Thermodynamics from the S-matrix reloaded, with applications to QCD & Flux Tubes.

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N.B. Today I will mostly cover the INTRO of our upcoming paper.

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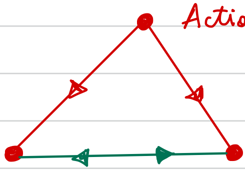
1.- Motivation
2.- Formal derivation
3.- Applications
- → QCD
- → Flux Tubes

Action $A = \int d^4x\, \mathcal{L}(x)$

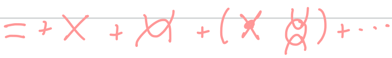


S-matrix

Thermal physics, partition function, spectrum, free energy, casimir energy, thermal mass, ...

e.g. $S(s) = 1 + i\frac{s}{4}\ell_s^2 - \frac{s^2}{32}\ell_s^4 + i\left(\gamma_3 - \frac{1}{384}\right) s^3 \ell_s^6 + \dots$

$E_0(R) = \frac{R}{\ell_s^2}\sqrt{1 - \frac{\pi \ell_s^2}{3R^2}} + O(R^{-9})$



some sort of tracing involved:

This intuition is indeed realized.

More than half a century ago Dashen, Ma, Bernstein [Phys. Rev. 187 (1969) 345-370] uncovered this beautiful formula

$$Z(\beta) = Z_0(\beta) - \frac{\beta}{2\pi i}\int_0^\infty dE\, e^{-\beta E}\, \overbrace{\mathrm{Tr}\log S(E)}^{\text{purely imaginary}}$$

$\mathrm{Tr}\log S = \log \underbrace{\det S}_{e^{i\phi}}$

Full (disconnected) S-matrix:

$\langle \alpha | S(E = E_\alpha) | \beta \rangle = \mathbb{1} + 2\pi i\, \delta(E_\alpha - E_\beta)\, T_{\alpha\beta}(E_\alpha)$

$T(z) = V + V G_0(z) T(z)$, w/ $G_0(z) = \frac{1}{z - H_0}$

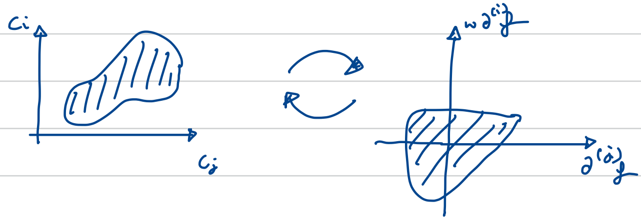
↳ $S(E)$ is the vacuum S-matrix, the integral with Boltzmann weights generates thermal physics.

↳ equation mostly used in non relativistic contexts.

↳ used to argue that long lived particles count as degrees of freedom in the entropy counting (obviously...)

↳ Why now?

* Many new results constraining the space of EFT's, typically expressed in terms of S-matrix w/o reference to $\mathcal{L}$. How can we transfer this info to other phys. obs.?



e.g. Ghost condensation or $P(X=\partial_\mu\phi\partial^\mu\phi)$ theories:
speed of sound < light
$\Rightarrow$ $P(X)$ convex.
(Proof: on $X=$ cte bckg., $c_s=\frac{P'(x)}{P'(x)+2XP''(x)}<1$)

* With the development of onshell business we have access to precise matrix element with arbitrary number of particles. Bad convergence of thermal QCD. New calculation scheme (no ghost, no gauge redundancies). Alternative treatment based on IR safe S-matrix?

* The method shines when the path integral formulation is not available: gravity, strings, spinning particles, ...

2.- Formal derivation:

straight forward to add, I'll omit from now on.

$$Z(\mu,\beta) = \mathrm{Tr}\; e^{-\beta(H-\mu Q)}$$

$$= \int_0^\infty dE\, e^{-\beta E}\,\mathrm{Tr}\underbrace{\left[\frac{1}{E-H+i0^+}-\frac{1}{E-H-i0^+}\right]}_{\equiv\, G^*-G}$$

Recall $\lim_{\epsilon\to 0} \mathrm{Im}\frac{i\pi}{x+i\epsilon}=\delta(x)$

Next we need to relate this kernel to the S-matrix.

Define $\quad G_0(E)=\frac{1}{E-H_0+i0^+} \quad \& \quad G(E)=\frac{1}{E-H+i0^+}$

Two step proof:

1.- Claim: $\quad S(E) = G_0^*\,(G^*)^{-1}\,G\,G_0^{-1}$ $\quad\leftarrow$ proof from Lippman - Schwinger
$|\psi_I\rangle = |\psi_{in}\rangle + G_0 V|\psi_I\rangle$

$\Omega(z)=G(z)\,G_0^{-1}(z)\;;\;\Omega^{-1}=1-G_0V$
$S=\Omega^{-1}\Omega$

simple algebra

2.– $\mathrm{Tr}\, S^{-1}\partial_E S = \mathrm{Tr}\left[-G_0^* + G^* - G + G_0\right]$

$$\Rightarrow Z(\beta) = \frac{1}{2\pi i}\int dE\, e^{-\beta E}\left\{ \mathrm{Tr}\left[S^{-1}(E)\partial_E S(E)\right] - \mathrm{Tr}\left[G_0(E) - G_0^*(E)\right]\right.$$

$$= Z_0(\beta) - \beta\int_0^\infty \frac{dE}{2\pi i}\, e^{-\beta E}\, \mathrm{Tr}\log S(E) = e^{-\beta F}$$

$f = \lim_{V\to\infty} \frac{F}{V}$. upon taking the log we get an extensive quantity, a single power of V.

$$\log Z = f\cdot V\times\beta$$

$$F(\beta) = F_0(\beta) - \frac{1}{2\pi i}\int_0^\infty dE\, e^{-\beta E}\, \mathrm{Tr}_c \log S(E)$$

↳ From the free-energy thermodynamic quantities follow:

$F = E - TS$, thus. $\quad S = -\partial_T F = -\frac{F}{T} + \frac{E}{T}$;

$p = -f$; $\rho = -\frac{\partial(\beta p)}{\partial\beta}$; $\omega = \frac{p}{\rho}$

↳ or derived quantities such as $c_v = T\frac{\partial^2 p}{\partial T^2}$; $c_s^2 = \frac{\partial e}{\partial p}$.

3.– Applications

* Simplest things first: free theory.

$$-\beta F_0 = \mathrm{Tr}_c\, e^{-\beta H_0} = \sum_{N=0}^{\infty}{}' \frac{1}{N!}\int d^3k_1\cdots d^3k_n \langle k_1\ldots k_n|\, e^{-\beta H_0}\,|k_1,\ldots,k_n\rangle$$

Next use that H_0 is diagonal and $\langle k_1,\ldots,k_n|k_1',\ldots,k_n'\rangle = \delta(k_1'-k_1)\cdots\delta(k_n'-k_n) \pm$ perms.
When we identify $k_i = k_i'$ through the trace we get many $\delta(0)$'s up to $\delta(0)^N$.
Here "c" enters crucially, leaving terms with a single $\delta(0)$. (no $\delta(0)$ for $N=0$)

connected

disconnected

For a given connected history there are $(N-1)!$ equivalent ones, due to cyclic perms.
All in all:

$$-\beta F_0 = 1 + \delta(0)\int d^3k \sum_{N=1}^{\infty} \frac{1}{N} e^{-N\beta E(k)} = 1 - \delta(0)\int d^3k \log\left(1 - e^{-\beta E(k)}\right)$$

$\delta(0) \to V/(2\pi)^3$ and thus

$$\underset{\substack{\uparrow \\ \text{gluons}}}{f_0} = \frac{T}{2\pi^3}\int_0^\infty dk\, k^2 \ln\left(1 - e^{-\beta k}\right) = -\frac{\pi^2 T^4}{90} \times \underbrace{2(N_c^2-1)}_{\#\,\text{gluons} \times \#\,\text{pols}}$$

(aside; the textbook calculation gives

gluon

$$= -\frac{\pi^2 T^4}{90}(N_c^2-1)\times 4$$

ghost

$$= -\frac{\pi^2 T^4}{90}(N_c^2-1)\times(-2)$$
)

* Interactions at leading order:

$$F - F_0 = \int dE\, e^{-\beta E}\, \mathrm{Tr}_c\left[T\,\delta(E-H)\right] + O(T^2)$$

a bit of algebra

$$= \underbrace{(2\pi)^3\delta(0)}_{V} \sum_{N=2}^{\infty} \frac{1}{N!} \int \prod_{i=1}^{N} d^3\tilde{k}_i\, n(k_i) \langle k_1 \dots k_N | T | k_1 \dots k_N \rangle + O(T^2)$$

w/ $d^3\tilde{k}_i = \dfrac{d^3k_i}{(2\pi)^3\, 2\pi E_i}$ & $n(k_i) = \dfrac{1}{e^{\beta E_k} - 1}$ the number density.

$f \equiv F/V$, free energy density.

* QCD at $O(\alpha_s)$

We need the $2 \to 2$ S-matrix element:

all incoming: $$\mathcal{M}(1^a_-, 2^b_+, 3^c_+, 4^d_-) = 2 g_s^2 \langle 14\rangle^2 [23]^2 \left(\frac{f^{abe} f^{cde}}{s_{12}\, s_{14}} + \frac{f^{ace} f^{bde}}{s_{13}\, s_{14}} \right)$$

contains all dynamical info for $++ \to ++$, $-- \to --$ & $+- \to +-$

Forward limit very simple for all of them:

$$\lim_{\text{forward}} \mathcal{M} = -2 g_s^2 f^{abc} f^{abc} = -2 g_s^2 N_c (N_c^2 - 1)$$

Therefore

$$f - f_0 = 2 g_s^2 \Big(\overset{++}{\frac{1}{2}} + \overset{--}{\frac{1}{2}} + \overset{+-}{1} \Big) N_c (N_c^2-1) \left[\int_0^\infty \frac{4\pi k^2 dk}{(2\pi)^3\, 2k\, (e^{\beta k}-1)} \right]^2 + O(g_s^4)$$

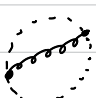
$$= \alpha_s N_c (N_c^2 - 1) \frac{\pi T^4}{36}$$

state of the art is g_s^4 + resummations.

To be compared w/ text book computation, e.g. Ch. 4.5 of Laine & Vuorinen, "Basics of Thermal Field Th."

$$= \frac{g^2}{4}\, 3 N_c \sum\!\!\!\!\!\!\int_K \frac{\delta^{ac} g_{\mu\nu}}{K^2} \sum\!\!\!\!\!\!\int_P \frac{1}{P^2} = 3 g^2 N_c (N_c^2-1) \frac{T^4}{(12)^2}$$

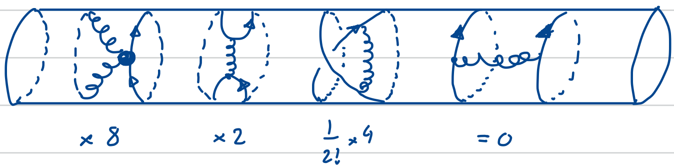
$$= -\frac{3 g^2 N_c}{12} \sum\!\!\!\!\!\!\int_K \frac{\delta^{ab}}{K^2} \sum\!\!\!\!\!\!\int_P \frac{K^2 + (K-P)^2 + P^2}{P^2 (K-P)^2} = -\frac{9}{4} g^2 N_c (N_c^2-1) \left(\frac{T^2}{12}\right)^2$$



$$= -\frac{1}{2}(-g^2 N_c) \sum\!\!\!\!\!\!\int \frac{\delta^{aa}}{K^2} \sum\!\!\!\!\!\!\int \frac{P^2 - K\cdot P}{P^2 (K-P)^2} = \frac{g^2}{4} N_c (N_c^2-1) \left(\frac{T^2}{12}\right)^2$$

obscures UV finiteness, gauge invariance, thermal v.s. vacuum loops.

* Adding QCD matter + comments



We need two amplitudes:

$$\lim_{\text{forward}} \mathcal{M}(1g^a_-, 3g^b_+, 3\bar{u}_i, 4u_j) = \lim_{\text{forward}} \mathcal{M}(1u^i_-, 3\bar{u}_+, 2\bar{u}_i, 4u_+) = -g^2(N_c^2-1)$$

$$-\Delta f_{\text{quarks}} = 4\int d\nu_g\, d\nu_q \left[\mathcal{M}(g_-u_+\to g_-u_+) + \mathcal{M}(g_+u_+\to g_+u_+)\right] +$$

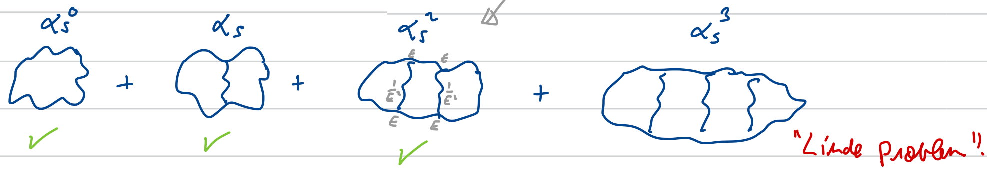
$$+\,2\int d\nu_u\, d\nu_u\, \mathcal{M}(\bar{u}_+u_-\to\bar{u}_+u_-) + 4\int \frac{d\nu_1 d\nu_2}{2}\mathcal{M}[u_-u_-\to u_-u_-]$$

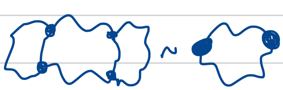
$$= \alpha_s (N_c^2-1)\frac{\pi T^4}{36}\,\frac{5}{4}N_f .$$

$$\int d\nu_{g/q} \equiv \int \frac{d^3k}{(2\pi)^3\, 2k\,(e^{\beta k}\mp 1)} = T^2\begin{cases}\frac{1}{24} & \text{boson}\\ \frac{1}{48} & \text{fermion.}\end{cases}$$

* QCD state of the art:

$$\left[\int d^3k\right]^3\left(\frac{1}{k^2}\right)^6 \epsilon^4 = \int d^6\epsilon\, \frac{1}{\epsilon^8} \quad \text{IR div.}$$





$$\sim \quad = T\int d^3k\, \frac{(g^2T^2)^2}{(k^2+g^2T^2)^2} \sim \frac{g^4T^5}{gT} = g^3T^4 \propto \alpha_s^{3/2}$$

$\Pi_{00} \propto g^2T^2$

↳ Debye mass, electric component, generates non-integer "α_s" powers.

EFT for zero modes. Matching at long distances $R \gg 1/T$

$$\mathcal{L}_{EQCD} = \frac{1}{4} G^a_{ij} G^a_{ij} + \frac{1}{2}\left(D_i A_0^a\right)^2 \frac{1}{2} m_E^2 A_0^a A_0^a + \frac{1}{8} \lambda_E \left(A_0^a\right)^2$$

scalar adjoint

$A^i(x)$, magnetostatic field. ; $D_i A_0^a = (\partial_i + g_E \epsilon^{abc} A_i^b A_0^c) A_0^a$

$m_E^2 = (1 + N_f/6) g^2 T^2$; $\lambda_E = \frac{9 - N_f}{12\pi^2} g^4 T$; $m_G \propto \alpha T^2 \ldots$

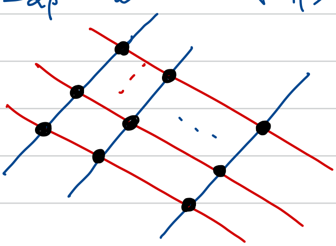
$$\mathcal{L}_{MQCD} = \frac{1}{4} G^a_{ij} G^a_{ij} + O(g^{12}); \quad -\frac{\log Z_{MQCD}}{V} = \left(a + 6 \log \frac{\Lambda}{g_M^2}\right) g_M^6$$

lattice.

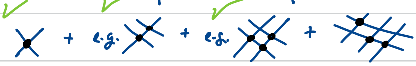
* Simpler set up, Yang-Mills confining flux tubes:

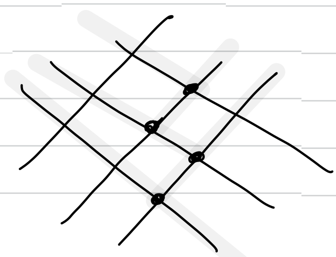
Write down the most general 2d action realizing (non-linearly) the symmetry breaking pattern $ISO(D-1,1) \mapsto ISO(1,1) \times SO(D-2)$. There are "$D-2$" goldstone bosons or transverse excitations. For $D=3$, there is an integrable realization of the system. (i.e. the Wilson coeficients of the action can be fixed to realized the following factorized S-matrix):

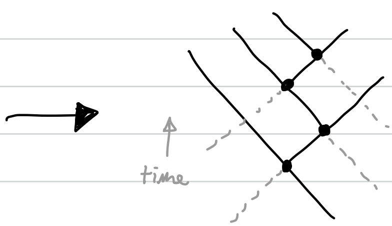
$$S_{\alpha\beta} = e^{\frac{i \ell_s^2}{4} \sum_{i<j} s_{ij}} \langle \alpha | \beta \rangle \longleftrightarrow f(\beta) = \frac{1}{\ell_s^2} \sqrt{1 - \frac{\ell_s^2}{\beta^2} \frac{\pi}{3}}$$



$$= \frac{1}{\ell_s^2} - \frac{\pi}{6\beta^2} - \frac{\ell_s^2 \pi^2}{72 \beta^4} - \frac{\ell_s^4 \pi^3}{432 \beta^6} - \frac{5 \ell_s^6 \pi^4}{10368 \beta^8} + \dots$$







Full binary tree, counted by Catalan numbers.

* Comments on thermal mass:

Think of Form Factors (FF) as perturbations of the S-matrix.

$$F \to F[h] = F_0 - \int \frac{dE}{2\pi i}\, e^{-\beta E}\, \mathrm{Tr}_c \ln S[h]$$

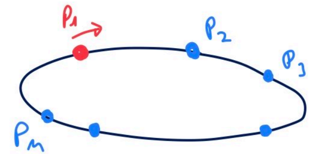
Action $S \to S + \int h \times \mathcal{O}$.

↑ overlap with elementary excitations.

$$\left.\frac{\delta F}{\delta h}\right|_{h=0} = \int dE\, \frac{e^{-\beta E}}{2\pi i}\, \mathrm{Tr}_c\left[S^{-1}[0] \times \int \mathcal{O}(x) \right]$$

@ leading order $\quad \Sigma_T(p) = -\int \frac{d^3k}{(2\pi)^3 2E_k}\, n(E_k)\, T(k+p \to k+p)$

Review of Asymptotic Bethe Ansatz



Two particle state in infinite volume

$$\psi(x_1, x_2) = \langle 0 | \, \phi(x_1) \, \phi(x_2) \, | p_1, p_2 \rangle$$

Consider $x_1 \ll x_2$ (mutatis mutandis $x_1 \gg x_2$)

$$\psi(x_1 \gg x_2) = e^{i p_1 x_1 + i p_2 x_2} + e^{i p_1 x_2 + i p_2 x_1} \, S_{12}(p_1, p_2)$$

Place the theory at finite volume w/ $x_1 \ll x_2 \ll x_1 + R$,

$$\psi(x_1, x_2) = \psi(x_1 + R, x_2) \implies e^{i p_1 R} \, S_{12}(p_1 \, p_2) = 1$$

Same logic for multiple rescatterings. all in all

$$p_i R + \sum_j 2 \, \delta_{ij} (p_i, p_j) = 2\pi N_i$$

Winding corrections exponentially suppressed for massive theories.