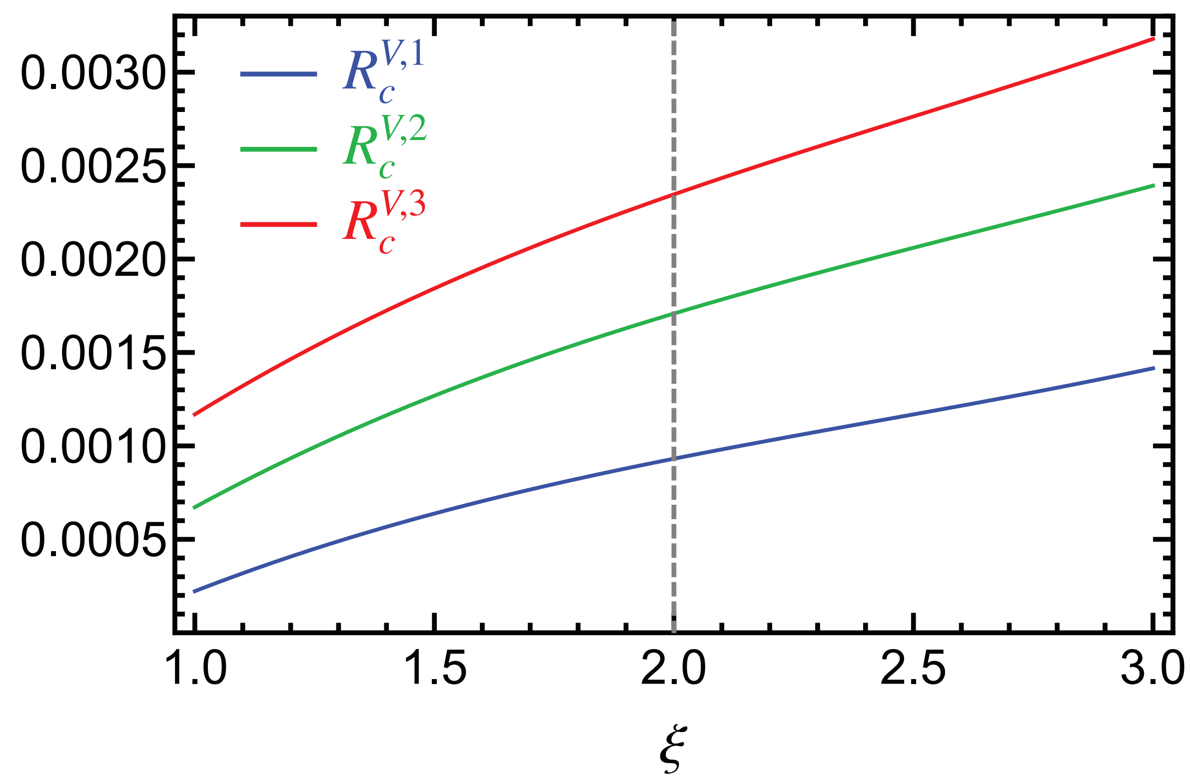
$$100 \times \left[\frac{\alpha_s^{(n_f=5)}(m_Z)}{\alpha_s^{(n_f=5)}(m_Z)\big|_{\xi=2}} - 1 \right] \text{ from } R_c^{V,i}$$



Small variations in the central values (~0.5%)

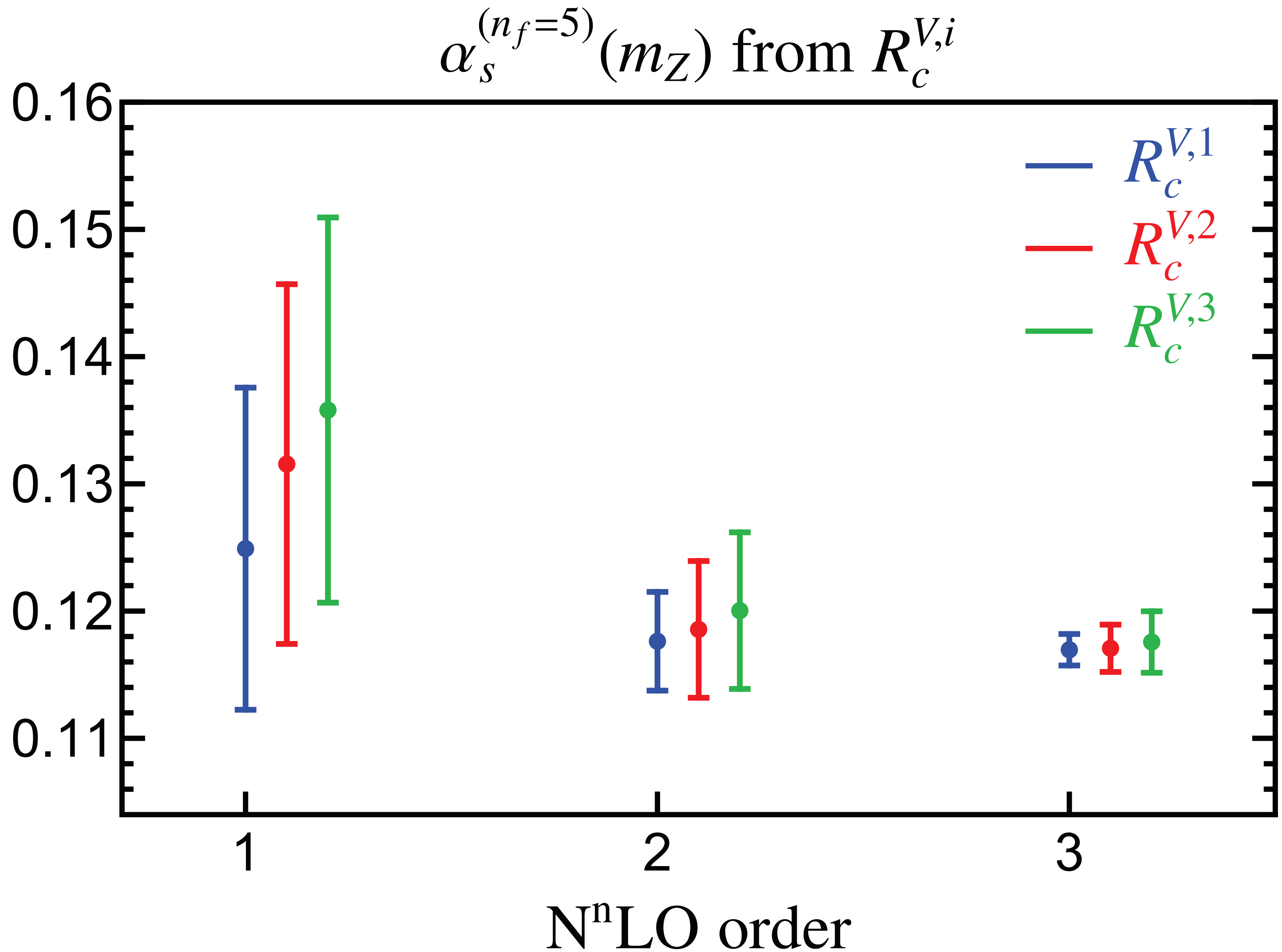
Diagonal variation: errors underestimated by a factor of up to ~2.0

$\Delta\alpha_s^{(n_f=5)}(m_Z)$ from $R_c^{V,i}$



Perturbative error analysis

Order by order convergence

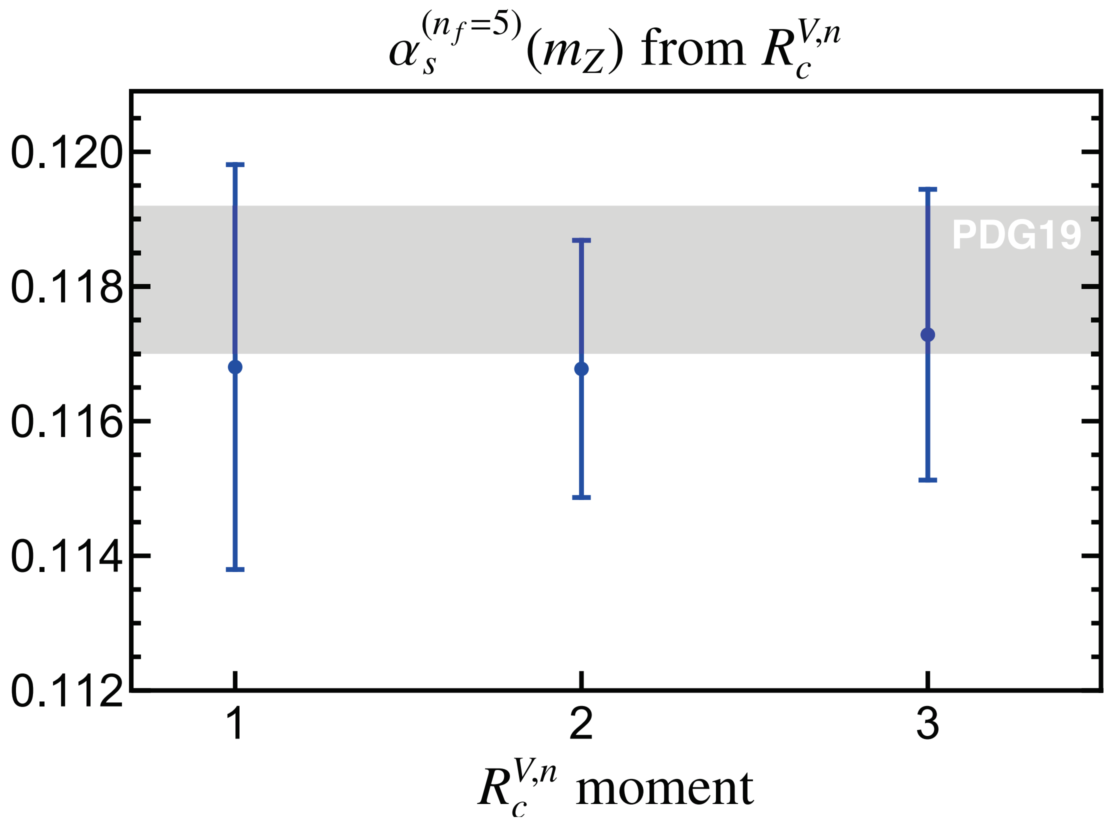


Results

Results from the charm moment ratios

flavor	n	$\alpha_s^{(n_f=5)}(m_Z)$	$\sigma_{\rm pert}$	$\sigma_{\rm exp}$	σ_{m_q}	$\sigma_{\rm np}$	$\sigma_{\rm total}$
charm	1	0.1168	0.0010	0.0028	0.0003	0.0006	0.0030
	2	0.1168	0.0015	0.0009	0.0003	0.0007	0.0019
	3	0.1173	0.0020	0.0005	0.0003	0.0006	0.0022

Trend to larger values



$$\alpha_s(m_Z) = 0.1168 \pm 0.0019$$

Results

Results from the charm and bottom moment ratios

flavor	n	$\alpha_s^{(n_f=5)}(m_Z)$	$\sigma_{\rm pert}$	$\sigma_{\rm exp}$	σ_{m_q}	$\sigma_{\rm np}$	$\sigma_{\rm total}$
bottom	1	0.1183	0.0011	0.0089	0.0002	0.0000	0.0090
	2	0.1186	0.0011	0.0046	0.0001	0.0000	0.0048
	3	0.1194	0.0013	0.0029	0.0001	0.0000	0.0032
charm	1	0.1168	0.0010	0.0028	0.0003	0.0006	0.0030
	2	0.1168	0.0015	0.0009	0.0003	0.0007	0.0019
	3	0.1173	0.0020	0.0005	0.0003	0.0006	0.0022

Trend to larger values

(Re)analysis of lattice data for pseudo-scalar charm-quark moments

Diogo Boito

Results from lattice correlators

Data for moments of the pseudo-scalar correnents are available from the lattice

moment	[6]	[9]	[10]	[11]	[12]
$M_c^{P,0}$	1.708(7)	1.709(5)	1.699(9)	1.705(5)	–
$R_c^{P,1}$	–	–	1.199(4)	1.1886(13)	1.188(5)
$R_c^{P,2}$	–	–	1.0344(13)	1.0324(16)	1.0341(19)

[6] HPQCD, Allison et al, Phys. Rev. D (2008)

[9] McNeile et al., Phys. Rev. D (2010)

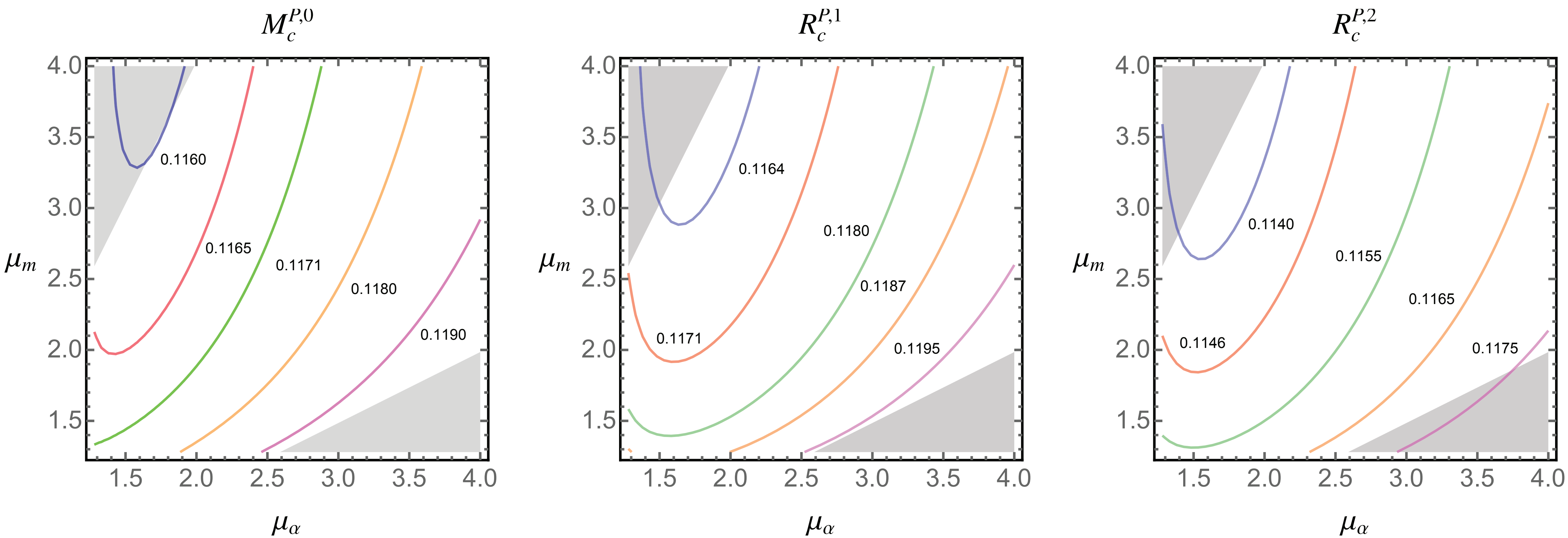
[10] Maezawa and Petreczky, Phys. Rev. D (2016)

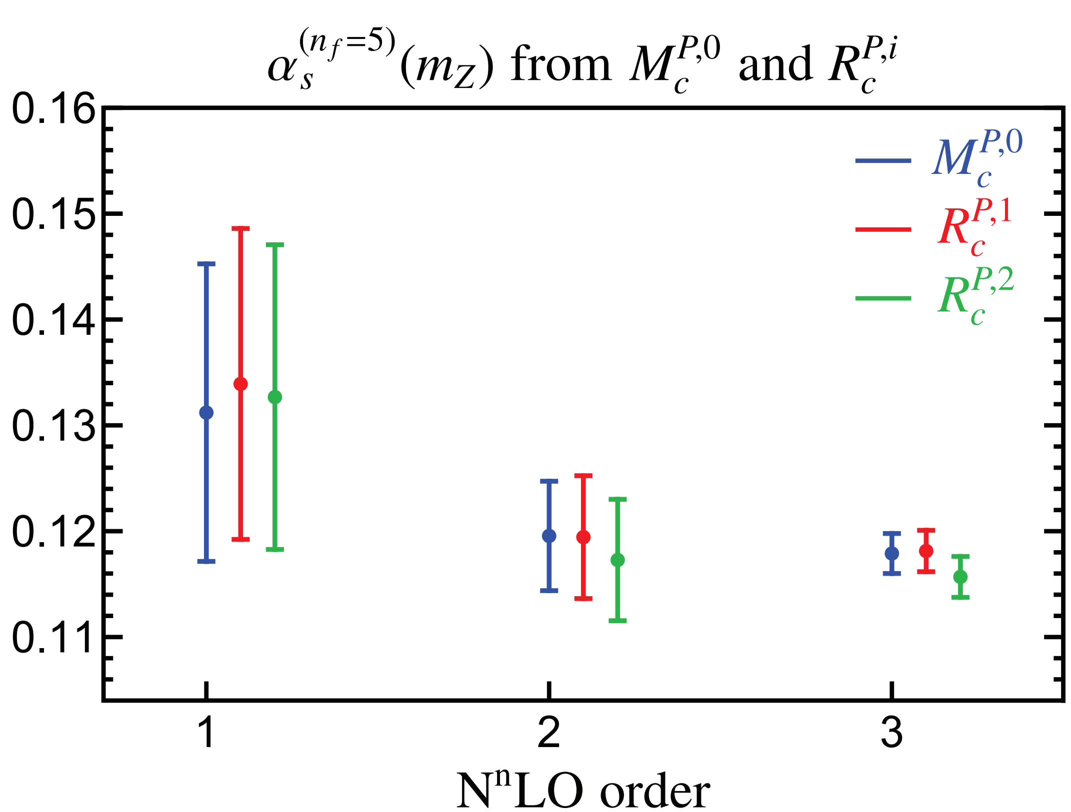
[11] Petreczky and Weber, Phys. Rev. D (2019)

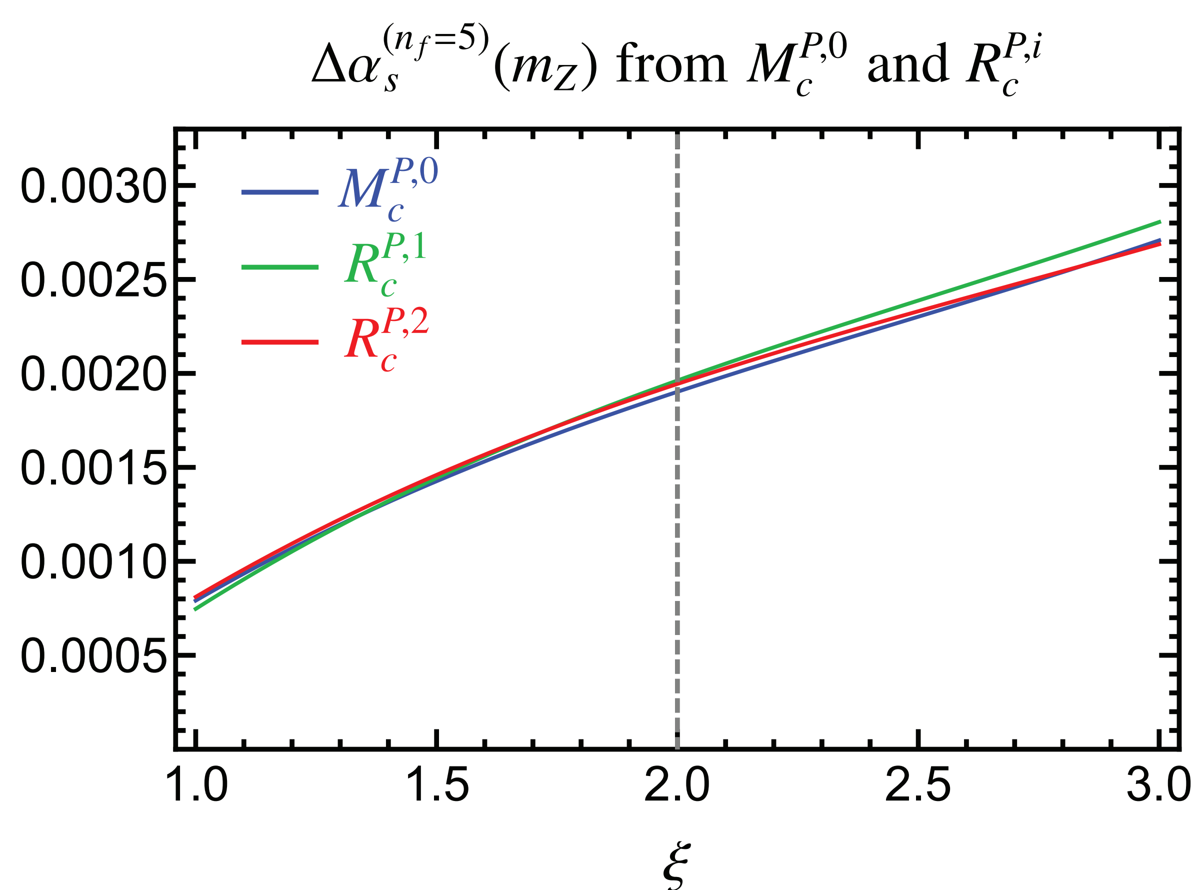
[12] Nakayama, Fahy, Hashimoto, Phys. Rev. D (2016)

We use the ratios and the 0-th moment to extract alpha_s and reassess pt. errors

Results from lattice correlators





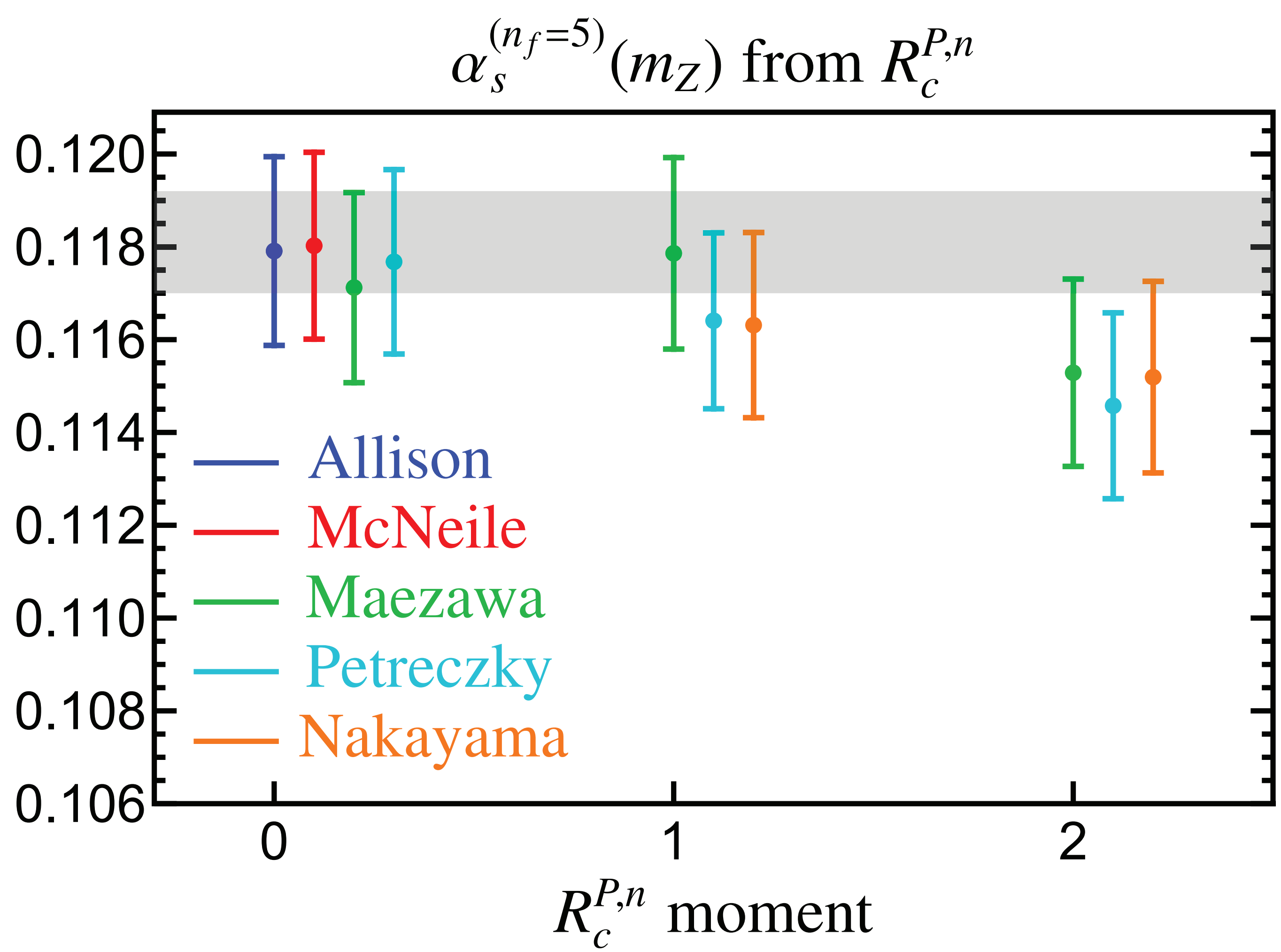


Results from lattice correlators

Larger errors due to more conservative pt. theory errors

Maezawa et al are even more conservative

Ref.	$\alpha_s^{(n_f=5)}(m_Z)$	$\sigma_{\rm pert}$	$\sigma_{\rm lattice}$	σ_{m_c}	$\sigma_{\rm NP}$	$\sigma_{\rm total}$
Allison et al. [6]	0.1179	0.0019	0.0006	0.0003	0.0004	0.0020
McNeile et al. [9]	0.1180	0.0019	0.0005	0.0003	0.0004	0.0020
Maezawa et al. [10]	0.1171	0.0018	0.0008	0.0003	0.0004	0.0020
Petreczky et al. [11]	0.1177	0.0019	0.0005	0.0003	0.0004	0.0020



Main result

Main result

Extraction from ***charm-quark vector-current moment ratios***:

Larger errors

$$\alpha_s(m_Z) = 0.1168(10)_{\rm pt}(28)_{\rm exp}(6)_{\rm np} = 0.1168(30)\ [R_c^{V,1}],$$

$$\alpha_s(m_Z) = 0.1168(15)_{\rm pt}(9)_{\rm exp}(7)_{\rm np} = 0.1168(19)\ [R_c^{V,2}],$$

$$\alpha_s(m_Z) = 0.1173(20)_{\rm pt}(5)_{\rm exp}(6)_{\rm np} = 0.1173(22)\ [R_c^{V,3}],$$

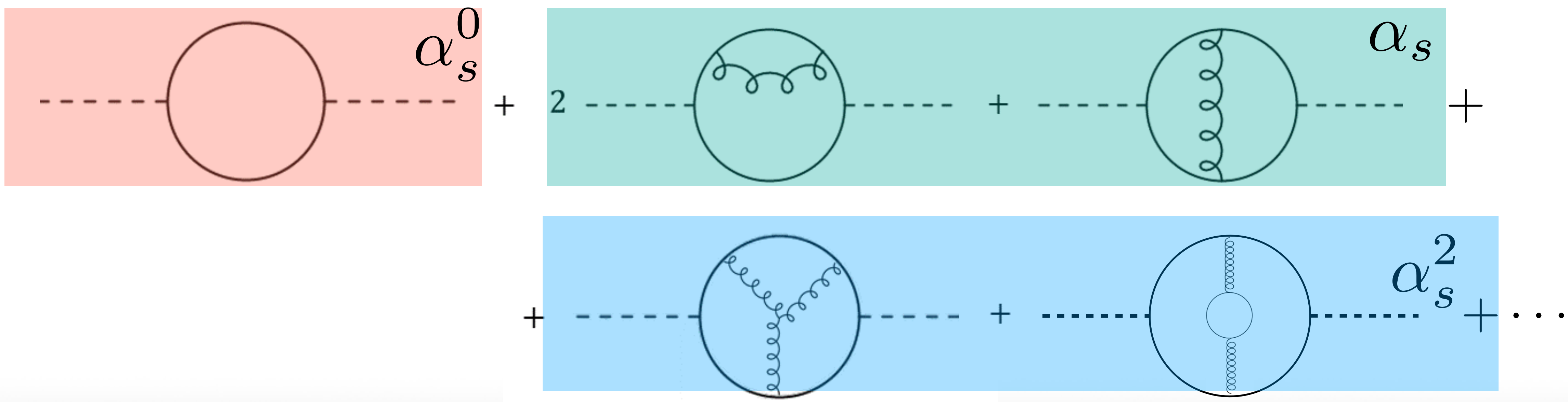
Large values of n

Very *conservative errors* (with diagonal scale variation error would be +/-0.0013)

Continuum contribution *treated self-consistently* (fixing it would give smaller errors).

Perturbative behaviour and renormalons

Standard perturbation theory



$$
\begin{aligned}
R = 1 & \quad \textbf{LO} \\
+ & \left[\, c_{10} \right] \alpha_s \quad \textbf{NLO} \\
+ & \left[\, c_{21} n_f + c_{20} \right] \alpha_s^2 \quad \textbf{NNLO} \\
+ & \left[\, c_{32} n_f^2 + c_{31} n_f + c_{30} \right] \alpha_s^3 \\
+ & \quad \cdots \qquad \cdots \qquad \cdots \\
+ & \left[\, c_{n,n-1} n_f^{n-1} + c_{n,n-2} n_f^{n-2} + \cdots + c_{n,0} \right] \alpha_s^n
\end{aligned}
$$

large-β_0

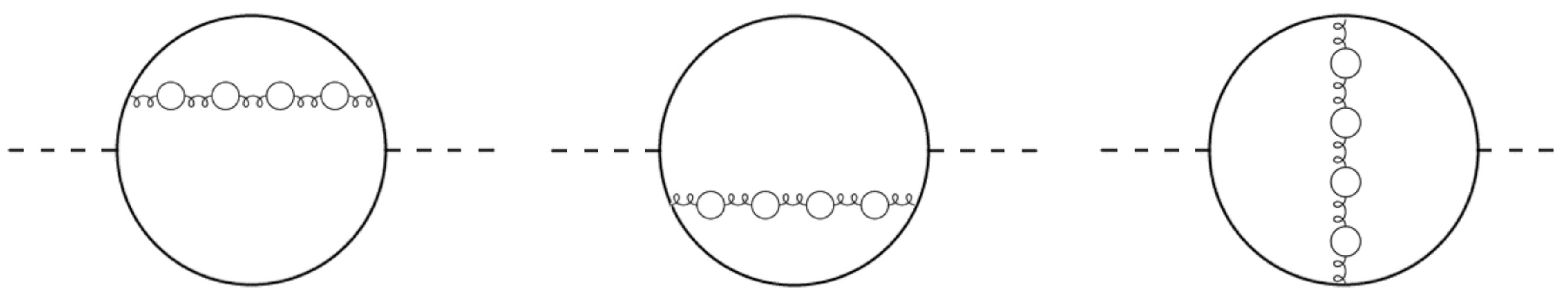
Gluon propagator with insertions of $q\bar{q}$ loops

$\alpha_s n_f$ $\qquad$ $(\alpha_s n_f)^2$

$$\alpha_s n_f \sim \mathcal{O}(1)$$

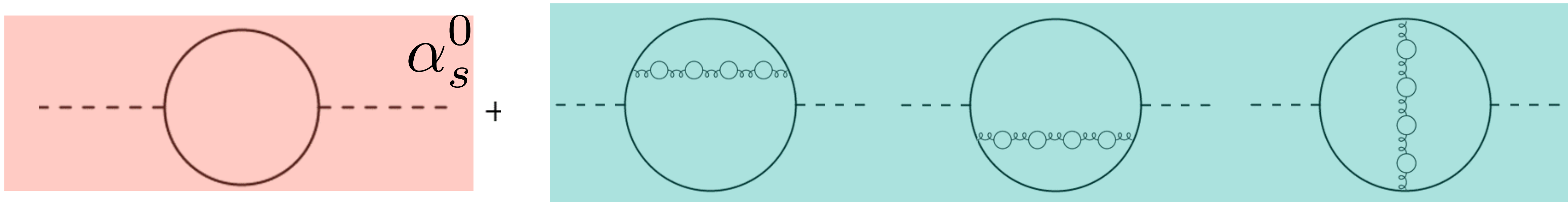
$$\beta_{0,f} = \frac{n_f}{6\pi}$$

Leading nf terms



Large-nf result

Standard perturbation theory



One chain | Two chains | Three chains

LO

NfLO

Nf-NLO

Nf-NNLO

$$
\begin{array}{llccccccl}
R = & \boxed{1} & & & & & & & \\
+ & [& c_{10} & & & & & &]\alpha_s \\
+ & [& c_{21} n_f & + & c_{20} & & & &]\alpha_s^2 \\
+ & [& c_{32} n_f^2 & + & c_{31} n_f & + & c_{30} & &]\alpha_s^3 \\
+ & & \cdots & & \cdots & & \cdots & & \\
+ & [& c_{n,n-1} n_f^{n-1} & + & c_{n,n-2} n_f^{n-2} & + & \cdots & + \quad c_{n,0} &]\alpha_s^n
\end{array}
$$

large-β_0

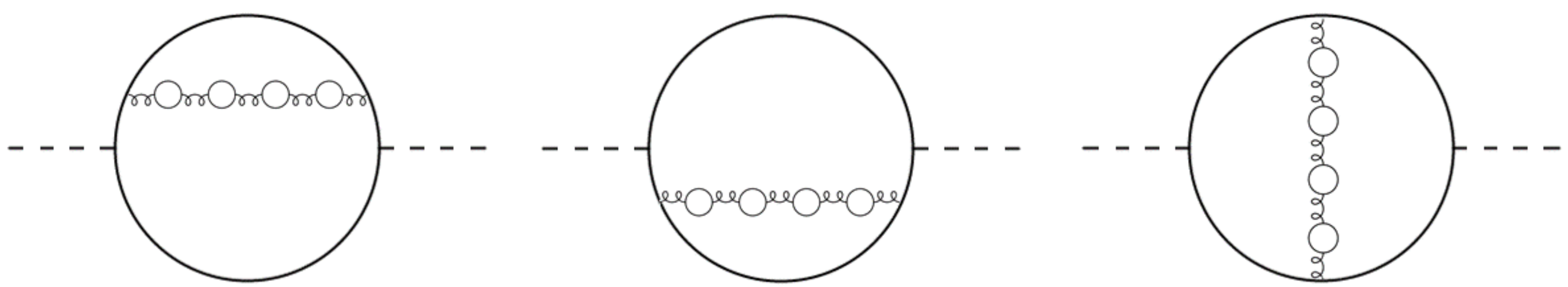
Gluon propagator with insertions of $q\bar{q}$ loops

$\alpha_s n_f$ $\qquad$ $(\alpha_s n_f)^2$

$$\boxed{\alpha_s n_f \sim \mathcal{O}(1)}$$

$$\beta_{0,f} = \frac{n_f}{6\pi}$$

Leading nf terms



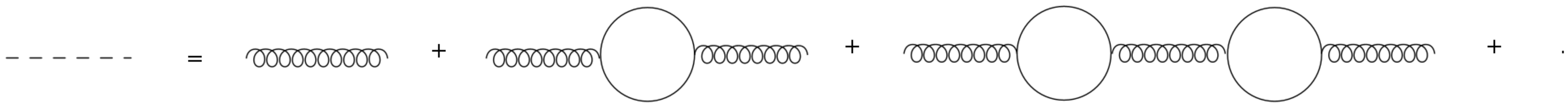
"Non-abelianization" of the result

$$\boxed{n_f \to 6\pi\beta_0}$$

A set of non-abelian diagrams included (running coupling)

large-β_0

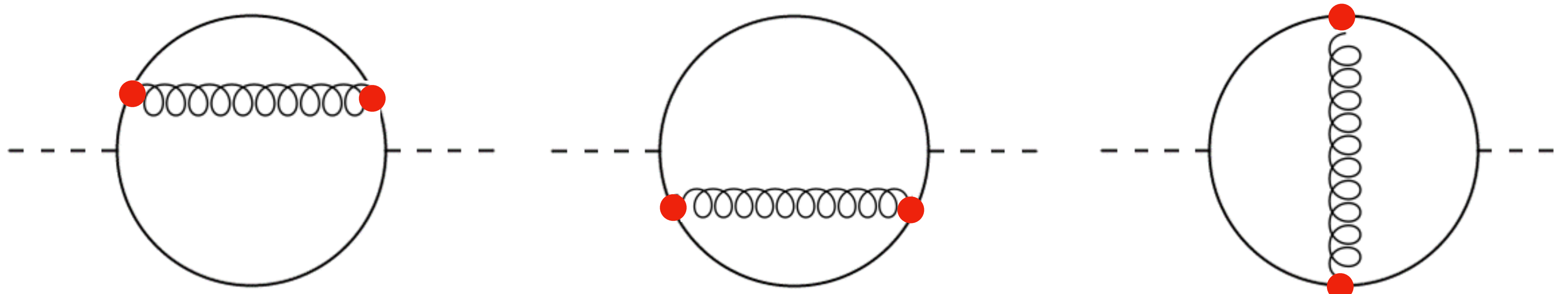
Gluon propagator with insertions of $q\bar{q}$ loops



$$\boxed{\alpha_s n_f \sim \mathcal{O}(1)}$$

$$\beta_{0,f} = \frac{n_f}{6\pi}$$

Leading nf terms



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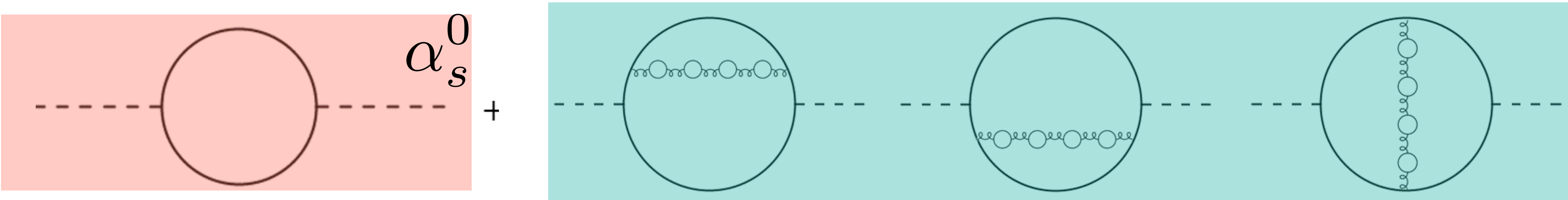


One chain

LO

NfLO

$$
\begin{array}{llll}
R = \boxed{1} & & & \\
+ & [\; c_{10} & &]\alpha_s \\
+ & [\; c_{21} n_f & + \; c_{20} &]\alpha_s^2 \\
+ & [\; c_{32} n_f^2 & + \; c_{31} n_f \; + \; c_{30} &]\alpha_s^3 \\
+ & \;\;\cdots & \;\;\cdots \quad\;\; \cdots & \\
+ & [\; c_{n,n-1} n_f^{n-1} & + \; c_{n,n-2} n_f^{n-2} \; + \; \cdots \; + \; c_{n,0} &]\alpha_s^n
\end{array}
$$

Renormalons

Perturbation theory is divergent Dyson '52 $\quad r_n \sim n!$

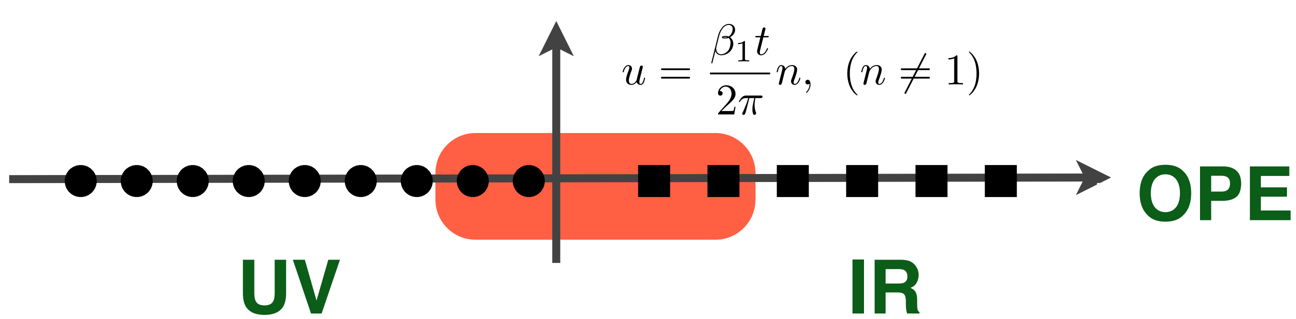
$$R \sim \sum_{n=0}^{n^*} r_n \alpha_s^{n+1} + e^{-p/\alpha_s} \xmapsto{\alpha_s(Q^2)} R \sim \sum_{n=0}^{n^*} r_n \alpha_s^{n+1} + \underbrace{\left(\frac{\Lambda^2}{Q^2}\right)^p}_{\text{OPE}}$$

Borel transform method

$$B[R](t) \equiv \sum_{n=0}^{\infty} r_n \frac{t^n}{n!}$$ which can be "summed" $\Longrightarrow$ $$\boxed{\tilde{R} \equiv \int_0^\infty dt\, \mathrm{e}^{-t/\alpha} B[R](t)}$$

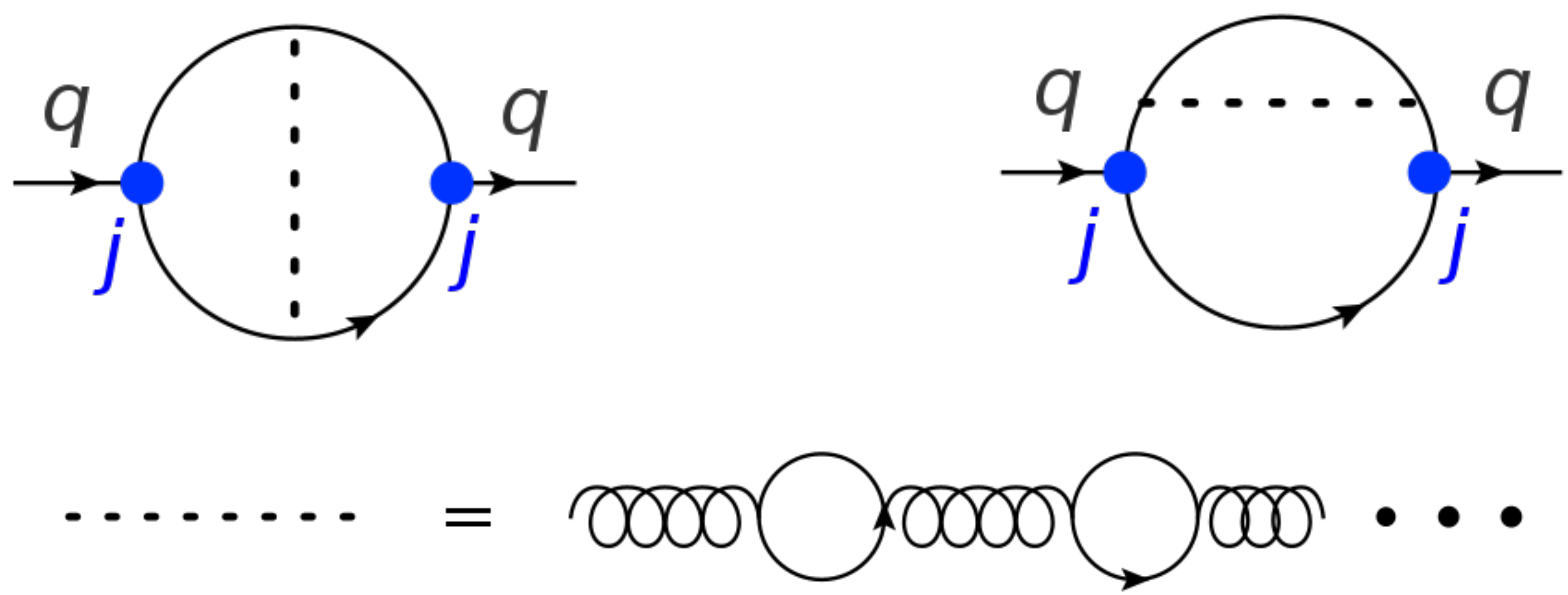
Singularities in the Borel plane: ***renormalons***

Beneke '99



$$B[R](t) = \frac{1}{(u-2)} \mapsto \left(\frac{\Lambda^2}{Q^2}\right)^2$$

Large-β_0 calculation of heavy-quark current correlators



Results available in the literature only for the moments of the *vector current*

Grozin & Sturm '04

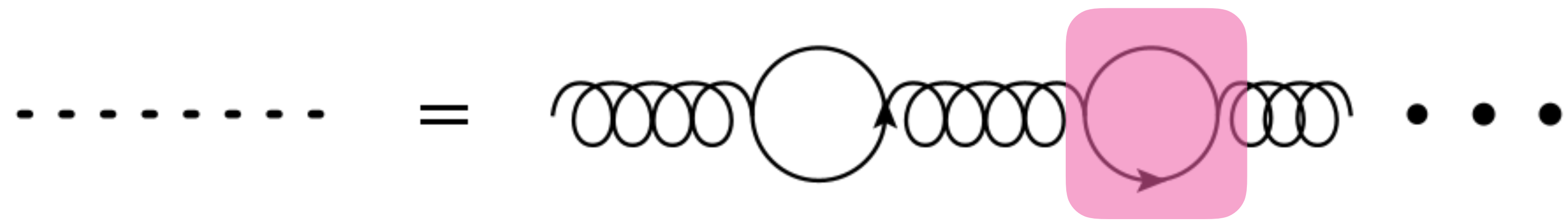
$$j_\mu^V = \bar{\psi}\gamma_\mu\psi, \quad j_\mu^A = \bar{\psi}\gamma_\mu\gamma_5\psi, \quad j^S = \bar{\psi}\psi \quad \text{and} \quad j^P = i\bar{\psi}\gamma_5\psi.$$

We have calculated for the first time the corresponding result for ***A***, ***S*** and ***PS*** cases

DB, V Mateu, M.V. Rodrigues, in preparation

large-β_0 results

Summing *n* bubbles in the gluon propagator (Landau gauge) $d = 4 - 2\epsilon$



$$D^{(n)}_{\mu\nu}(k) = \frac{-i}{(-k^2)^{(1+n\epsilon)}}\left(g_{\mu\nu} - \frac{k_{\mu\nu}}{k^2}\right)[I_B(\epsilon)]^n$$

(Continuous) shift in the power of the momentum in the denominator

Result of loop integration

A. Grozin, '03

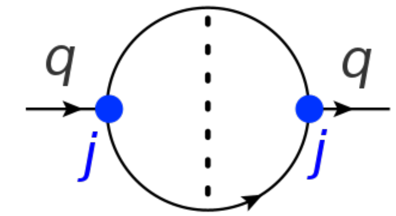
- Extend γ_5 to D dim.

Larin '93

- Renormalization
- Expansion in $\frac{q^2}{4m^2} \sim 0$

large-β_0 results

- Expansion in $\frac{q^2}{4m^2} \sim 0$



$$J_2(n_1, \cdots, n_5) = \int \frac{\mathrm{d}^d k_1 \, \mathrm{d}^d k_2}{[(k_1+q)^2 - m_0^2]^{n_1} [(k_2+q)^2 - m_0^2]^{n_2} [k_1^2 - m_0^2]^{n_3} [k_2^2 - m_0^2]^{n_4} [(k_2 - k_1)^2]^{n_5}}$$

After expanding and setting $q^2 = 0$

$$J_2(n_1, \cdots, n_5)|_{q=0} = \int \frac{\mathrm{d}^d k_1 \, \mathrm{d}^d k_2}{[k_1^2 - m_0^2]^{n_1+n_3} [k_2^2 - m_0^2]^{n_2+n_4} [(k_2 - k_1)^2]^{n_5}}$$

$$= -\pi^d (-1)^{\lambda_1+\lambda_2+\lambda_3} (m_0^2)^{d-\lambda_1-\lambda_2-\lambda_3}$$

$$\frac{\Gamma(\lambda_1+\lambda_3-d/2)\Gamma(\lambda_2+\lambda_3-d/2)\Gamma(d/2-\lambda_3)\Gamma(\lambda_1+\lambda_2+\lambda_3-d)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_1+\lambda_2+2\lambda_3-d)\Gamma(d/2)}$$

$\lambda_1 \equiv n_1 + n_3$, $\lambda_2 \equiv n_2 + n_4$ and $\lambda_3 \equiv n_5$

large-β_0 results

Structure of the results for the moments *M*

one-loop normalizations

$$M_n^V = \left[12\pi^2 Q_q^2 \frac{3}{16\pi^2}\right] \frac{g_n^V(0)}{(4m^2(\mu))^n} A_n^V(\mu)$$

After renormalization and expressing everything in terms of the $\overline{\mathrm{MS}}$ Mass

$$\hat{A}_n^\delta = 1 + \frac{1}{\beta_0}\int_0^\infty \mathrm{d}u\, e^{-u/\beta(\alpha_s(\mu_0))} \boxed{S_n^\delta(u)} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

Borel transform of the moments

General structure of the Borel transform of the moments

$$S_n^V(u) = \frac{8n}{u} + \left(\frac{e^{5/3}\mu_0^2}{\tilde{m}^2}\right)^u \frac{\mathrm{Csc}(\pi u)\Gamma(n+u)}{4^u\Gamma(3/2+n+u)} \pi^{3/2}(-1+u)(u+1+n)N_n^V(u)$$

large-β_0 results

Anatomy of the result

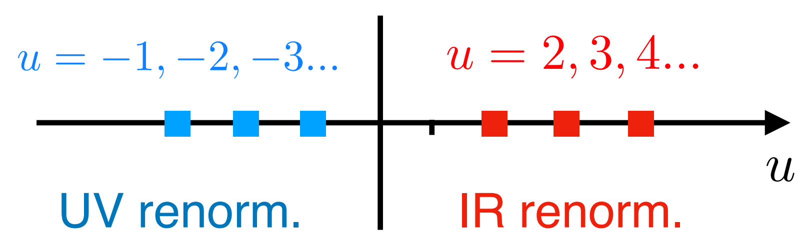
Scheme and scale independent

$$S_n^V(u) = \frac{8n}{u} + \left(\frac{e^{5/3}\mu_0^2}{\tilde{m}^2}\right)^u \frac{\mathrm{Csc}(\pi u)\Gamma(n+u)}{4^u\Gamma(3/2+n+u)}\pi^{3/2}(-1+u)(u+1+n)N_n^V(u)$$

Residual scale and scheme dependence

Renormalons are in the singularities of these functions

Polynomial of u



Non-trivial polynomials in u for each value of n

$$N_1^V(u) = \frac{u^3}{9} + \frac{29u^2}{27} + \frac{92u}{27} + 3,$$

$$N_2^V(u) = \frac{u^5}{96} + \frac{7u^4}{54} + \frac{2887u^3}{2592} + \frac{7393u^2}{1296} + \frac{2095u}{162} + 10$$

$\ldots$

Similar results for A, S, PS cases

DB, V Mateu, M.V. Rodrigues, in preparation

large-β_0 results

Non-trivial checks of the correctness of the results

- Reproduce all known leading-n_f terms in the QCD results.
- First IR renormalon at $u = 2$ (gluon condensate).
- PS $n = 3$ moment does not have a $u = 2$ renormalon which confirms that calculation of the gluon condensate coefficient, which vanishes in this one case.

Broadhurst, Baikov, Ilyin, Fleischer, Tarasov, and Smirnov '94

large-β_0 results

Ratios of moments

$$R_q^{X,n} \equiv \frac{\left(M_q^{X,n}\right)^{\frac{1}{n}}}{\left(M_q^{X,n+1}\right)^{\frac{1}{n+1}}}$$

Borel representation of the ratios of moments

$$R_n^V = \left(\frac{9}{4}Q_q^2\right)^{\frac{1}{n(n+1)}} \frac{\left(g_n^V(0)\right)^{\frac{1}{n}}}{\left(g_{n+1}^V(0)\right)^{\frac{1}{n+1}}}\left[1+\frac{1}{\beta_0}\int_0^\infty du e^{-u/\hat{a}(\mu)} B_n^V(u) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)\right]$$

DB,V Mateu, M.V. Rodrigues, in preparation

Borel transform

$$B_n^V(u) = \frac{S_n^V(u)}{n} - \frac{S_{n+1}^V(u)}{n+1}$$

Explicitly

Scheme and scale independent

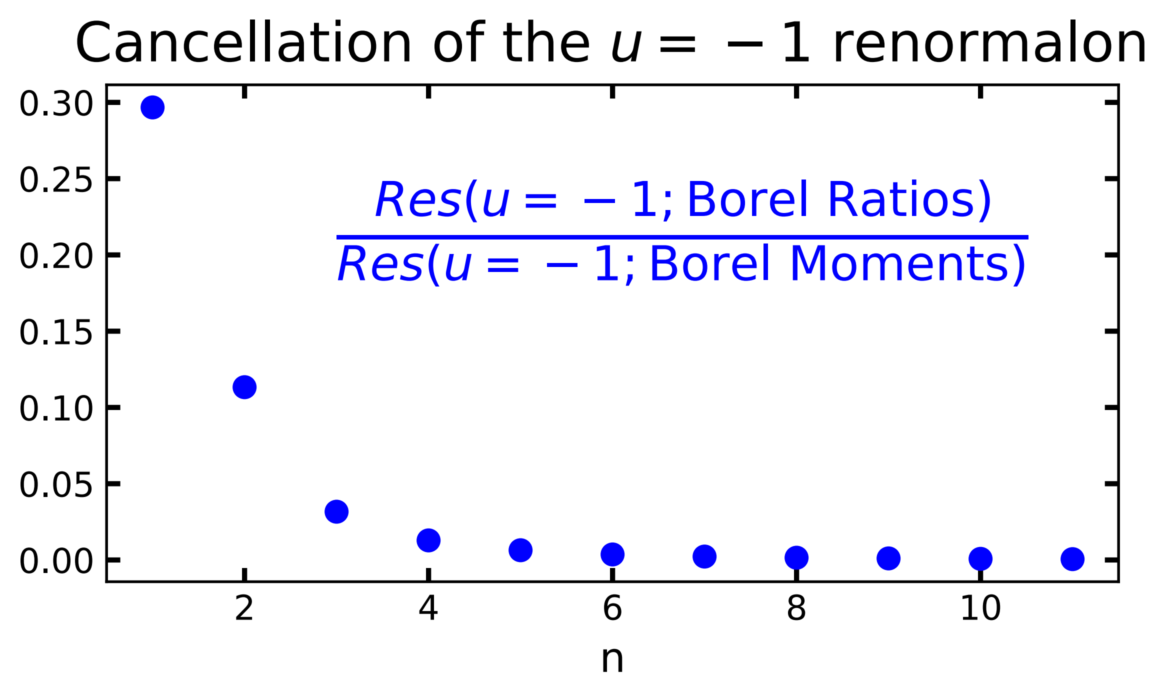
Partial cancelation of the u=-1 renormalon

$$B_n^V(u) = \boxed{\left(\frac{e^{5/3}\mu^2}{m^2}\right)^u} \frac{\mathrm{Csc}(\pi u)\pi^{3/2}(-1+u)}{4^u}$$

$$\left[\frac{\Gamma(n+u)(u+1+n)N_n^V(u)}{n\,\Gamma(3/2+n+u)} - \frac{\Gamma(n+u+1)(u+2+n)N_{n+1}^V(u)}{(n+1)\,\Gamma(5/2+n+u)}\right]$$

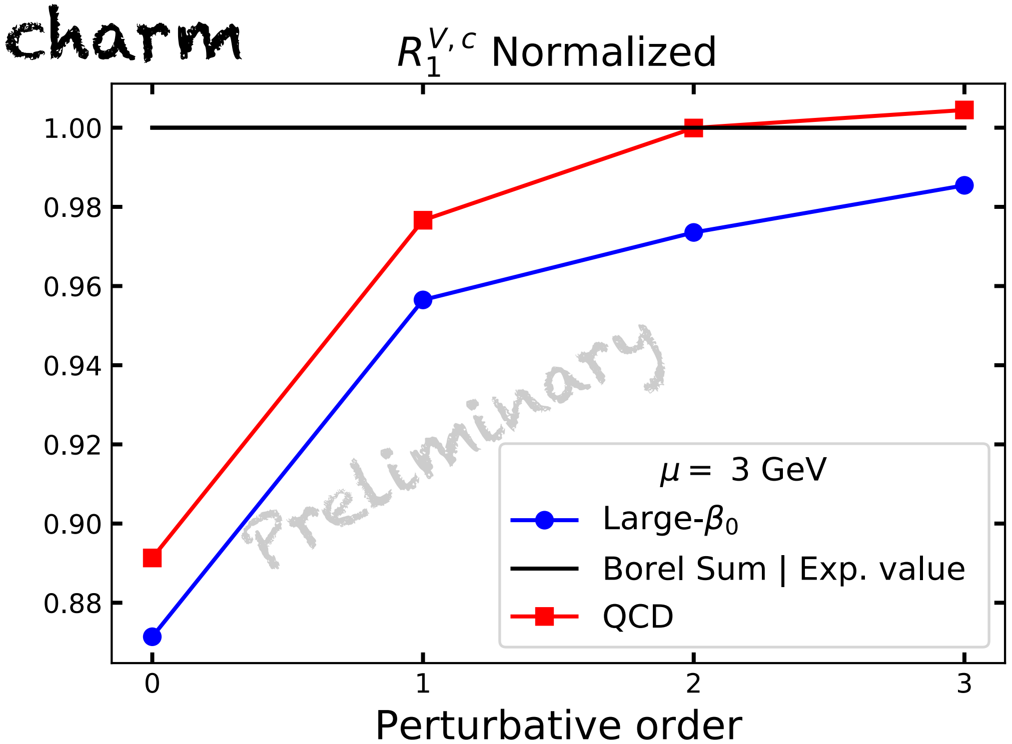
large-β_0 results

Renormalon cancelation in the R_n ratios (vector case)



QCD vs Large-beta 0

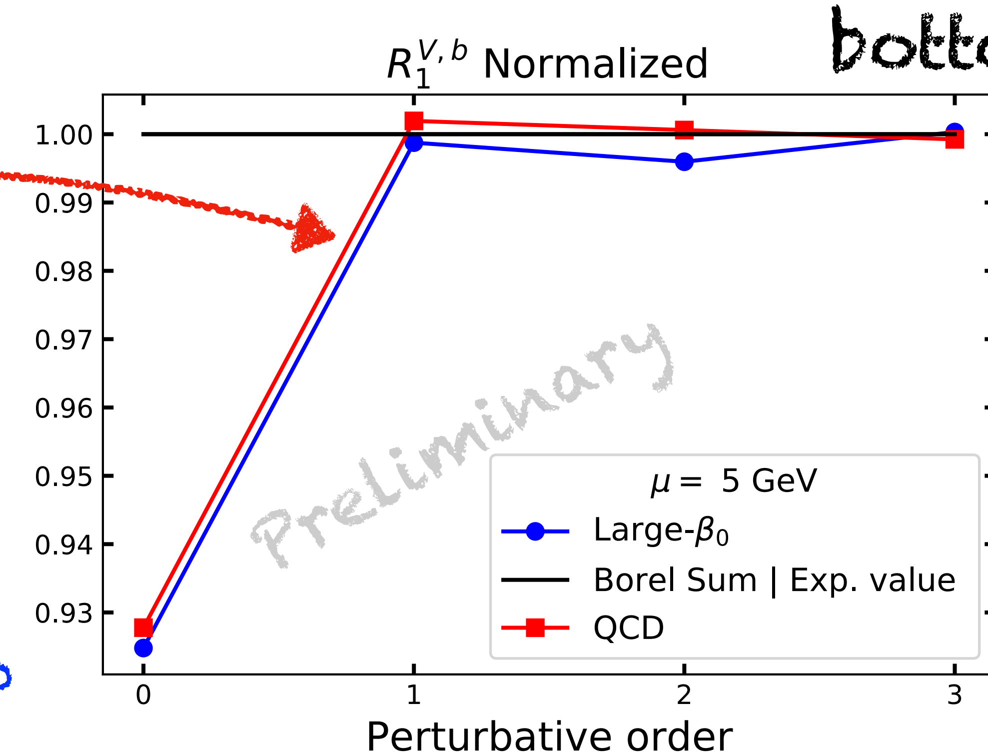
charm



Similar behaviour but QCD is actually better than large-beta_0

QCD vs Large-beta 0

bottom

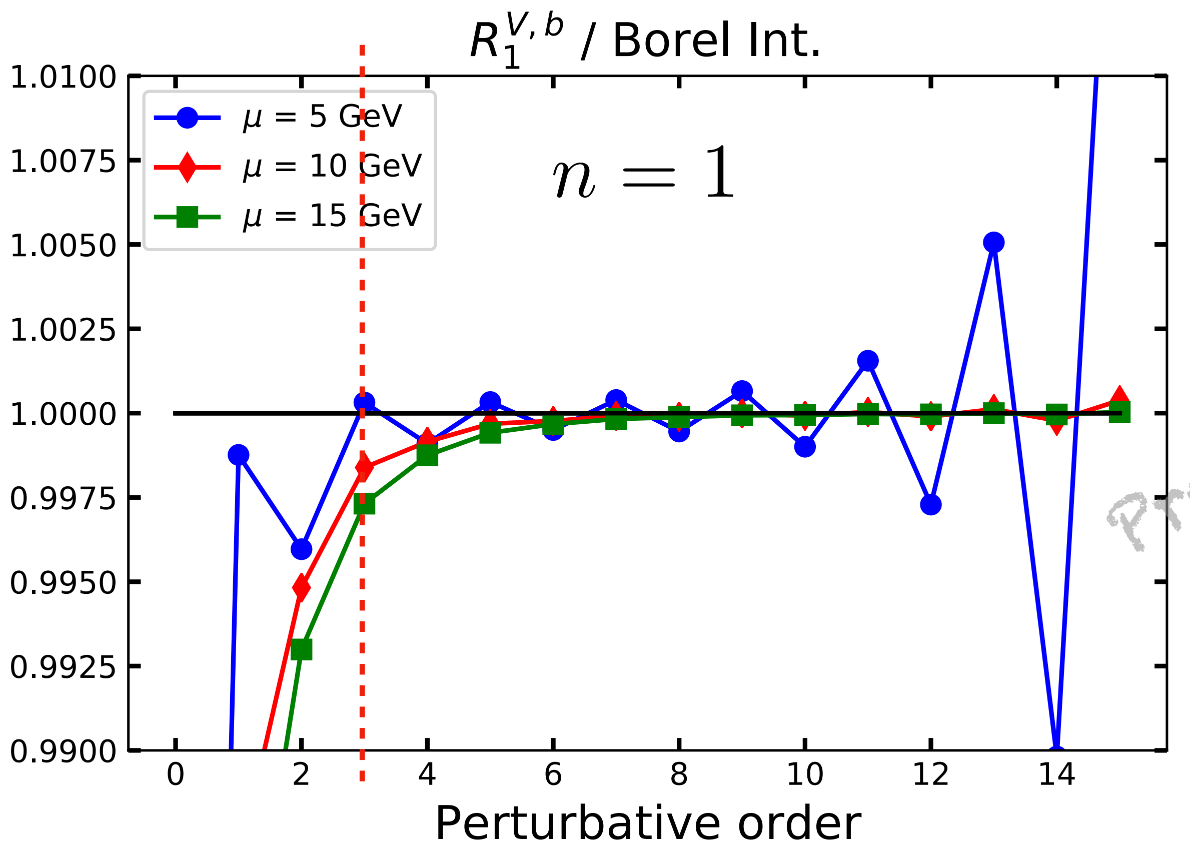


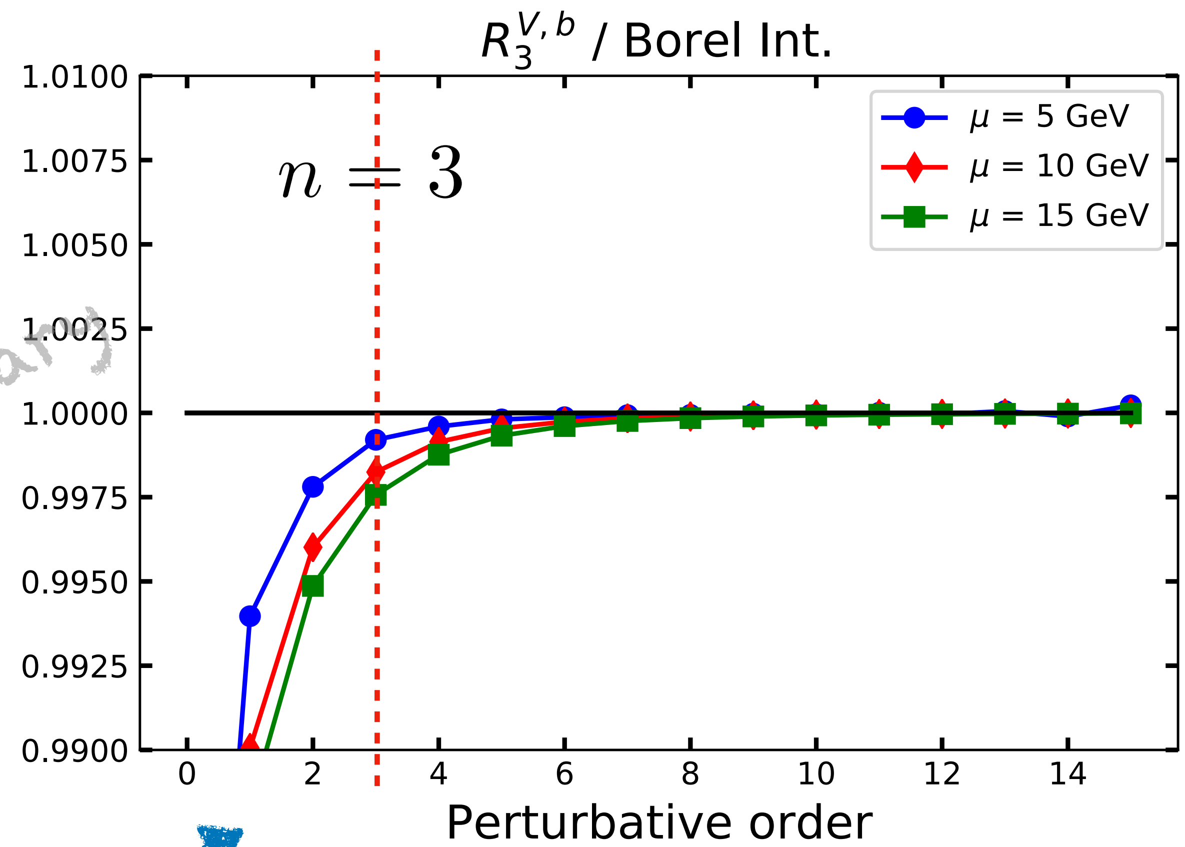
large-β_0 results

$$R_n^V$$

bottom

Good pt behaviour but somewhat slow convergence





UV renormalon strongly suppressed with higher n

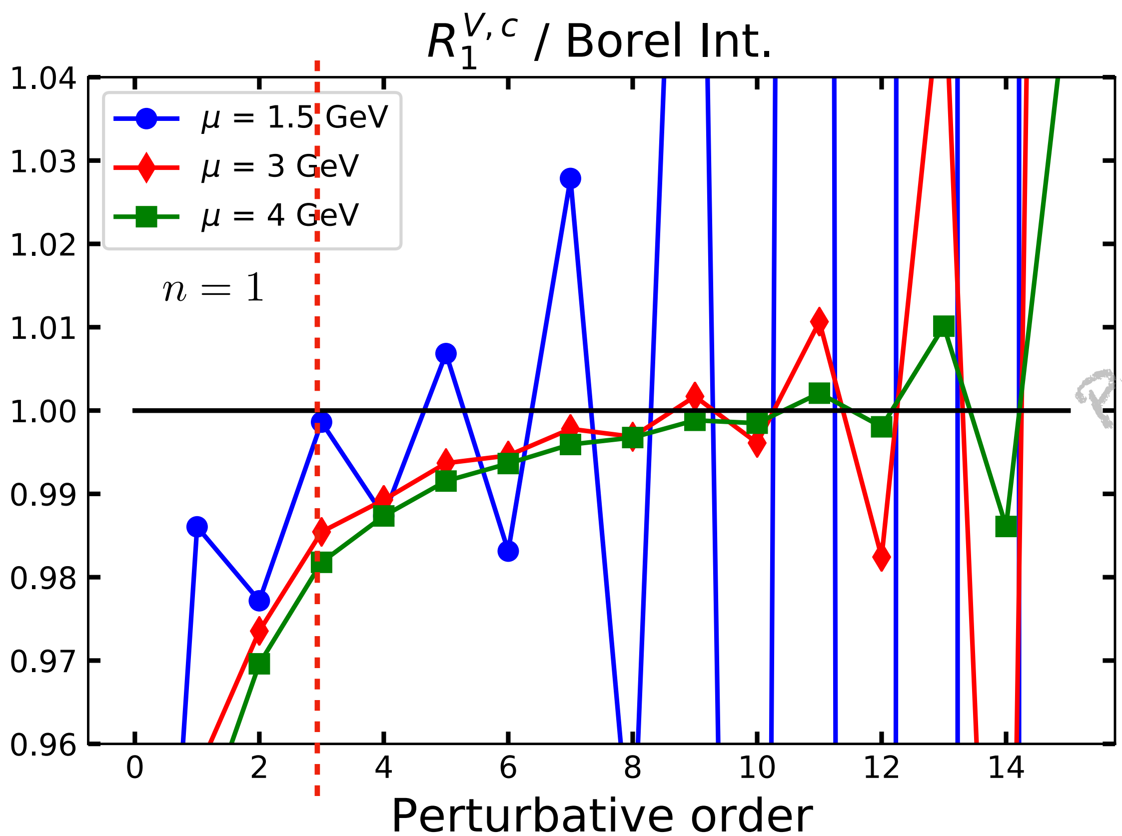
large-β_0 results

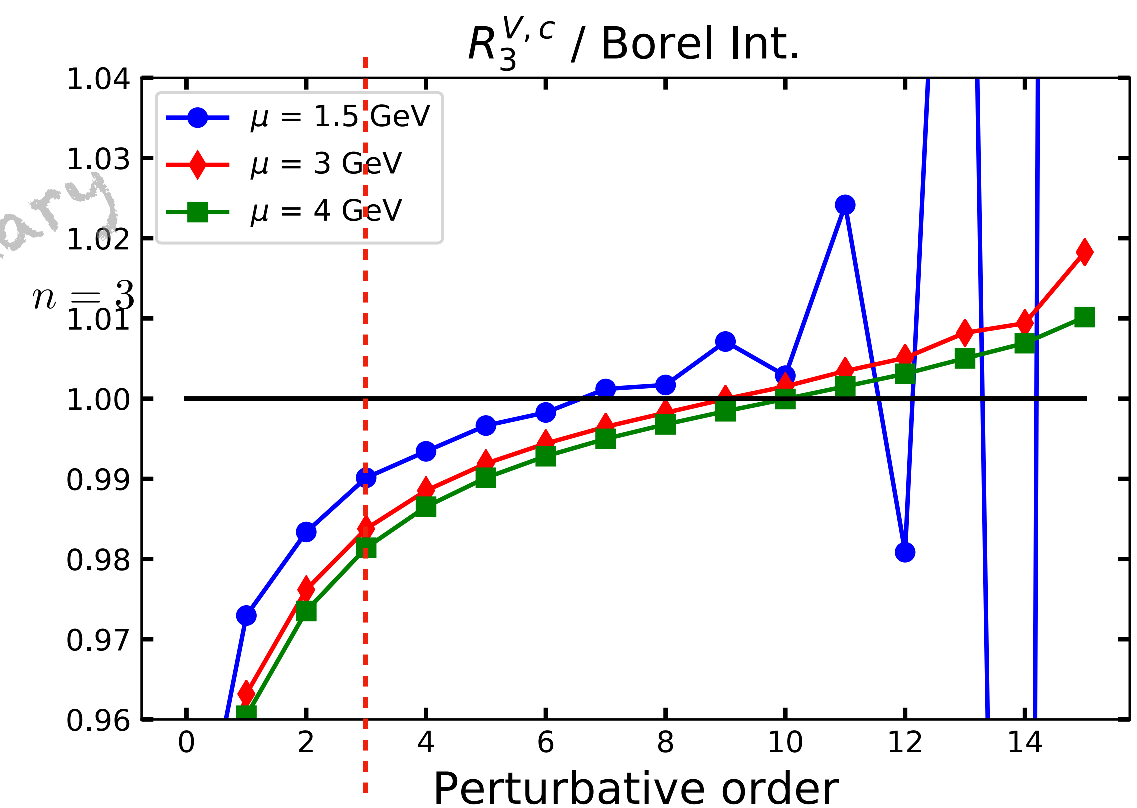
$$R_n^V$$

charm

Signs of the leading IR renormalon

Beneke, DB, Jamin. ’12
DB, Oliani ‘20

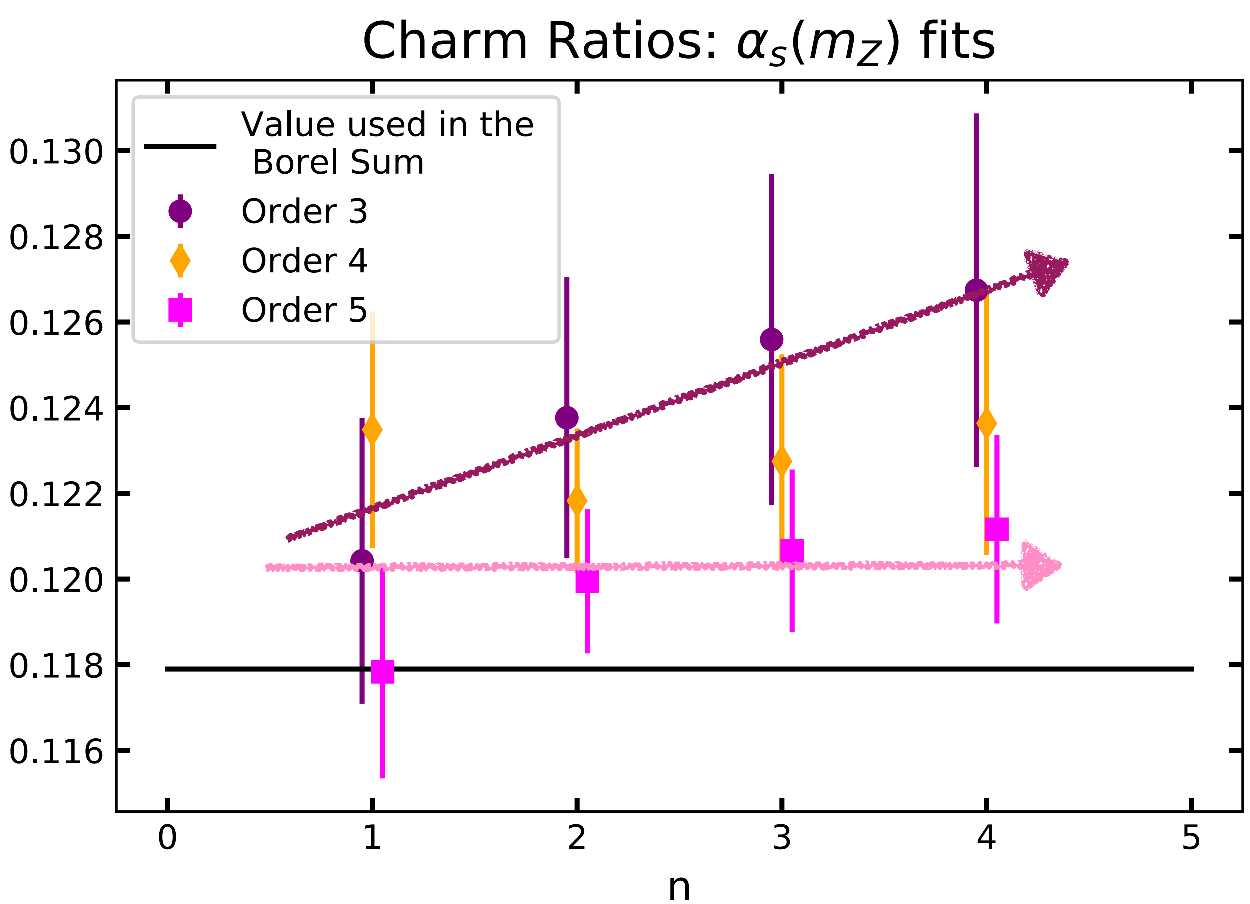


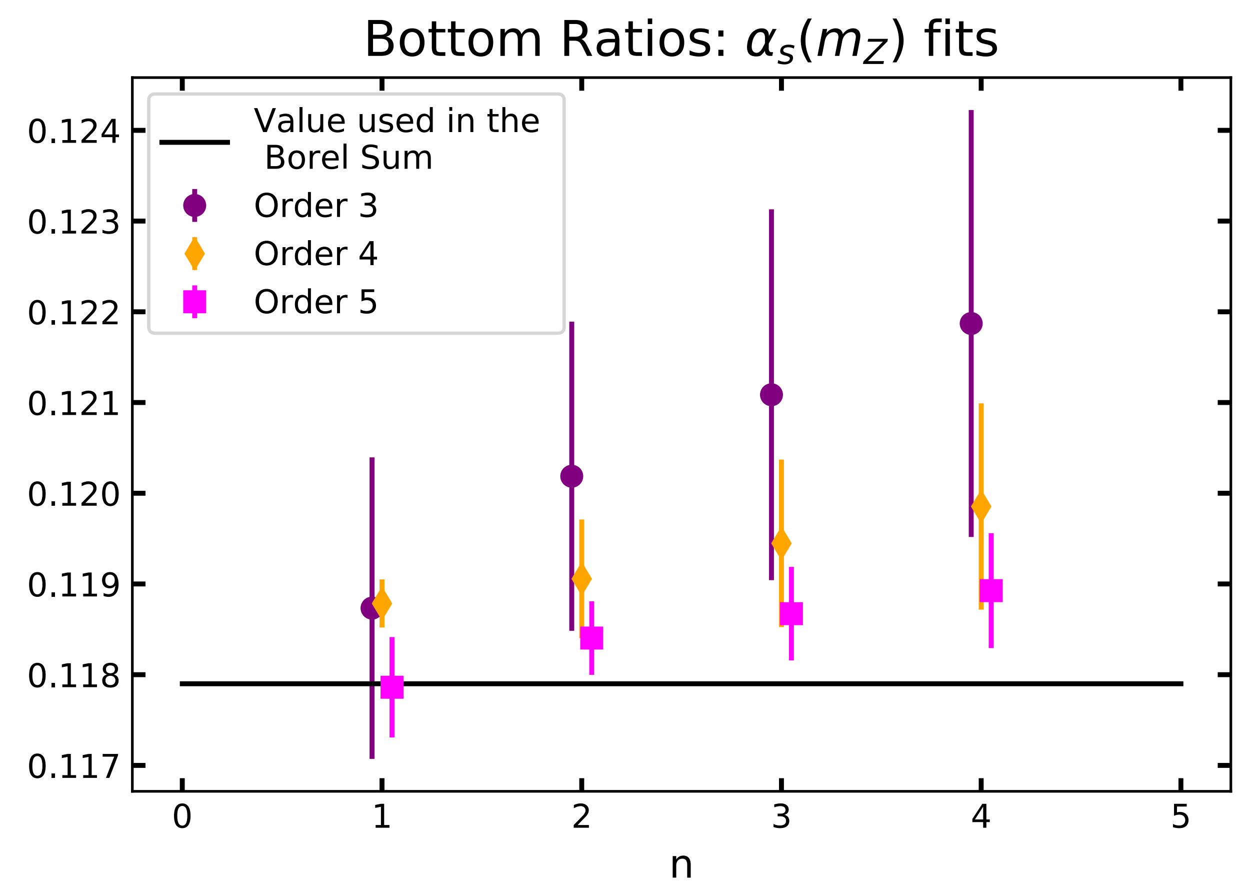


UV renormalon less strongly suppressed with higher n

large-β_0 results

Toy extraction of α_s in large-β_0 with the Borel sum as “experiment”





Trends in alpha_s values qualitatively corroborated by large-beta0 results.

One order more in the pt. series should lead to more stable results.

Conclusions

Conclusions

α_s can be extracted reliably from *R* data with 4, and 5 active flavours.

Ratios of moments of bottomonium vector-current correlators ideal from the theory view point, but larger exp. errors.

Ratios tend to have good perturbative expansion (renormalon cancelations).

The five loop result would still improve our results (stability and pt. errors)

At present, best determination from charm ratio with *n*=2:

$$\alpha_s(m_Z) = 0.1168 \pm 0.0019$$

Our results are obtained with a conservative error estimate.

PS current moments (from lattice) give stable results but with larger uncertainty.

Our analysis of the perturbative error is more conservative than some of the original studies