Scalar resonances in the hadronic light-by-light contribution to the muon $(g-2)_\mu$: an holographic approach

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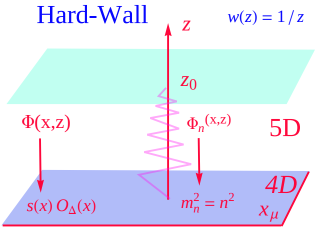
IFAE Virtual Seminar, Barcelona, February, 24, 2022

Plan of the talk

- Holographic models of QCD
- 2-point functions: VV, AA and SS Current-Current Correlators
- 3-point functions: The Pion Transition Form Factor
- 4-point functions: The HLbL Tensor
  - The pion and axial vectors exchange contributions [C., Cata, D'Ambrosio, Greynat, Iyer]
  - Short distance constraints: Quark loop and Melnikov-Vainshtein
- The scalars contribution [C., Cata, D'Ambrosio]
  - 3-point functions: SVV currents correlator
  - Asymptotic behaviour and SD constraints
  - Numerical results
- Conclusions and Outlook

A first glimpse: Scalar field in a flat 5D slice



The Conjecture

$$\exp(iW_4[\mathbf{s}(\mathbf{x})])$$
$$\equiv \left\langle \exp\left(i \int d^4x\, \mathbf{s}(\mathbf{x}) O_\Delta(x)\right)\right\rangle_{\text{strong coupled}}$$
$$= \exp\left(i\, S_5^{\text{gravity}}(\Phi_0(z,x))\right)$$

For simplicity take a singe flat 5D slice, $w(z) = 1$

$$S_X = \int d^4x \int_0^{z_0} dz\, \left[\eta^{MN}\partial_M X(x,z)\partial_N X(x,z) + V(X)\right], \qquad \eta_{MN} = (\eta_{\mu\nu}, -1)$$

Free case $V(X) = 0$. By parts and assuming $X\partial_\mu X \to 0$, *for* $|x| \to \infty$:

$$S_X = -\int d^4x\, X(x,z)\partial_z X(x,z)|_0^{z_0} - \int d^4x \int_0^{z_0} dz\, X(x,z)\underbrace{\Box X(x,z)}_{=0 \text{ if } \Box X(x,z)=0}$$

On-shell $S_X \to$ 4D term depending on the boundary values $X(0)$ and $X(z_0)$ of the 5D scalar field.

A first glimpse: Scalar field in a flat 5D slice cont'd

Let $X(x,z) = s(q)e^{iq\cdot x}f(z,q)$, with $f(z,q)$ solving the 5D EOM, with b.c. Dirichlet: $X(x,z_0) = 0$, (or Neumann $\partial_z X(x,z_0) = 0$) to get rid of b. terms at z_0 and $f(0) = 1$.

$$f(z,q) = \cos(qz) - \cot(qz0)\sin(qz), \Rightarrow S_X^{on-shell} = \int d^4k\ s(q)\Pi_X(q^2)s(-q)$$

$$\Pi_X(q^2) \equiv q\tan(q\,z_0)$$

is a 2-point correlator in 4D momentum space, with single poles at $q_n = n\pi/z_0$ due to the presence of the infinite KK tower of 4D scalar resonances of increasing masses, and normalizable eigenfunctions

$$\phi_n(z) \propto \sin(m_n z0), \quad m_n = n\pi/z_0$$

Notice that the Large Euclidean limit $Q^2 \equiv -q^2 \to \infty$ is not the good one to match with pQCD

$$\Pi_X(-Q^2) \to -Q$$

Clearly the $z \to 0$ behaviour of $f(z)$ is wrong. More on this later.
Interacting case $V(x) \neq 0$. If $V(X) = c_3X^3 + c_4X^4 + \cdots$ we have local interaction terms in 5D space. 5D EOM can be solved only perturbatively, using the 5D Green function.

$$\Box_{(x,z)}G_X(x,z;x',z') = \delta^4(x-x')\delta(z-z')$$

The analytic expressions, can be more easily understood , using the socalled Witten diagramms in 5D, as we shall see.

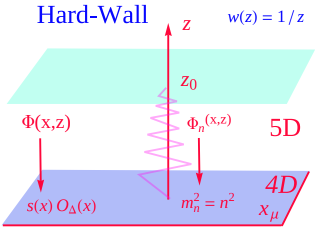
Holographic models of QCD: SS, HW1, HW2, SW

SS:	[Sakai,Sugimoto(05)]
HW1:	[Erlich, Katz, Son, Stephanov(05)],[Da Rold, Pomarol(05)]
HW2:	[Hirn,Sanz(05)]
SW:	[Karch, Katz, Son, Stephanov(06)]

Holographic models of QCD: recipes & ingredients

HQCD models inspired by AdS/CFT duality between a 4D (conformal) (Large-N_c) gauge theory at strong coupling and a (classical) 5D field theory in a curved Anti-de Sitter space

$$\exp(iW[\mathbf{s}(\mathbf{x})]) \equiv \left\langle \exp\left(i \int d^4x\, \mathbf{s}(\mathbf{x}) O_\Delta(x) \right) \right\rangle_{QCD} = \exp\left(i\, S_5(\Phi_0(z,x)) \right)$$



4D	5D
operator $O_\Delta(x)$	dual field $\Phi(x,z)$
source $\mathbf{s}(\mathbf{x})$ coupled to $O_\Delta(x)$	on-shell $\Phi_0(x,z) \to \mathbf{s}(\mathbf{x})$
conformal dimension Δ	mass m_Φ:
	$m_\Phi^2 = (\Delta - p)(\Delta + p - 4)$
$U(N_f)_L \times U(N_f)_R$	$U(N_f)_L \times U(N_f)_R$
global symmetry	gauge symmetry
vector current $\bar{q}\gamma^\mu t^a q$ $\mathbf{v}_\mu^\mathbf{a}(\mathbf{x}) \leftarrow$	gauge field $V_\mu^a(x,z)$
axial current $\bar{q}\gamma^\mu\gamma_5 t^a q$ $\mathbf{a}_\mu^\mathbf{a}(\mathbf{x}) \leftarrow$	gauge field $A_\mu^a(x,z)$
quark bilinear $\bar{q}t^a q$ $\mathbf{s}(\mathbf{x}) \leftarrow$	scalar field $X^a(x,z)$

confinement	Hard-Wall: sharp cut-off $0 \le z \le z_0$ Soft-Wall: dilaton potential
Chiral Symmetry Breaking	5D profile $X(z)$ 5D parity/ ChSB boundary conditions

Holographic models of QCD: *minimal* 5D Lagrangian

$$S_5 = \int d^5x\sqrt{g}\left(\mathcal{L}_{\rm YM} + \mathcal{L}_{\rm CS} + \mathcal{L}_{\rm X}\right)$$

$$\mathcal{L}_{YM+CS} = -\lambda\,\mathrm{tr}\left[F^{MN}_{(L)}F_{(L)MN} + F^{MN}_{(R)}F_{(R)MN}\right] + c\,\mathrm{tr}\left[\omega_5(L) - \omega_5(R)\right]$$
$$\mathcal{L}_X = \rho\,\mathrm{tr}\left[D^M X^\dagger D_M X - m_X^2 X^\dagger X - z\delta(z-z_0)V(X)\right]$$

- 5D metric $ds_5^2 = w(z)^2\left(dx_\mu^2 - dz^2\right)$. For AdS, $w(z) = 1/z$.
- X transforms as a bifundamental of $U(3)_L \times U(3)_R$: $X \to g_L X g_R^\dagger$
- $\mathcal{F}_{MN} = \partial_M \mathcal{A}_N - \partial_N \mathcal{A}_M - i[\mathcal{A}_M, \mathcal{A}_N]$ and $\mathcal{A}_{L,R} = V \mp A$,
- In the HW1 models the 5D scalar field $X(x,z)$, dual to $\bar{q}q$, induces ChSB, by acquiring a non trivial 5D profile

$$X = X_0(z) = m_q\left[\left(\frac{z}{z_0}\right) - \left(\frac{z}{z_0}\right)^3\right] + s_0\left(\frac{z}{z_0}\right)^3.$$

- In HW2 there is no 5D scalar field. ChSB broken by different boundary conditions for V_μ and A_μ on the IR wall z_0 and the 4D chiral field $U(x)$ appears as the remnant of non trivial 5D Wilson line of A_z.

Holographic models of QCD: *non minimal* 5D Lagrangian

$$S_5 = \int d^5x \sqrt{g} \left(\mathcal{L}_{\rm YM} + \mathcal{L}_{\rm CS} + \mathcal{L}_{\rm X} + \mathcal{L'}_{\rm X} \right)$$

$$\mathcal{L}_{YM+CS} = -\lambda \, \mathrm{tr}\left[F^{MN}_{(L)} F_{(L)MN} + F^{MN}_{(R)} F_{(R)MN} \right] + c \, \mathrm{tr}\left[\omega_5(L) - \omega_5(R) \right]$$

$$\mathcal{L}_X = \rho \, \mathrm{tr}\left[D^M X^\dagger D_M X - m_X^2 X^\dagger X - z\delta(z - z_0) V(X) \right]$$

$$\mathcal{L}'_X = \zeta_+ \, \mathrm{tr}\left[X^\dagger X F^{MN}_{(R)} F_{(R)MN} + X X^\dagger F^{MN}_{(L)} F_{(L)MN} \right] + \zeta_- \mathrm{tr}\left[X^\dagger F^{MN}_{(L)} X F_{(R)MN} \right] .$$

- $V(x) = \frac{1}{2}\mu^2 \, \mathrm{tr}\left[X^\dagger X\right] - \eta \, \mathrm{tr}\left[(X^\dagger X)^2\right]$ is a scalar potential on the boundary, used it to enforce b.c. on the scalar field.
- **Very important!** $\mathcal{L}'_X$ generates $s\gamma\gamma$ vertices from $X_0(z) X(x,z) F^{MN}_{(V)} F_{(V)MN}$, depending on the parameter $\zeta = \zeta_+ + \frac{1}{2}\zeta_-$.

2-point Functions: VV, AA and SS Current-Current Correlators

$$\left\langle T\left\{J_V^\mu(x)J_V^\nu(y)\right\}\right\rangle \Longleftrightarrow \frac{\delta^2 S_5}{\delta v^\mu(x)\delta v^\nu(y)}$$

$$\left\langle T\left\{J_A^\mu(x)J_A^\nu(y)\right\}\right\rangle \Longleftrightarrow \frac{\delta^2 S_5}{\delta a^\mu(x)\delta a^\nu(y)}$$

$$\left\langle T\left\{J_S(x)J_S(y)\right\}\right\rangle \Longleftrightarrow \frac{\delta^2 S_5}{\delta s(x)\delta s(y)}$$

2-point Function: Fixing the parameters of HW2

$$2i\int d^4x\, e^{iq\cdot x}\left\langle T\left\{J^{a,\mu}_{V,A}(x)J^{b,\nu}_{V,A}(0)\right\}\right\rangle = \delta^{ab}\left(q^\mu q^\nu - q^2 g^{\mu\nu}\right)\Pi_{V,A}(q^2)$$

$$\Pi_V(q^2) = \overset{L_{10},\,H_1}{\otimes\otimes} + \frac{q^2}{g_5^2}\int_0^{z0} dz \int_0^{z0} dz' \underset{\rho,\,\rho',\,\ldots}{\overset{V}{\otimes\!=\!=\!=\!\otimes}} = \frac{1}{q^2 g_5^2}\,\partial_z\partial_{z'}\overset{V}{\otimes\!=\!=\!=\!\otimes}|_{z=z'=0}$$

$$\Pi_A(q^2) = \overset{f_\pi^2}{\otimes\!\bullet\!\otimes} + \underset{\pi}{\overset{f_\pi \qquad f_\pi}{\otimes\!\bullet - - \bullet\!\otimes}} + \overset{L_{10},\,H_1}{\otimes\otimes} + \frac{q^2}{g_5^2}\int_0^{z0} dz \int_0^{z0} dz' \underset{a_1,\, a_1',\,\ldots}{\overset{A}{\otimes\!=\!=\!=\!\otimes}}$$

$$= \frac{1}{q^2 g_5^2}\,\partial_z\partial_{z'}\overset{A}{\otimes\!=\!=\!=\!\otimes}|_{z=z'=0}$$

QCD OPE for Large Euclidean momentum $Q^2 = -q^2$

$$\Pi_{V,A,S}(-Q^2) \propto N_c\left(\log\frac{Q^2}{\mu^2}\right) + \cdots; \Longrightarrow \lambda = \frac{N_c}{48\pi^2},\quad \rho = \frac{N_c}{8\pi^2}$$

Low momenta: pion field canonical normalization: $f_\pi^2 = \dfrac{N_c}{6\pi^2 z_0^2}$

$$m_\rho = \frac{\gamma_{0,1}}{z_0} = \frac{2.405}{z_0} \Longrightarrow m_\rho = 776 MeV \text{ fixes the } \textit{size} \text{ of the extra-dim.} z_0 = 3.103 GeV^{-1}$$

3-point Functions: The Pion Transition Form Factor

$$\langle \pi(x) | \, T \left\{ J^{\mu}_{e.m.}(y) J^{\nu}_{e.m.}(z) \right\} | \rangle \Longleftrightarrow \frac{\delta^3 S_5}{\delta \pi(x) \delta v^{\mu}_0(y) \delta v^{\nu}_0(z)}$$

3-point Function: The Pion TFF from HW2

$$\int d^4x\, e^{-iq_1\cdot x}\, \langle P(q_1+q_2)|\, T\left\{J^{\mu}_{e.m.}(x)J^{\nu}_{e.m.}(0)\right\}|\rangle = \epsilon^{\mu\nu\rho\sigma} q_{1\,\rho} q_{2\,\sigma} \mathcal{F}_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$$

where $Q^2_{1,2} = -q^2_{1,2}$

For $P = \pi^0$, real photons normalization

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0) = \frac{N_c}{12\pi^2 f_\pi} \quad \text{(pointlike WZW vertex)}$$

Normalized TFF $K(Q_1^2, Q_2^2) \equiv \mathcal{F}_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)/\mathcal{F}_{P\gamma^*\gamma^*}(0,0) \rightarrow K(0,0) = 1$

Where is the pion field in HW2?

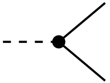
$$V_\mu(x,z) = v_\mu(x) + V^{(reson)}_\mu(x,z)$$
$$A_\mu(x,z) = \left(a_\mu(x) + \frac{\partial_\mu \pi(x)}{f_\pi}\right)\alpha(z) + A^{(reson)}_\mu(x,z)$$

Anomalous *AVV* amplitudes from trilinear terms in the CS action

$$S^{(3)}_{CS} = \frac{N_c}{24\pi^2}\int \mathrm{tr}\left(L(dL)^2 - R(dR)^2\right) \quad \text{with } L = V + A,\ R = V - A$$

3-point Functions: The Pion Transition Form Factor cont'nd

$$K(Q_1^2, Q_2^2) = -\int_0^{z_0} v(Q_1, z) v(Q_2, z) \partial_z \alpha(z) dz \quad \Longrightarrow$$



Vector bulk-to-boundary propagator $v(q^2, z) = -w(z')\partial_{z'} G_V(z, z'; q^2)|_{z' \to 0}$

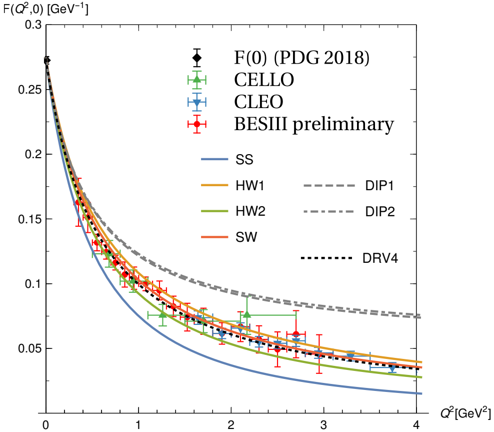
Low-Q^2

$$K(Q_1^2, Q_2^2) = 1 + \widehat{\alpha}\,(Q_1^2 + Q_2^2) + \widehat{\beta}\, Q_1^2 Q_2^2 + \widehat{\gamma}\,(Q_1^4 + Q_2^4) + \ldots$$

CELLO(91) : $\widehat{\alpha} = -1.76(22) GeV^{-2}$

NA62(17) : $\widehat{\alpha} = -1.76(22) GeV^{-2}$

$\longrightarrow$ W.A. : $\widehat{\alpha} = -1.84(17) GeV^{-2}$



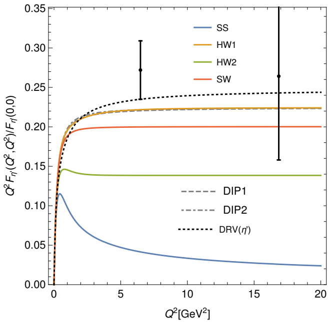
[Leutgeb,Mager,Rebhan(19)]

3-point Functions: The Pion Transition Form Factor cont'nd

Large Euclidean momentum $Q^2 \gg \Lambda_{QCD}$

$$K^{pQCD}(Q^2,0) = \frac{8\pi^2 f_\pi^2}{Q^2} \qquad K^{pQCD}(Q^2,Q^2) = \frac{8\pi^2 f_\pi^2}{3Q^2}$$

- The same expressions obtained in HW1 and HW2 and SW due to AdS metric
- However, with $z_0 = 3.103 GeV^{-1}$, in order to reproduce the value of the ρ meson mass, f_π is underestimated in HW2, and since $8\pi^2 f_\pi^2 = 4/z_0$, one gets 61.6% of the pQCD result, as shown in the figure.
- Possible solution: shrinking $z_0 = 3.103 GeV^{-1}$ one gets the physical value of $f_\pi = 92.4 MeV$, at the cost of overestimating $m_\rho = 987 MeV$



Double-virtual TFF with experimental data for η' from BaBar rescaled by f_π / f'_η [Leutgeb,Mager,Rebhan(19)]

3-point Func.: TFF and one-pion exchange HLbL diagrams

Ansätze for $\mathcal{F}_{P\gamma^*\gamma^*}(q_1^2, q_2^2)$

$$WZW: -\frac{N_c}{12\pi^2 f_\pi}$$

$$VMD: -\frac{N_c}{12\pi^2 f_\pi}\frac{m_V^2}{(q_1^2 - m_V^2)}\frac{m_V^2}{(q_2^2 - m_V^2)}$$

$$LMD: \frac{f_\pi}{3}\frac{q_1^2 + q_2^2 - (N_c m_V^4/(4\pi^2 f_\pi^2))}{(q_1^2 - m_V^2)(q_2^2 - m_V^2)}$$

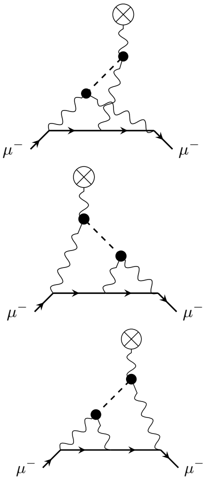
$$LMD + V: \frac{f_\pi}{3}\frac{P_6(q_1^2, q_2^2, M_{V_1}^2, M_{V_2}^2; h_1, h_2, h_5)}{(q_1^2 - m_{V_1}^2)(q_2^2 - m_{V_1}^2)(q_1^2 - m_{V_2}^2)(q_2^2 - m_{V_2}^2)}$$

[Knecht, Nyffeler(01))]

$$DIP: -\frac{N_c}{12\pi^2 f_\pi}\left(1 + \lambda\left(\frac{q_1^2}{(q_1^2 - m_{V_1}^2)} + \frac{q_2^2}{(q_2^2 - m_{V_2}^2)}\right)\right.$$
$$\left. + \eta \sum_{i=1,2}\frac{q_1^2 q_2^2}{(q_1^2 - m_{V_i}^2)(q_2^2 - m_{V_i}^2)}\right)$$

[C, Cata,D'Ambrosio(10))]

HLbL One-pion exchange diagrams.



3-point Func.: $a_\mu^{\mathrm{HLbL},\pi^0}$ estimates

$$a_\mu^{\mathrm{HLbL},\pi^0} = -\frac{e^6}{48m_\mu}\int\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{1}{q_1^2q_2^2(q_1+q_2)^2}\frac{1}{(p+q_1)^2-m_\mu^2}\frac{1}{(p-q_2)^2-m_\mu^2}$$

$$\times\left[\frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_3^2,0)}{q_3^2-m_\pi^2}T_1(q_1,q_2;p)\right.$$

$$\left.+\frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_3^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_2^2,0)}{q_2^2-m_\pi^2}T_2(q_1,q_2;p)\right]$$

Using Gegenbauer polynomials techniques [Knecht Nyffeler 01] only a triple integral remains

$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1\int_0^\infty dQ_2\int_{-1}^1 d\tau\sqrt{1-\tau^2}Q_1^3Q_2^3\sum_{i=1}^2\bar{T}_i(Q_1,Q_2,\tau)\bar{\Pi}_i(Q_1,Q_2,\tau),$$

where $Q_1 := |Q_1|$, $Q_2 := |Q_2|$. $\bar{\Pi}_i$ evaluated for the reduced kinematics

$$q_1^2 = -Q_1^2,\quad q_2^2 = -Q_2^2,\quad q_3^2 = -Q_3^2 = -Q_1^2 - 2Q_1Q_2\tau - Q_2^2,\quad q_4^2 = 0.$$

3-point Func. : $a_\mu^{\mathrm{HLbL},\pi^0}$ estimates contn'd

$a_\mu^{\mathrm{HLbL},\pi^0} \times 10^{-9}$		
VMD	5.7	KN(01)
LMD+V	6.3	KN(01)
DIP	6.58	CCD(11)
⟨HQCD's⟩	5.9(2)	LMR(19)
DVR interp.	5.64(25)	DVR(19)
Lattice	5.97± 0.23	GMN(19)

⟨HQCD's⟩LMR(19) $a_\mu^{\mathrm{HLbL},\pi^0} \times 10^{-9}$	
SS	4.83
HW1	6.13
HW2	5.66
SW	5.92

[Danilkin,Redmer,Vanderaeghen(19)], [Gérardin,Meyer, Nyffeler(19)]

However, there is a problem: The value for HW2 is obtained with the physical value $f_\pi = 92,4 MeV$ while taking $N_c = 3$ and $m_\rho = 776 MeV$, but as we already saw, **these three parameters are not independent in HW2 !** Different choices of fixing two of the parameters to their physical values (but not the third) all lead to a sensible increase of the value of $a_\mu^{\mathrm{HLbL},\pi^0} \sim 30\%$

4-point Function: The Hadronic Light-by-Light Tensor

$$\left\langle \left| T\left\{ J^{\mu}_{e.m.}(x) J^{\nu}_{e.m.}(y) J^{\lambda}_{e.m.}(z) J^{\sigma}_{e.m.}(w) \right\} \right| \right\rangle$$

$$\Longleftrightarrow \frac{\delta^4 S_5}{\delta v_0^{\mu}(x)\delta v_0^{\nu}(y)\delta v_0^{\lambda}(z)\delta v_0^{\sigma}(w)}$$

4-point Function I:The Hadronic Light-by-Light Tensor

$$\Pi^{\mu\nu\lambda\sigma}(q_1,q_2,q_3) = -i\int d^4x d^4y d^4z\, e^{-i(q_1\cdot x+q_2\cdot y+q_3\cdot z)} < |T\{j^\mu_{\rm em}(x)j^\nu_{\rm em}(y)j^\lambda_{\rm em}(z)j^\sigma_{\rm em}(0)\}| >$$

$$q_4 = q_1+q_2+q_3$$

138 Lorentz structures

$$\Pi^{\mu\nu\lambda\sigma} = g^{\mu\nu}g^{\lambda\sigma}\,\Pi^1 + g^{\mu\lambda}g^{\nu\sigma}\,\Pi^2 + g^{\mu\sigma}g^{\nu\lambda}\,\Pi^3$$
$$+\sum_{i,j=1,2,3}\Big(g^{\mu\nu}q_i^\lambda q_j^\sigma\,\Pi^4_{ij} + g^{\mu\lambda}q_i^\nu q_j^\sigma\,\Pi^5_{ij} + g^{\mu\sigma}q_i^\nu q_j^\lambda\,\Pi^6_{ij}$$
$$+g^{\nu\lambda}q_i^\mu q_j^\sigma\,\Pi^7_{ij} + g^{\nu\sigma}q_i^\mu q_j^\lambda\,\Pi^8_{ij} + g^{\lambda\sigma}q_i^\mu q_j^\nu\,\Pi^9_{ij}\Big)$$
$$+\sum_{i,j,k,l=1,2,3} q_i^\mu q_j^\nu q_k^\lambda q_l^\sigma\,\Pi^{10}_{ijkl}$$

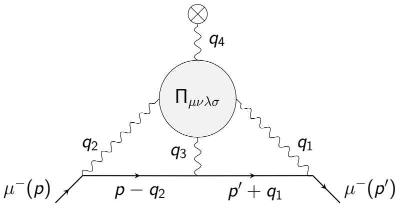
95 linearly independent relations from gauge invariance $\Longrightarrow$

$$\{q_{1\mu}, q_{2\nu}, q_{3\rho}, q_{4\sigma}\}\Pi^{\mu\nu\lambda\sigma}(q_1,q_2,q_3) = 0$$

Complete crossing symmetric, e.g. under

$$\mathcal{C}_{14} = \{q_1 \leftrightarrow -q_4, \mu \leftrightarrow \sigma\}, \quad \mathcal{C}_{13} = \{q_1 \leftrightarrow q_3, \mu \leftrightarrow \lambda\}$$

The HLbL tensor in the HLbL diagram



43 linearly independent tensor structures

BTT basis: 54 (redundant) tensor structures, with scalar functions Π_i free of kinematic singularities [Colangelo et al.15]

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma}\Pi_i,$$

4-point Function: The Master Formula for $a_\mu^{\rm HLbL}$

$$a_\mu^{\rm HLbL} = -\frac{e^6}{48 m_\mu}\int \frac{d^4 q_1}{(2\pi)^4}\frac{d^4 q_2}{(2\pi)^4}\frac{1}{q_1^2 q_2^2 (q_1+q_2)^2}\frac{1}{(p+q_1)^2-m_\mu^2}\frac{1}{(p-q_2)^2-m_\mu^2}$$
$$\times \,\mathrm{Tr}\left((\not p + m_\mu)[\gamma^\rho,\gamma^\sigma](\not p + m_\mu)\gamma^\mu(\not p + \not q_1 + m_\mu)\gamma^\lambda(\not p - \not q_2 + m_\mu)\gamma^\nu\right)$$
$$\times \sum_{i=1}^{54}\left(\frac{\partial}{\partial q_4^\rho} T^i_{\mu\nu\lambda\sigma}(q_1,q_2,q_4-q_1-q_2)\right)\Bigg|_{q_4=0}\Pi_i(q_1,q_2,-q_1-q_2).$$

Only 19 independent linear combinations of the 54 $T_i^{\mu\nu\rho\lambda}$ contribute to $a_\mu^{\rm HLbL}$. Using Gegenbauer polynomials techniques [Knecht Nyffeler 01], the symmetry of the loop integral and the propagators, there remain 12 different integrals containing 12 coefficients $\bar\Pi_i(q_1,q_2,-q_1-q_2)$.

$$a_\mu^{\rm HLbL} = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1\int_0^\infty dQ_2\int_{-1}^{1} d\tau\sqrt{1-\tau^2}\,Q_1^3 Q_2^3\sum_{i=1}^{12}\bar T_i(Q_1,Q_2,\tau)\bar\Pi_i(Q_1,Q_2,\tau),$$

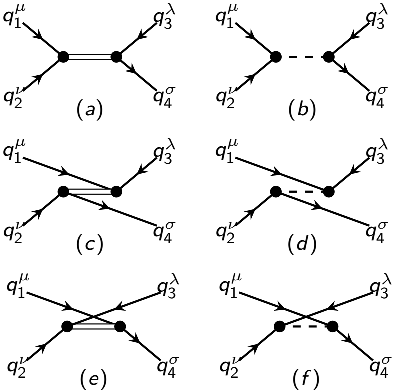
where $Q_1 := |Q_1|$, $Q_2 := |Q_2|$. $\bar\Pi_i$ evaluated for the reduced kinematics

$$q_1^2=-Q_1^2,\quad q_2^2=-Q_2^2,\quad q_3^2=-Q_3^2=-Q_1^2-2Q_1Q_2\tau-Q_2^2,\quad q_4^2=0.$$

Integral kernels expressions $\bar T_i(Q_1,Q_2,\tau)$, in [Colangelo et al.15&17]

4-point Function: HLbL tensor from HW2

[Cata, C., D'Ambrosio, Greynat, Iyer]

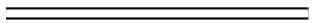


Pion and Massive axial resonances anomalous AVV vertices from S_{CS}

Propagators (from S_{YM})

(Massive) axial resonances

$G_A^{\mu\nu}$



5D axial Green function

$$G_A^{\mu\nu}(z,z';q^2) =$$
$$G_A^T(z,z';q^2)P_T^{\mu\nu}(q) + G_A^L(z,z')P_L^{\mu\nu}(q)$$
$$P_T^{\mu\nu}(q) = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right),$$
$$P_L^{\mu\nu}(q) = \frac{q^\mu q^\nu}{q^2}$$

Pion propagator

π

$$\frac{i}{q^2 - m_\pi^2}$$

4-point Function: HLbL tensor from HW2 contn'd

$$\Pi^{\mu\nu\lambda\sigma} = \underbrace{\Pi_L^{(\pi,\,A)\,\mu\nu\lambda\sigma}}_{\text{pion \& massive axial reson.}} + \underbrace{\Pi_T^{(A)\,\mu\nu\lambda\sigma}}_{\text{massive axial reson.}}$$

where, for the massive resonances contributions

$$\Pi_{L,T}^{(A)\,\mu\nu\lambda\sigma} = \underbrace{\left(g^{\mu\mu'} - \frac{q_1^\mu q_1^{\mu'}}{q_1^2}\right)\left(g^{\nu\nu'} - \frac{q_2^\nu q_2^{\nu'}}{q_2^2}\right)\left(g^{\lambda\lambda'} - \frac{q_3^\lambda q_3^{\lambda'}}{q_3^2}\right)\left(g^{\sigma\sigma'} - \frac{q_4^\sigma q_4^{\sigma'}}{q_4^2}\right)}_{\text{transverse projectors on external vector legs}}$$

$$\times \underbrace{\varepsilon_{\mu'\nu'\alpha\beta}\,\varepsilon_{\lambda'\sigma'\gamma\delta}}_{\text{anomalous couplings}} \times \underbrace{P_{L,T}^{\alpha\gamma}}_{\text{L,T proj. in } G_A} \times \underbrace{A_{L,T}^{\beta\delta}}_{z \text{ and } z' \text{ integrals}}$$

$A_{L,T}^{\beta\delta}$ contains combinations of the form $q_a^\beta q_c^\delta\, \mathcal{G}_A^{L,T}(q_a, q_b; q_c, q_d)$ with the convolution integrals

$$\mathcal{G}_A^L(q_a, q_b; q_c, q_d) = \int_0^{z_0} dz \int_0^{z_0} dz'\, v(z, q_a^2)\partial_z v(z, q_b^2) G_A^L(z, z') v(z', q_c^2)\partial_{z'} v(z', q_d^2)$$

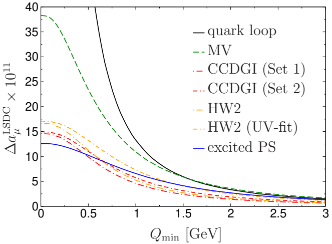
$$\mathcal{G}_A^T(q_a, q_b; q_c, q_d) = \int_0^{z_0} dz \int_0^{z_0} dz'\, v(z, q_a^2)\partial_z v(z, q_b^2) G_A^T(z, z'; q_a+q_b) v(z', q_c^2)\partial_{z'} v(z', q_d^2)$$

4-point Function: Short distance constraints

Asymptotic behaviour of the HW2 4-point amplitude for large Euclidean momenta

- Main result: Melnikov-Vainshtein [Melnikov,Vainshtein(04)] QCD OPE constraints are satisfied by the sole contributions of pions and the whole tower of massive axial vectors. No contributions from other fields, at least in the chiral limit.
  While the pion contribution is dominating at low momenta, the massive axial resonance contribution gives the MV OPE behaviour for Large Euclidean momenta.
- In the literature the MV constraint
  - lead to an increase of the accepted estimate of the HLbL
  - was difficult to implement in models:
    For instance MV proposed a model with pointlike WZW at the vertex with physical photon, while [JegerlehnerNyffeler(09)] got the MV behaviour using LMD+V TFF's, with an elaborate choice of the parameters.
- the HW2 seems the first model to satisfy MV, without any of the above assumptions dispite its simplicity
- axial anomaly plays a fundamantal role incontrolling the MV constraint. Nothing similar for other SD constraints, such as the quark loop limit.

Pions and axial vector contrib's: Numerical results



Estimates of corrections to the HLbL from SD constraints on the asymptotic behaviour (from White Paper), using different models.

	Set 1	**Set 2**
$a_\mu^{\mathrm{PS}}(\pi^0+\eta+\eta')$	8.1 (5.7+1.4+1.0)	11.2 (7.5+2.1+1.6)
$a_\mu^{A_L}(a_1+f_1+f_1^*)$	1.4 (0.4+0.4+0.6)	1.4 (0.4+0.4+0.6)
$a_\mu^L(a_\mu^{\mathrm{PS}}+a_\mu^{A_L})$	9.6	12.6
$a_\mu^T(a_1+f_1+f_1^*)$	1.4 (0.4+0.4+0.6)	1.4 (0.4+0.4+0.6)
a_μ	11.0	14.0

Table: Results for the longitudinal and transverse contributions to $a_\mu^{\mathrm{HLbL}} \times 10^{10}$. (In good agreement with the HQCD results of [Leutgeb, Rebhan] (HW2, HW2 (UV-fit) curves in the Figure).

Averaging out the results from the two sets of parameters and using the spread as an estimate of the uncertainty, our final number for the contribution of Goldstone modes and axial-vector states is

$$a_\mu^{(\mathrm{AV+PS})} = 12.5(1.5)\cdot 10^{-10}\,.$$

HLbL tensor: One scalar exchange contribution

$$\Pi_{\mu\nu\lambda\rho}(q_1,q_2,q_3,q_4) = \int_\epsilon^{z_0} dz \int_\epsilon^{z_0} dz' [T_{12}^{\mu\nu(a)} G^{(a)}(z,z';s) T_{34}^{\lambda\rho(a)} + T_{13}^{\mu\lambda(a)} G^{(a)}(z,z';t) T_{24}^{\nu\rho(a)} + T_{14}^{\mu\rho(a)} G^{(a)}(z,z';u) T_{23}^{\nu\lambda(a)}] ,$$

where $s=(q_1+q_2)^2$, $t=(q_1+q_3)^2$, $u=(q_1+q_4)^2$ and

$$T_{ij}^{\mu\nu(a)}(z) = \mathcal{P}_{ij}^{(a)}(z) P^{\mu\nu}(q_i,q_j) + \mathcal{Q}_{ij}^{(a)}(z) Q^{\mu\nu}(q_i,q_j) ,$$

where the two gauge-invariant tensors

$$P^{\mu\nu}(q_1,q_2) = q_2^\mu q_1^\nu - (q_1\cdot q_2)\eta^{\mu\nu}$$
$$Q^{\mu\nu}(q_1,q_2) = q_2^2 q_1^\mu q_1^\nu + q_1^2 q_2^\mu q_2^\nu - (q_1\cdot q_2) q_1^\mu q_2^\nu - q_2^2 q_1^2 \eta^{\mu\nu} ,$$

and the holographic form factors

$$\mathcal{P}_{ij}^{(a)}(z) = 8\zeta \hat{d}^{a\gamma\gamma} \frac{X_0(z)}{z} v(z,q_i) v(z,q_j) ,$$
$$\mathcal{Q}_{ij}^{(a)}(z) = 8\zeta \hat{d}^{a\gamma\gamma} \frac{X_0(z)}{z} \frac{\partial_z v(z,q_i)}{q_i^2} \frac{\partial_z v(z,q_j)}{q_j^2} . \qquad (1)$$

HLbL tensor: One scalar exchange contribution cont'd

Non vanishing dynamical coefficients for $(g-2)$ from scalar exchange

$$\bar{\Pi}_3(Q_1, Q_2, \tau) = \int_\epsilon^{z_0} dz \int_\epsilon^{z_0} dz' \left[\mathcal{P}_{12}^{(a)} + (Q_1^2 + Q_2^2 + Q_1 Q_2 \tau)\mathcal{Q}_{12}^{(a)}\right] G_{(a)}(z, z'; s)\mathcal{P}_{34}^{(a)} ,$$
$$\bar{\Pi}_4(Q_1, Q_2, \tau) = \int_\epsilon^{z_0} dz \int_\epsilon^{z_0} dz' \left[\mathcal{P}_{13}^{(a)} + (Q_1^2 + Q_2^2 + Q_1 Q_2 \tau)\mathcal{Q}_{13}^{(a)}\right] G_{(a)}(z, z'; t)\mathcal{P}_{24}^{(a)} ,$$
$$\bar{\Pi}_8(Q_1, Q_2, \tau) = \int_\epsilon^{z_0} dz \int_\epsilon^{z_0} dz' \mathcal{P}_{14}^{(a)} G_{(a)}(z, z'; u)\mathcal{Q}_{23}^{(a)} ,$$
$$\bar{\Pi}_9(Q_1, Q_2, \tau) = \int_\epsilon^{z_0} dz \int_\epsilon^{z_0} dz' \mathcal{Q}_{12}^{(a)} G_{(a)}(z, z'; s)\mathcal{P}_{34}^{(a)} . \qquad (2)$$

Scalar 3-point funct.: Asymptotic behaviour

$$\Gamma_{\mu\nu}^{(n,a)}(q_1,q_2) = i\int d^4x\; e^{-iq_1\cdot x}\langle 0|T\{j^\mu_{\rm em}(x)j^\nu_{\rm em}(0)\}|S^a_n\rangle$$
$$= F_1^{(n,a)}(q_1^2,q_2^2)P_{\mu\nu}(q_1,q_2) + F_2^{(n,a)}(q_1^2,q_2^2)Q_{\mu\nu}(q_1,q_2)\,,$$

with transition form factors for each scalar meson:

$$F_1^{(n,a)}(q_1^2,q_2^2) = 8\zeta\hat{d}^{a\gamma\gamma}\int_\epsilon^{z_0} dz\,\frac{X_0(z)}{z}\varphi_n^S(z)v_1(z)v_2(z)\,,$$
$$F_2^{(n,a)}(q_1^2,q_2^2) = 8\zeta\hat{d}^{a\gamma\gamma}\int_\epsilon^{z_0} dz\,\frac{X_0(z)}{z}\varphi_n^S(z)\frac{\partial_z v_1(z)}{q_1^2}\,\frac{\partial_z v_2(z)}{q_2^2}\,.$$

The decay width of the scalar into two on-shell photons can be expressed in terms of $F_1^{(n,a)}(0,0)$ alone as

$$\Gamma_{\gamma\gamma}^{(n,a)} = \frac{\pi\alpha^2}{4}m_n^3\left|F_1^{(n,a)}(0,0)\right|^2\,.$$

with

$$F_1^{(n,a)}(0,0) = 8\zeta\hat{d}^{a\gamma\gamma}\int_\epsilon^{z_0} dz\,\frac{X_0(z)}{z}\varphi_n^S(z) = 8s_0z_0^2\zeta\hat{d}^{a\gamma\gamma}\,\frac{A_n}{\omega_n^2}\,[4J_3(\omega_n)-\omega_nJ_4(\omega_n)]$$

Scalar 3-point funct.: Asymptotic behaviour cont'd

For highly virtual photons, *i.e.* for large Q, $v(z,Q) \sim QzK_1(Qz)$. In terms of the variables $Q^2 = \frac{1}{2}(Q_1^2 + Q_2^2)$ and $w = (Q_1^2 - Q_2^2)(Q_1^2 + Q_2^2)^{-1}$, such that $Q_{1,2} = Q\sqrt{1 \pm w}$ the model then predicts

$$\lim_{Q^2\to\infty} F_1^{(n,a)}(Q_1^2, Q_2^2) = \frac{1536}{35}\zeta \hat{d}^{a\gamma\gamma}\frac{s_0}{z_0^4}\frac{A_n\omega_n}{Q^6} f_1(w)\,,$$

$$\lim_{Q^2\to\infty} F_2^{(n,a)}(Q_1^2, Q_2^2) = \frac{1152}{35}\zeta \hat{d}^{a\gamma\gamma}\frac{s_0}{z_0^4}\frac{A_n\omega_n}{Q^8} f_2(w)\,,$$

with

$$\begin{aligned} f_1(w) &= \frac{35}{384}\sqrt{1-w^2}\int_0^\infty dyy^7 K_1(y\sqrt{1+w})K_1(y\sqrt{1-w}) \\ &= \frac{35}{32w^7}\left[30w - 26w^3 - 3(w^4 - 6w^2 + 5)\log\left(\frac{1+w}{1-w}\right)\right] \\ f_2(w) &= \frac{35}{288}\int_0^\infty dyy^7 K_0(y\sqrt{1+w})K_0(y\sqrt{1-w} \\ &= \frac{35}{12w^7}\left[-15w + 4w^3 - \frac{9w^2-15}{2}\log\left(\frac{1+w}{1-w}\right)\right] \end{aligned}$$

Notice that that the model does not match with pQCD, which predicts the asymptotic scalings

$$F_1(Q^2, Q^2) \sim Q^{-2}, \quad \text{and} \quad F_2(Q^2, Q^2) \sim Q^{-4}$$

and the identity $f_1(w) = f_2(w)$

Scalar 3-point funct.: Asymptotic behaviour cont'd

The model however shows the right asymptotic pQCD scaling for the case of the $< SVV >$ correlator

$$\Gamma^{(a)}_{\mu\nu}(q_2, q_2) = i^2 \int d^4x \int d^4y \; e^{-i(q_1\cdot x + q_2 \cdot y)} \langle 0 | T\{ j^{\mu}_{\rm em}(x) j^{\nu}_{\rm em}(y) j^{a}_{S}(0)\} | 0 \rangle$$
$$= \bar{\mathcal{P}}^{(a)}(q_1^2, q_2^2) P_{\mu\nu}(q_1, q_2) + \bar{\mathcal{Q}}^{(a)}(q_1^2, q_2^2) Q_{\mu\nu}(q_1, q_2),$$

- All momenta much larger than Λ_{QCD}, e.g.$q1 = q2 = q_3/2 \equiv q$

  $$\lim_{q^2\to\infty} \Gamma^{(a)}_{\mu\nu}(q,q) = \frac{16 s_0 \zeta}{z_0^3} \frac{\hat{d}^{a\gamma\gamma}}{Q^4} (q_\mu q_\nu - q^2 \eta_{\mu\nu}) \int_0^\infty dy y^6 K_1(2y) \left[K_1^2(y) - K_0^2(y) \right] .$$

  To be compared with the QCD OPE result

  $$\lim_{q^2\to\infty} \Gamma^{(a)}_{\mu\nu}(q,q) = 2 \hat{d}^{a\gamma\gamma} \frac{\langle \bar{q} q \rangle}{Q^4} (q_\mu q_\nu - q^2 \eta_{\mu\nu}) \,. \qquad (3)$$

- Vector momenta hard and the scalar one soft. To leading order, $q_1 = -q_2 \equiv q$

  $$\lim_{q^2\to\infty} \Gamma^{(a)}_{\mu\nu}(q,-q) = \frac{64 s_0 \zeta}{15 z_0^3} \frac{\hat{d}^{a\gamma\gamma}}{Q^4} (q_\mu q_\nu - q^2 \eta_{\mu\nu}) + \mathcal{O}(Q^{-6}) \,. \qquad (4)$$

  Again, the scaling is the one expected from the OPE.

Numerical results for the scalars: Fixing the parameters

- The starting action has nine parameters, namely the coefficients of the different bulk operators (λ, c, ρ, $\zeta_\pm$, m_X), the size of the fifth dimension z_0 and the parameters from the scalar boundary potential (μ, η).
- For the scalar contributions to the HLbL, only a subset of them are relevant, namely ρ, z_0, the combination $\zeta = \zeta_+ + \frac{1}{2}\zeta_-$, m_X, and the parameters of the boundary potential, which can be traded for the quark condensate $\langle \bar{q}q \rangle$ and γ.The value of the 5-dimensional scalar mass m_X is the one dictated by the AdS/CFT correspondence, $m_X^2 = -3$
- We will require that ζ and ρ match the $\langle SVV \rangle$ short-distance constraint of eq. (3) and the decay width of the lowest-lying scalars into two photons
- We need to introduce flavour breaking, as we did in our paper for the Goldstone and axial-vector towers, and generate independent copies of the original Lagrangian for each of the different light scalar states. Only γ, ρ and ζ, will be flavour-dependent.

Final results for the scalar contribution

- We have studied also the dependence of γ from the mass range (e.g. $m_\sigma = (450 - 550)$ MeV). Our estimate for the $\sigma(500)$ contribution to the HLbL is

$$a_\mu^S(\sigma) = (-8.5 \pm 2.0) \cdot 10^{-11}$$

orientative, but should correctly captures the right order of magnitude for the uncertainty.

- The contributions of $a_0(980)$ and $f_0(990)$ can be computed in a less problematic way: both states are rather narrow.

$$a_\mu^S(a_0) = -0.29(13) \cdot 10^{-11}; \quad a_\mu^S(f_0) = -0.27(13) \cdot 10^{-11}$$

- Effect of higher massive states are found very small due to the peak of of kinematic kernels around 1 GeV

	$n=1$	$n=2$	**Total**
$a_\mu^{\mathrm{S}}(\sigma)$	-8.5(2.0)	-0.07(2)	-8.7(2.0)
$a_\mu^{\mathrm{S}}(a_0)$	-0.29(13)	-0.025(10)	-0.32(14)
$a_\mu^{\mathrm{S}}(f_0)$	-0.27(13)	-0.025(9)	-0.29(14)
a_μ^{S}	-9(2)	-0.12(4)	-9(2)

Our final result is $a_\mu^S = -9(2) \cdot 10^{-11}$, rather close to previous estimates.

Conclusions and outlook

- We have provided an estimate of the scalar contribution to the HLbL, including the $\sigma(500)$, $a_0(980)$ and $f_0(980)$ states together with an infinite tower of excited scalar states with a holograpkic model of QCD.
- In our final result $a_\mu^S = -9(2) \cdot 10^{-11}$, we think that we have given conservative estimate for the uncertainty is given. This includes the uncertainty on the $\sigma(500)$ parameters, which overwhelmingly dominates.
- Our result agrees with previous inclusive scalar estimates and points at a neatly negative contribution for the scalar contribution to the HLbL.
- One of the advantages of the model is that it is minimal with a small number of free parameters. However, this also entails some limitations. Scalar transition form fac- tors, with some mismatches with QCD expectations. We have argued that these shortcomings have a limited im- pact on the HLbL and are in any case taken into account in the final error band.
- The estimate of the contribution of scalar resonances beyond 1 GeV is in general hindered by the rather uncer- tain knowledge of their couplings to two photons. with a rather poor description of the states populating the 1-2 GeV energy window
- Adding to other contributions (e.g. pion and axial vectors) errors linearly, one would find aHLbL $a_\mu = 116(17) \cdot 10^{-11}$ which is in agreement with all the recent estimates of the HLbL.
- Caveat. The addition of scalar fields into the action, has an effect on the axial-vector and Goldstone sectors. This will affect the estimate of the axial-vector and Goldstone contributions to the HLbL and it calls for a re-analysis of Goldstone, axial-vector and scalar exchange.This is left for a future work.