Update on the multi-photon Monte Carlo event generator KKMCee

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(A) Monte Carlo programs `Bhlumi` for Bhabha scattering luminosity measurements, `KKMC` for lepton pair productions, demonstrated that predictions for observables, of complex cuts , sub permille level (0.043% Opal lumi. measurement) are possible.

(B) Main building blocks — aspects enabling such precision: (i) Phase space (issues of iterations or better solutions) (ii) Matrix Elements and spin (issues of iterations, separation into parts) (iii) Tests for programs and of development process.

(C) `KKMC` for $e^+e^- \to l^+l^-(n\gamma)$, including τ spin, decays and radiative corrections in decays. represent good example. Hopefully its development will continue and precision required by FCC will be reached.

(D) My aim: underline difficulties, challenges, and breakthrough steps, status of today.

(E) I will talk about lifetime project at the time when change of main authors is pressing $\to$

(F) expertise survival, New People (NP) for long term involvement.

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Talk plan

Sentimental... For me, first international step toward Monte Carlo for accelerator physics, was my stay in Marseille in 80's; first trip from there was Barcelona.

From Barcelona, early testbed for spin amplitudes with multileg amplitudes.

My talk supplements, but is of different, more <u>work to do</u>, perspective than *Standard model theory for the FCC-ee Tera-Z stage* 1809.01830

1. KKMCee presentation and daily service. See other talk.
2. Recovering old f77 solutions, like attributing helicity states or polarimetric vectors.
3. KKMC-f77 in use by Belle collaboration. Input for future.
4. Anomalous couplings, extra algorithms.
5. All of above does not help to improve on precision. CHALLENGES
6. Toward higher order 3 hard photon amplitudes
7. Toward higher order 3 loop QED amplitudes
8. Toward two loop electroweak corrections.

1, 2 KKMCee and old solutions

- The development of KKMCee was managed by Stanislaw Jadach. His disappearance is not only the great loss for the project, but also underline importance of man-power issues.

- At present, physics content of `KKMCee` does not evolve fast. It user interface is managed through `https://github.com/KrakowHEPSoft/KKMCee`,

- Please have a look at Alan Price slides for details.

- Exclusive exponentiation open way to manage interface of theoretical predictions with detailed experimental acceptance, which is irregular: rectangular detector cells, backgrounds etc.

— some cuts may arrive in the middle of experimental runs $\rightarrow$ dead detector cells

1, 2 KKMCee and old solutions

There are two modifications under imminent implementation:

• Attribution of tau helicities tags for each event. That is inconsistent with picture of quantum entanglement, but for many applications can be used.

— Precision tag for this attribution is at the level of $\sim m_\tau/E_\tau$

• The tau decay polarimetric vectors are stored after being boosted to laboratory frame.

— They can be boosted back to τ frames of users choice and introduced into event weights emulating input from anomalous interaction. No need to use internal boosting routine of KKMCee. Use of event stored on a disk is OK.

— Example algorithm (it is straightforward to re-code into any language) is explained in Phys.Rev.D 109 (2024) 1, 013002

3,4 KKMC-f77 Anomalous interactions

- FORTRAN version of KKMC is in daily use by Belle collaboration.

- Valuable source of New Physics applications, such as: anomalous dipole moments, dark photon emissions, light exotic scalars. Bulk of the work for KKMCee is completed (polarimetric vectors in generated event data-files)

- In future, path to solutions for ZH intermediate state processes, and many other.

- Even if high precision can not be achieved before investing into Matrix Elements, prototypes source and benchmarks codes are needed.

- In the past we were developing such solutions (for start) based on fixed order Monte Carlo combined with parallel simulations based on collinear resummations. Examples: • early time: OldBab+LumLog for luminosity. • Later: KORALW+ YSFWW for four fermion production.

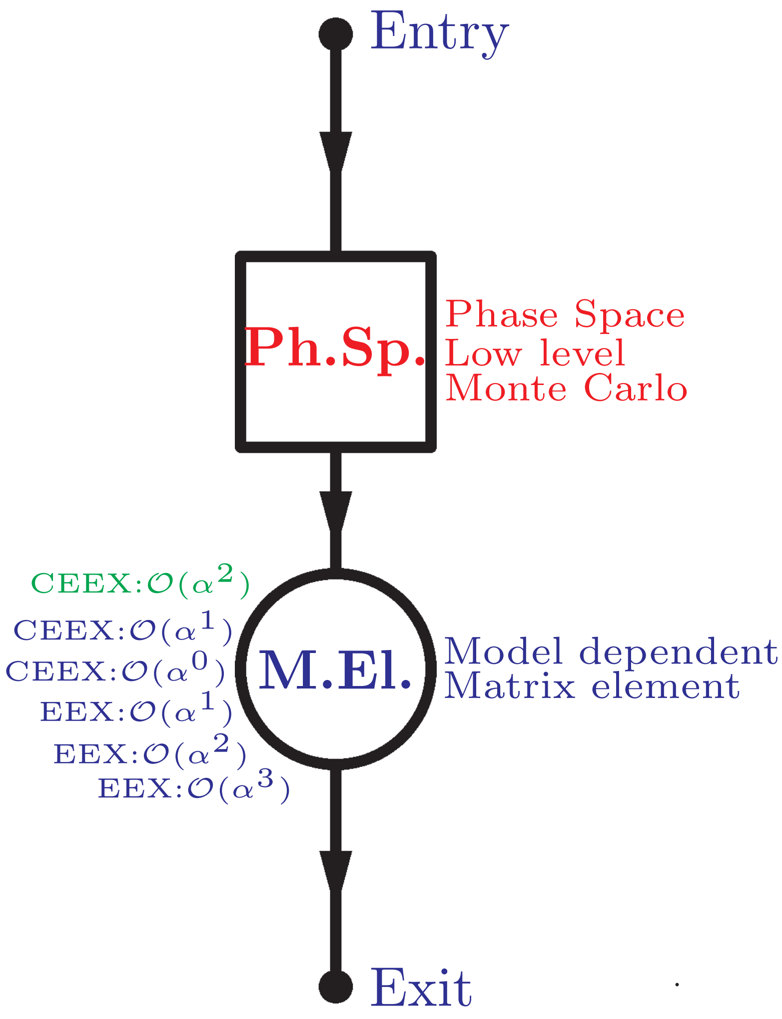
  Disadvantage difficult to use and/or loss of detector response details.

  Advantage Helpful for fits.

Proven dead-end for precision below 0.5% level. Breakthrough of exclusive exponentiation, game changer, but ... came with a price.

3,4 KKMC-f77 Anomalous interactions

KKMC or photos rigorous "matrix element × full phase space" implementation



- Phase-space Monte Carlo simulator is a module producing "raw events" (including importance sampling for possible intermediate resonances/singularities)
- Library of Matrix Elements; input for "model weight"; independent module
- KKMC for $e^+e^- \to \tau^+\tau^- n\gamma$ and photos for radiative corrections in decays are non-Markovian algorithms, photons are generated independently first, phase space constraints are added later, thanks to conformal symmetry of eikonal QED part KKMC or iteratively, Kinoshita-Lee-Nauenberg theorem, for photos.
- KKMC handle initial state radiation, photos massive states emission too.

3,4 KKMC-f77 Anomalous interactions

I will use `KKMC` *and* `photos` *as examples*

- **KKMC** precision Monte Carlo for $e^+e^- \to \bar{l}l(n\gamma)$. Non markovian algorithm, exploits conformal symmetry. Clear way for consecutive higher orders of matrix element implementation.
- **photos** for radiative corrections of resonance and particles decays, also non markovian, use iterative procedure to implement phase space limits and Jacobians. Convenient for generation massive states, but not developed for second order matrix elements.
- And pairs emission will be needed for precision required at FCC.
- In both cases complete exact phase-space, where number of of photons is part of crude distribution.
- Solution used in `KKMC`, based on conformal symmetry, is better suited **for beyond NL order** matrix element implementation,
  also for ISR when intermediate s-channel resonances/peaks are present.

5, Lessons, messages of the past.

Both for KKMC and photos algorithms are non Markovian

1. For photos fixed order algorithm is preserved for tests. It is good for comparisons with fixed order orthodox calculations, for KKMC such option is abandoned. It was present in its predecessor KORALZ.
2. For fixed order MC binomial distribution for number of photons candidates has to be used. To regulate infrared singularity, soft photon region need to be integrated out and combined with virtual correction. That leads to technical approximation, photons below threshold value are not generated. Alternatively negative weight events can be used. Both solutions are not good for experiments.
3. If one goes to second order, things are getting worse for technical approximation.
4. Here exponentiation helps, primary distribution of photon number is poissonian and any value can be used for minimal energy of the photon to be generated.
5. That is all I can say before entering discussions of matrix elements.
6. May be the only comment can be that in case of KKMC phase space constraints are introduced in one step λ factor re-scaling of photon momenta. In case of photos phase space constraints are introduced iteratively.

5, Lessons, messages of the past.

1. Both KKMC and photos generation starts from non-markovian generation of photon candidates accordingly to poisonian distribution, number of photons and independently each photon energy, θ and ϕ.

2. Solution of KKMC phase space, is prepared for use with second order or higher matrix elements. This is thanks to conformal symmetry of multi-photon phase space. This enables independent generation of line-shape and beamstrahlung.

3. Solution of photos could have been extended to generation of additional massive particles (lepton pair). Kinoshita-Lee-Nauenberg theorem is used to relate phase-space slots of given multiplicity. It is less convenient for higher order matrix element implementation, because of iterative nature of phase space, jacobians and boundaries, implementation.

4. In both cases extensions to QCD higher orders, etc are possible, but there is a question of man power.

5. Training takes time and other domains value expertise higher...

5, Lessons, messages of the past.

Phase-space alone makes no sense:

1. Matrix elements and spin degrees of freedom must match that. For spin, language of matrix elements is more convenient than work with distributions

2. Especially important is matching enhancements of matrix elements; collinear and soft, with phase space parametrisations and pre-samplers.

3. I will drop virtual corrections and mixed real-virtual ones. Nothing about complex masses, how they affect parametric ambiguities $\to$ non-analytic nature of dispersion relations. That is valid at one loop level, what is beyond?

4. Kleiss-Stirling formalism for spin amplitudes. We had to revise reference frames, common definition independently of number of particles (photons) in final state: S. Jadach, B.F.L. Ward, Z. Was, *Global positioning of spin GPS scheme for half spin massive spinors* Eur.Phys.J.C 22 (2001) 423. For `photos` (working on distributions) reference frame orientation was essential too. Matrix elements required re-do: divide them in parts corresponding to: crude level distribution build from eikonal parts and parts which could have been identified in higher order amplitudes.

Easiest case – $\mathcal{M}$atrix $\mathcal{E}$lement for Z decay:

- **Single photon amplitude**, current J, nearly like Born level amplitude, **but its parts must match what is in higher order amplitudes.** Momenta p, q, k_1 of outgoing fermions and photon.
- The same is true for amplitudes of other processes.
  Details, will be covered later, fermion spinors dropped lot hidden in J.
- Notation useful for Kleiss-Stirling techniques input.
- $$I = I^A \qquad\qquad + I^B \qquad\qquad + I^C$$
- $$I = \not{J}\left[\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{q\cdot e_1}{q\cdot k_1}\right)\right] - \left[\frac{1}{2}\frac{\not{e}_1\not{k}_1}{p\cdot k_1}\right]\not{J} + \not{J}\left[\frac{1}{2}\frac{\not{e}_1\not{k}_1}{q\cdot k_1}\right]$$

three gauge invariant parts, I^A is eikonal; I^B, I^C carry collinear contrib from p and q

MIRACLE?

If I^A is taken only, all double collinear-infrared logarithms appear after phase space integration.

If parts I^B and I^C are added to spin amplitudes, and integration is repeated, nothing change in double logarithms, but collinear ones are now reproduced also.

Fermion masses are taken into account.

This is of course no miracle, but one of the properties at the bottom of Yennie-Frautchi-Suura exponentiation.

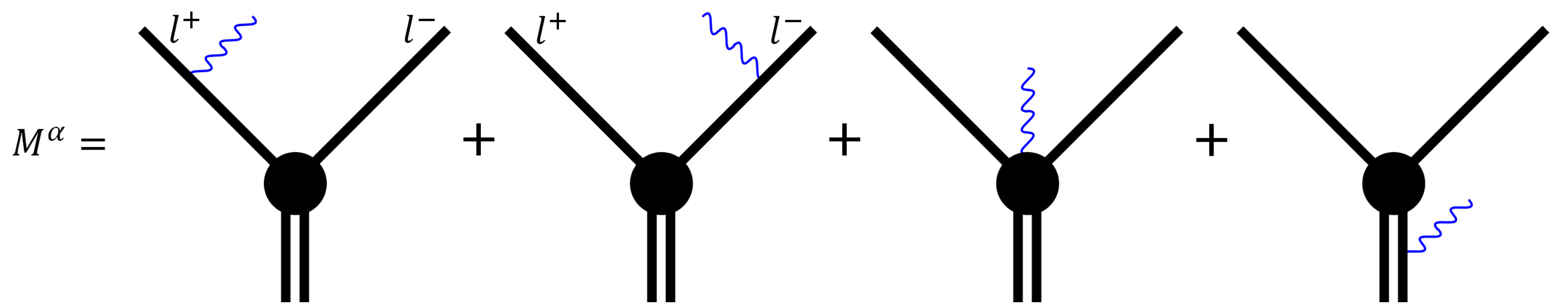
How does it work for other processes and first order effects?

Different languages but separation at the amplitude level for parts corresponding (after integration) to logarithms will appear.

For single gluon emission structure of amplitude is essentialy the same as for QED.

I will go through single photon emission amplitudes of W decay and of scalar QED processes too. Patterns origin is more general than YFS.

5, Lessons, messages of the past.



- *Feynman diagrams for FSR in Z/γ^* decays*
- *Out of the first two diagrams distribution for Z/γ decay was obtained.*
- *Other two diagrams appear e.g. in scalar QED, and/or in decays of W's or B mesons.*
- *Let us look into sub-structure of these amplitudes.*

The Feynman diagrams for $e^+e^- \to \bar{\nu}_e\nu_e\gamma$.

We will start with the process $e^+e^-\nu_e\bar{\nu}_e\gamma$

For the details of notation see later ...

Next two slides: five diagrams, five lines in formula (last 3 for diagram no.5) ...

The four momenta p_a, p_b, p_c, p_d, k_1 denote respectively momenta of incoming electron, positron, outcoming neutrino, antineutrino and finally photon.

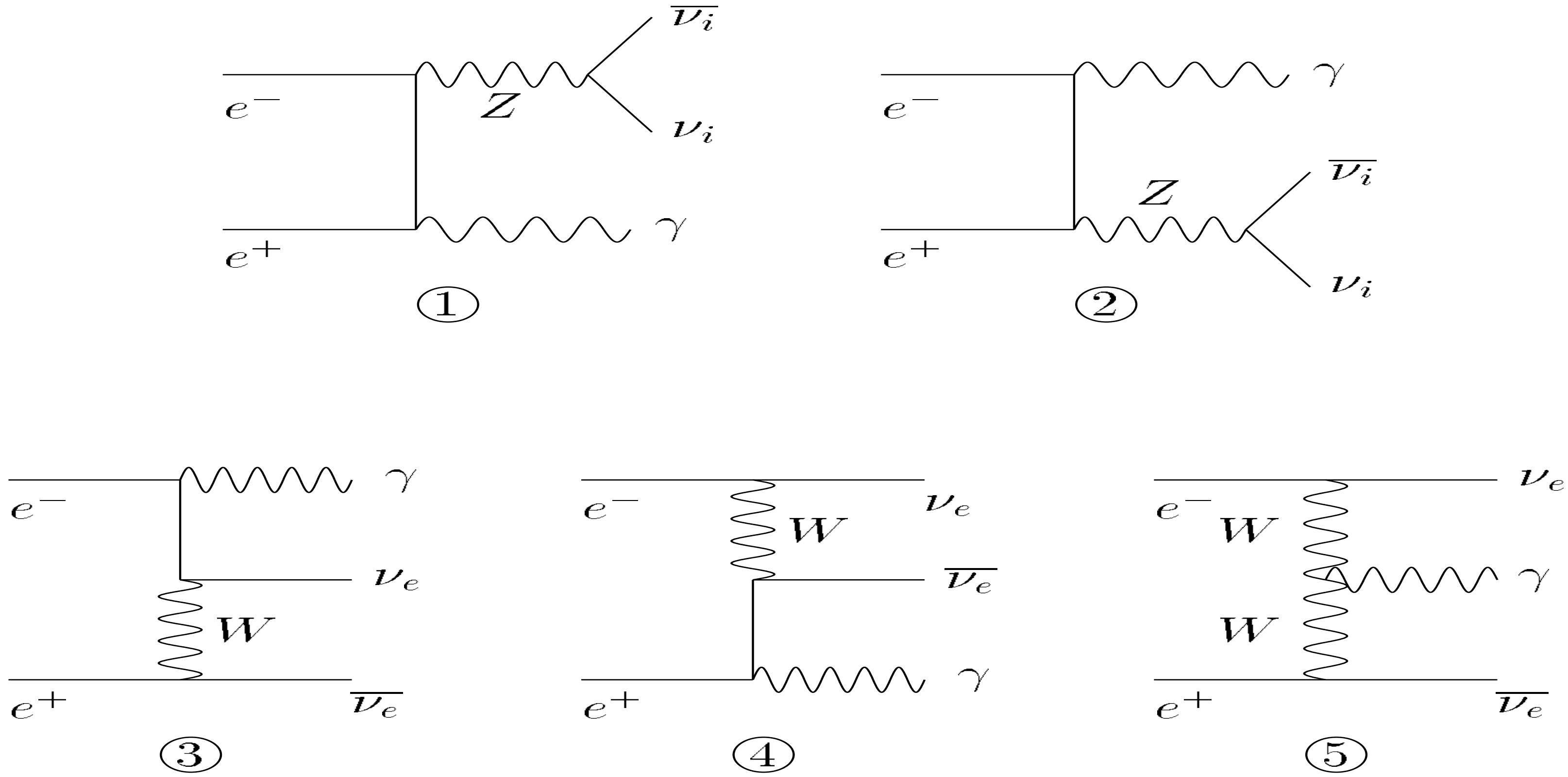
The indices for the spin states for the fermions are denoted respectively as λ_a, λ_b, λ_c, λ_d and for photon σ_1.

The photon polarization vector is denoted as ϵ_{σ_1}.

We will search for gauge invariant parts. Expansion with respect to contact interaction helpful.

Lots of details (sorry) follows, necessary to keep gauge invariant parts in the code, useful for using first order amplitudes parts as parts of higher order amplitudes as well.

The Feynman diagrams for $e^+e^- \to \bar{\nu}_e\nu_e\gamma$.



The first-order matrix element[a] obtained from the Feynman diagrams depicted in fig. 14, can be written in a rather straightforward way:

$$
\begin{aligned}
\mathcal{M}_{1\{I\}}\binom{p\,k_1}{\lambda\,\sigma_1} =& eQ_e\; \bar{v}(p_b,\lambda_b)\; \mathbf{M}^{bd}_{\{I\}}\; \frac{\not{p}_a + m - \not{k}_1}{-2k_1p_a}\; \not{\epsilon}^\star_{\sigma_1}(k_1)\; u(p_a,\lambda_a) \\
&+ eQ_e\; \bar{v}(p_b,\lambda_b)\; \not{\epsilon}^\star_{\sigma_1}(k_1)\; \frac{-\not{p}_b + m + \not{k}_1}{-2k_1p_b}\; \mathbf{M}^{ac}_{\{I\}}\; u(p_a,\lambda_a) \\
&+ e\; \bar{v}(p_b,\lambda_b)\; \mathbf{M}^{bd,ac}_{\{I\}}\; u(p_a,\lambda_a)\; \frac{\epsilon^\star_{\sigma_1}(k_1)\cdot(p_c - p_a + p_b - p_d)}{(t_a - M_W^2)(t_b - M_W^2)} \\
&+ e\; \frac{\bar{v}(p_b,\lambda_b) g^{We\nu}_{\lambda_b,\lambda_d}\; \not{\epsilon}^\star_{\sigma_1}(k_1)\; v(p_d,\lambda_d)\bar{u}(p_c,\lambda_c) g^{We\nu}_{\lambda_c,\lambda_a}\; \not{k}_1\; u(p_a,\lambda_a)}{(t_a - M_W^2)(t_b - M_W^2)} \\
&- e\; \frac{\bar{v}(p_b,\lambda_b) g^{We\nu}_{\lambda_b,\lambda_d}\; \not{k}_1\; v(p_d,\lambda_d)\bar{u}(p_c,\lambda_c) g^{We\nu}_{\lambda_c,\lambda_a}\; \not{\epsilon}^\star_{\sigma_1}(k_1)\; u(p_a,\lambda_a)}{(t_a - M_W^2)(t_b - M_W^2)},
\end{aligned}
\tag{1}
$$

[a] $\mathcal{M}_{1\{I\}}\binom{p\,k_1}{\lambda\,\sigma_1}$ The subscripts 1 and $\{I\}$ denote respectively, that the amplitudes are of the first order and are included as part of the initial state bremsstrahlung. This spurious notation is however convenient for the reader interested in ref. Jadach:1998wp.

or, equivalently:

$$\mathcal{M}_{1\{I\}}\binom{p\ k_1}{\lambda\ \sigma_1} = \mathcal{M}^0 + \mathcal{M}^1 + \mathcal{M}^2 + \mathcal{M}^3$$

$$\mathcal{M}^0 = eQ_e\ \bar{v}(p_b,\lambda_b)\ \mathbf{M}^{bd}_{\{I\}}\ \frac{\not{p}_a + m - \not{k}_1}{-2k_1 p_a}\ \not{\epsilon}^\star_{\sigma_1}(k_1)\ u(p_a,\lambda_a)$$

$$+\ eQ_e\ \bar{v}(p_b,\lambda_b)\ \not{\epsilon}^\star_{\sigma_1}(k_1)\ \frac{-\not{p}_b + m + \not{k}_1}{-2k_1 p_b}\ \mathbf{M}^{ac}_{\{I\}}\ u(p_a,\lambda_a)$$

$$\mathcal{M}^1 = \mathcal{M}^{1'} + \mathcal{M}^{1''}$$

$$\mathcal{M}^{1'} = +e\ \bar{v}(p_b,\lambda_b)\ \ \mathbf{M}^{bd,ac}_{\{I\}}\ u(p_a,\lambda_a)\epsilon^\star_{\sigma_1}(k_1)\cdot(p_c - p_a)\frac{1}{t_a - M_W^2}\,\frac{1}{t_b - M_W^2},$$

$$\mathcal{M}^{1''} = +e\ \bar{v}(p_b,\lambda_b)\ \ \mathbf{M}^{bd,ac}_{\{I\}}\ u(p_a,\lambda_a)\epsilon^\star_{\sigma_1}(k_1)\cdot(p_b - p_d)\frac{1}{t_a - M_W^2}\,\frac{1}{t_b - M_W^2},$$

$$\mathcal{M}^2 = +e\ \bar{v}(p_b,\lambda_b)g^{We\nu}_{\lambda_b,\lambda_d}\ \not{\epsilon}^\star_{\sigma_1}(k_1)\ \ v(p_d,\lambda_d)\bar{u}(p_c,\lambda_c)g^{We\nu}_{\lambda_c,\lambda_a}\ \not{k}_1\ \ u(p_a,\lambda_a)\frac{1}{t_a - M_W^2}\,\frac{1}{t_b - M_W^2}$$

$$\mathcal{M}^3 = -e\ \bar{v}(p_b,\lambda_b)g^{We\nu}_{\lambda_b,\lambda_d}\ \not{k}_1\ \ v(p_d,\lambda_d)\bar{u}(p_c,\lambda_c)g^{We\nu}_{\lambda_c,\lambda_a}\ \not{\epsilon}^\star_{\sigma_1}(k_1)\ \ u(p_a,\lambda_a)\frac{1}{t_a - M_W^2}\,\frac{1}{t_b - M_W^2}, \tag{2}$$

where, the part of the amplitude, consisting of bosonic couplings ($g_\lambda^{Z,f}$ denote coupling constant of Z with fermion f and handedness λ, in electric charge units), final state fermion spinors and boson propagators reads as

$$\mathbf{M}^{xy}_{\{I\}} = ie^2(\mathcal{R}_Z + \mathcal{R}_W) = ie^2 \sum_{B=W,Z} \Pi^{\mu\nu}_B(X)\, G^B_{e,\mu}\, (G^B_{f,\nu})_{[cd]} \tag{3}$$

with

$$\begin{aligned} G^B_{e,\mu} &= \gamma_\mu \sum_{\lambda=\pm} \frac{1}{2}(1+\lambda\gamma_5) g_\lambda^{B,e} \\ (G^B_{f,\nu})_{[cd]} &= \bar{u}(p_c,\lambda_c) G^B_{f,\nu} v(p_d,\lambda_d) \\ \Pi^{\mu\nu}_{B=Z}(X) &= \frac{g^{\mu\nu}}{X^2 - M_Z^2 + i\Gamma_Z X^2/M_Z} \\ \Pi^{\mu\nu}_{B=W}(X) &= \frac{g^{\mu\nu}}{t - M_W^2}. \end{aligned} \tag{4}$$

The final-state spinors are explicitly included, and Fierz transformation is applied for the part of W exchange. The W coupling constant reads

$$g^{We\nu}_{\lambda_c,\lambda_a} = \frac{1}{\sqrt{2}\sin\theta_W}\delta^{\lambda_c}_{\lambda_a}\delta^{\lambda_c}_{+}. \tag{5}$$

Only for the W contribution, the superscripts xy in $\mathbf{M}_{\{I\}}$ have the meaning, they define the momentum transfer in the W propagator $\Pi^{\mu\nu}_W(X)$: for $xy = ac$ the transfer [a] is $t_a = (p_a - p_c)^2$, for bd it is $t_b = (p_b - p_d)^2$. If both are explicitly marked, then the expression

$$\mathbf{M}^{bd,ac}_{\{I\}} = ie^2 G^W_{e,\mu}\,(G^{W,\,\mu}_{\nu})_{[cd]} \tag{6}$$

is used. For that parts of formula (2) W propagators are explicitly given. The notations $\mathcal{R}_Z$ and $\mathcal{R}_W$ will be used later.

Let us start now to rewrite expression (2). It is straightforward to notice that the first term $\mathcal{M}^0$ can be split into soft IR parts proportional to $(\not p \pm m)$ and non-IR parts proportional to $\not k_1$. The non-IR parts are individually gauge invariant by construction. The soft part of $\mathcal{M}^0$, with Z couplings only, is gauge invariant as well.

[a] Transfers can be expressed also as $t_a = (p_b - k_1 - p_d)^2$ and $t_b = (p_a - k_1 - p_c)^2$, this make difference if extrapolation procedures are used for the configurations off mass shell where $p_a + p_b \neq p_c + p_d + k_1$, otherwise $\mathcal{M}^{1'} = \mathcal{M}^{1''}$ of course.

Employing the completeness relations of eq. (A14) we obtain the different form of (2):

$$\mathcal{M}_{1\{I\}}\begin{pmatrix} p & k_1 \\ \lambda & \sigma_1 \end{pmatrix} = -\frac{eQ_e}{2k_1p_a}\sum_{\rho_a}\mathfrak{B}\begin{bmatrix} p_b & p_a \\ \lambda_b & \rho_a \end{bmatrix}_{[cd]} U\begin{bmatrix} p_a & k_1 & p_a \\ \rho_a & \sigma_1 & \lambda_a \end{bmatrix} + \frac{eQ_e}{2k_1p_b}\sum_{\rho_b} V\begin{bmatrix} p_b & k_1 & p_b \\ \lambda_b & \sigma_1 & \rho_b \end{bmatrix}\mathfrak{B}\begin{bmatrix} p_b & p_a \\ \rho_b & \lambda_a \end{bmatrix}_{[cd]}$$
$$+\frac{eQ_e}{2k_1p_a}\sum_{\rho}\mathfrak{B}\begin{bmatrix} p_b & k_1 \\ \lambda_b & \rho \end{bmatrix}_{[cd]} U\begin{bmatrix} k_1 & k_1 & p_a \\ \rho & \sigma_1 & \lambda_a \end{bmatrix} - \frac{eQ_e}{2k_1p_b}\sum_{\rho} V\begin{bmatrix} p_b & k_1 & k_1 \\ \lambda_b & \sigma_1 & \rho \end{bmatrix}\mathfrak{B}\begin{bmatrix} k_1 & p_a \\ \rho & \lambda_a \end{bmatrix}_{[cd]}$$
$$+\mathcal{M}^{1'} + \mathcal{M}^{1''} + \mathcal{M}^2 + \mathcal{M}^3. \tag{7}$$

The terms $\mathcal{M}^{1'}$ to $\mathcal{M}^3$ correspond to the last three lines[b] of eq. (1). These contributions are also IR-finite. In the next step let us remove the sum in the first two terms thanks to the diagonality of U and V (ref.Jadach:2000ir). The matrices

[b] The term $\mathcal{M}^1 + \mathcal{M}^2 + \mathcal{M}^3$ originates from the $WW\gamma$ vertex

$$-ie\big[g_{\mu\nu}(p-q)_\rho + g_{\nu\rho}(q-r)_\mu + g_{\mu\rho}(r-p)_\nu\big]$$

where all momenta are outcoming, and indices on outgoing lines are paired with momenta as p^μ, q^ν r^ρ; $\mathcal{M}^1$ originates from the term where $g^{\mu\nu}$ connects the e^-–ν_e, e^+–$\bar{\nu}_e$ fermion lines.

$\mathfrak{B}$ are also defined in this reference. We obtain

$$
\begin{aligned}
\mathcal{M}_{1\{I\}}\begin{pmatrix} p\, k_1 \\ \lambda\, \sigma_1 \end{pmatrix} =& \mathfrak{s}^{\{I\}}_{\sigma_1}(k_1)\hat{\mathfrak{B}}\begin{bmatrix} p \\ \lambda \end{bmatrix} + (r^{B'}_{\{I\}} + \mathcal{M}^{1'}) + (r^{B'}_{\{I\}} + \mathcal{M}^{1''}) \\
& + r^{A'}_{\{I\}} + r^{A''}_{\{I\}} + (\mathcal{M}^2 + \mathcal{M}^3) \\
r^{B'}_{\{I\}}\begin{pmatrix} p\, k_1 \\ \lambda\, \sigma_1 \end{pmatrix} =& - \frac{eQ_e}{2k_1 p_a} \sum_\rho \bar{\mathfrak{B}}\begin{bmatrix} p_b\, p_a \\ \lambda_b\, \rho_a \end{bmatrix}_{[cd]} U\begin{bmatrix} p_a\, k_1\, p_a \\ \rho_a\, \sigma_1\, \lambda_a \end{bmatrix} \\
r^{B''}_{\{I\}}\begin{pmatrix} p\, k_1 \\ \lambda\, \sigma_1 \end{pmatrix} =& + \frac{eQ_e}{2k_1 p_b} \sum_\rho V\begin{bmatrix} p_b\, k_1\, p_b \\ \lambda_b\, \sigma_1\, \rho_b \end{bmatrix} \bar{\mathfrak{B}}\begin{bmatrix} p_b\, p_a \\ \rho_b\, \lambda_a \end{bmatrix}_{[cd]} \\
r^{A'}_{\{I\}}\begin{pmatrix} p\, k_1 \\ \lambda\, \sigma_1 \end{pmatrix} =& + \frac{eQ_e}{2k_1 p_a} \sum_\rho \mathfrak{B}\begin{bmatrix} p_b\, k_1 \\ \lambda_b\, \rho \end{bmatrix}_{[cd]} U\begin{bmatrix} k_1\, k_1\, p_a \\ \rho\, \sigma_1\, \lambda_a \end{bmatrix}, \\
r^{A''}_{\{I\}}\begin{pmatrix} p\, k_1 \\ \lambda\, \sigma_1 \end{pmatrix} =& - \frac{eQ_e}{2k_1 p_b} \sum_\rho V\begin{bmatrix} p_b\, k_1\, k_1 \\ \lambda_b\, \sigma_1\, \rho \end{bmatrix} \mathfrak{B}\begin{bmatrix} k_1\, p_a \\ \rho\, \lambda_a \end{bmatrix}_{[cd]}, \\
\mathfrak{s}^{\{I\}}_{\sigma_1}(k_1) =& - eQ_e \frac{b_{\sigma_1}(k_1, p_a)}{2k_1 p_a} + eQ_e \frac{b_{\sigma_1}(k_1, p_b)}{2k_1 p_b} \quad .
\end{aligned}
\tag{8}
$$

The soft part is now clearly separated from the remaining non-IR part, used in the CEEX exponentiation for construction of $\mathcal{O}(\alpha)$ corrections. We have ordered the expression, with the help of expansion similar to the contact interaction for W propagator as well.

Second order matrix elements.

$\mathcal{E}$xact $\mathcal{M}$atrix $\mathcal{E}$lement: $e^+e^- \to \nu_\mu\bar{\nu}_\mu\gamma\gamma$ explicitly;

- Expressions are valid for any current J,
- For complete amplitude add fermionic fields, eg. $\bar{u}(p)$ and $v(q)$; 1-st/2-nd photon momenta/polarizations are: k_1/k_2 e_1/e_2.

$$I_1^{\{1,2\}} = \frac{1}{2} \not{J} \left(\frac{p \cdot e_1}{p \cdot k_1} - \frac{q \cdot e_1}{q \cdot k_1} \right) \left(\frac{p \cdot e_2}{p \cdot k_2} - \frac{q \cdot e_2}{q \cdot k_2} \right) \qquad eikonal$$

$$I_{2l}^{\{1,2\}} = -\frac{1}{4} \left[\left(\frac{p \cdot e_1}{p \cdot k_1} - \frac{q \cdot e_1}{q \cdot k_1} \right) \frac{\not{e}_2 \not{k}_2}{p \cdot k_2} + \left(\frac{p \cdot e_2}{p \cdot k_2} - \frac{q \cdot e_2}{q \cdot k_2} \right) \frac{\not{e}_1 \not{k}_1}{p \cdot k_1} \right] \not{J} \qquad \beta_1$$

$$I_{2r}^{\{1,2\}} = \frac{1}{4} \not{J} \left[\left(\frac{p \cdot e_1}{p \cdot k_1} - \frac{q \cdot e_1}{q \cdot k_1} \right) \frac{\not{k}_2 \not{e}_2}{q \cdot k_2} + \left(\frac{p \cdot e_2}{p \cdot k_2} - \frac{q \cdot e_2}{q \cdot k_2} \right) \frac{\not{k}_1 \not{e}_1}{q \cdot k_1} \right] \qquad \beta_1$$

$$I_3^{\{1,2\}} = -\frac{1}{8} \left(\frac{\not{e}_1 \not{k}_1}{p \cdot k_1} \not{J} \frac{\not{k}_2 \not{e}_2}{q \cdot k_2} + \frac{\not{e}_2 \not{k}_2}{p \cdot k_2} \not{J} \frac{\not{k}_1 \not{e}_1}{q \cdot k_1} \right) \qquad start for \beta_2 ...$$

Second order matrix elements.

$$I_{4p}^{\{1,2\}} = \frac{1}{8}\frac{1}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2}\left(\frac{\not e_1 \not k_1 \not e_2 \not k_2}{p\cdot k_1} + \frac{\not e_2 \not k_2 \not e_1 \not k_1}{p\cdot k_2}\right)\not J$$

$$I_{4q}^{\{1,2\}} = \frac{1}{8}\not J \frac{1}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\left(\frac{\not k_2 \not e_2 \not k_1 \not e_1}{q\cdot k_1} + \frac{\not k_1 \not e_1 \not k_2 \not e_2}{q\cdot k_2}\right)$$

$$I_{5pA}^{\{1,2\}} = \frac{1}{2}\not J \frac{k_1\cdot k_2}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2}\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1}\right)\left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right)$$

$$I_{5pB}^{\{1,2\}} = -\frac{1}{2}\not J \frac{1}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2}\left(\frac{k_1\cdot e_2 k_2\cdot e_1}{k_1\cdot k_2} - e_1\cdot e_2\right)$$

$$I_{5qA}^{\{1,2\}} = \frac{1}{2}\not J \frac{k_1\cdot k_2}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\left(\frac{q\cdot e_1}{q\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1}\right)\left(\frac{q\cdot e_2}{q\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right)$$

$$I_{5qB}^{\{1,2\}} = -\frac{1}{2}\not J \frac{1}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\left(\frac{k_1\cdot e_2 k_2\cdot e_1}{k_1\cdot k_2} - e_1\cdot e_2\right)$$

$$I_{6B}^{\{1,2\}} = -\frac{1}{4}\frac{k_1\cdot k_2}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2}\left[+\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_1\cdot k_2}\right)\frac{\not e_2 \not k_2}{p\cdot k_2} + \left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right)\frac{\not e_1 \not k_1}{p\cdot k_1}\right]\not J$$

Second order matrix elements.

$$I_{7B}^{\{1,2\}} = -\frac{1}{4}\, J\, \frac{k_1\cdot k_2}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\left[+\left(\frac{q\cdot e_1}{q\cdot k_1} - \frac{k_2\cdot e_1}{k_1\cdot k_2}\right)\frac{\not k_2 \not e_2}{q\cdot k_2} + \left(\frac{q\cdot e_2}{q\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right)\frac{\not k_1 \not e_1}{q\cdot k_1}\right]$$

- for the **exponentiation** we have used **separation** into 3 parts only. It is **crystal clear**, also in case of contributions with t-channel W, was very useful for KKMC,
- for PHOTOS kernel, parts $I_3^{\{1,2\}}$, $I_{4p}^{\{1,2\}}$, $I_{4q}^{\{1,2\}}$ studied separately as well.
- In fact older works on spin amplitdues were used E. Richter-Was Z.Phys.C64:227-240,1994, Z.Phys.C61:323-340,1994.
- Other parts clearly visible but not used. Further separation of β_2 terms possible ...
- IMPORTANT: higher order amplitudes could be constructed from parts of lower order ones plus numerically rather insignificant corrections.
- IMPORTANT: Gauge cancellation at spin amplitudes level open path for interface to elelectroweak effects through form-factors.

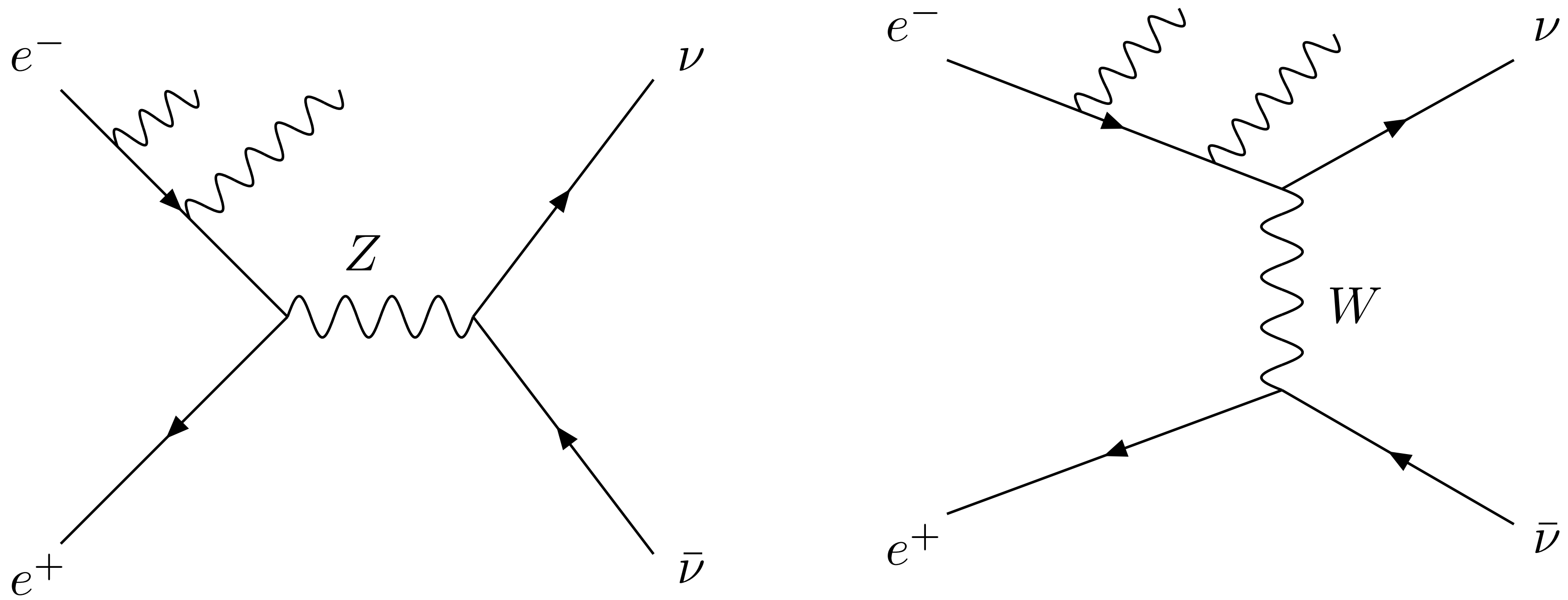


Figure 1: Double emission from electron

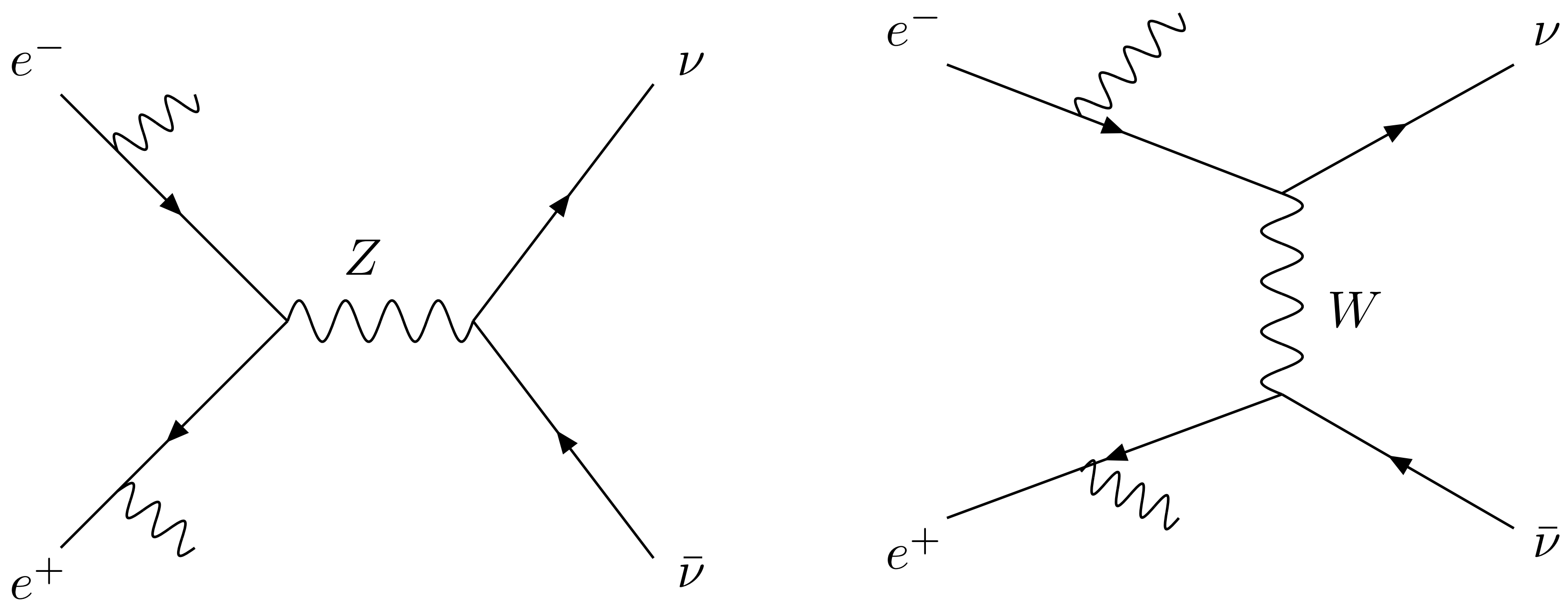


Figure 2: Single emission from electron and positron

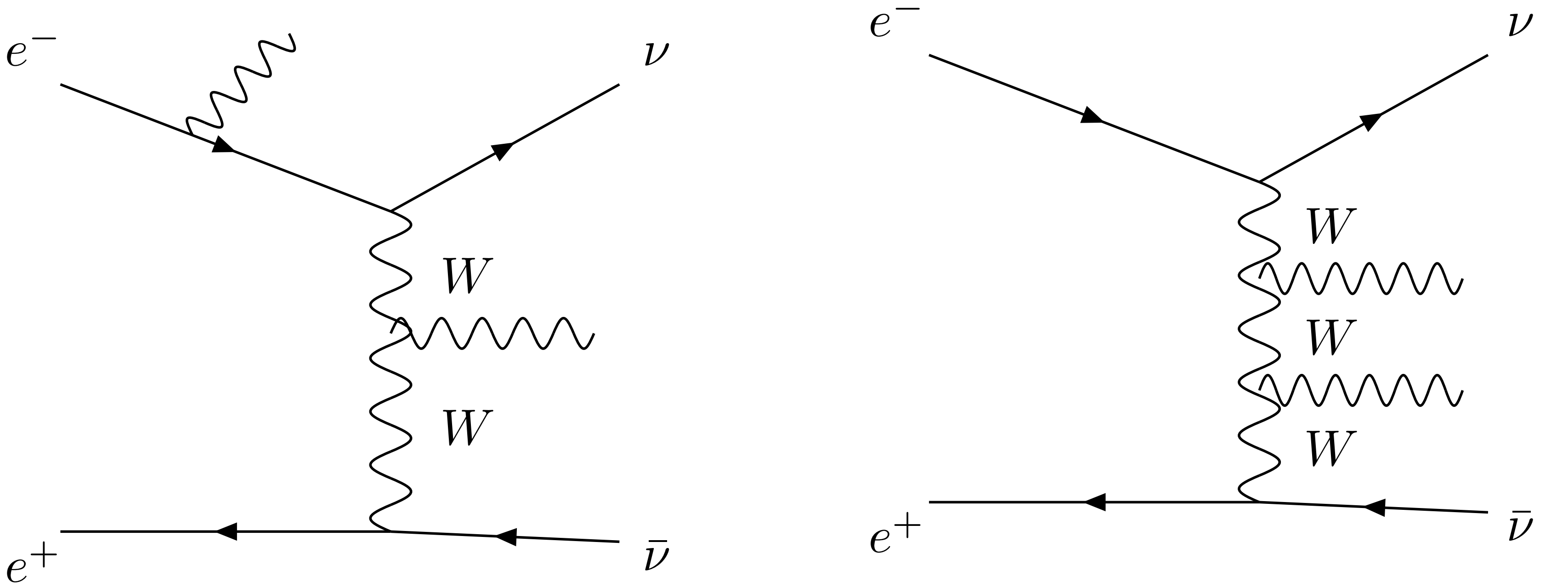


Figure 3: Single and double emission from W

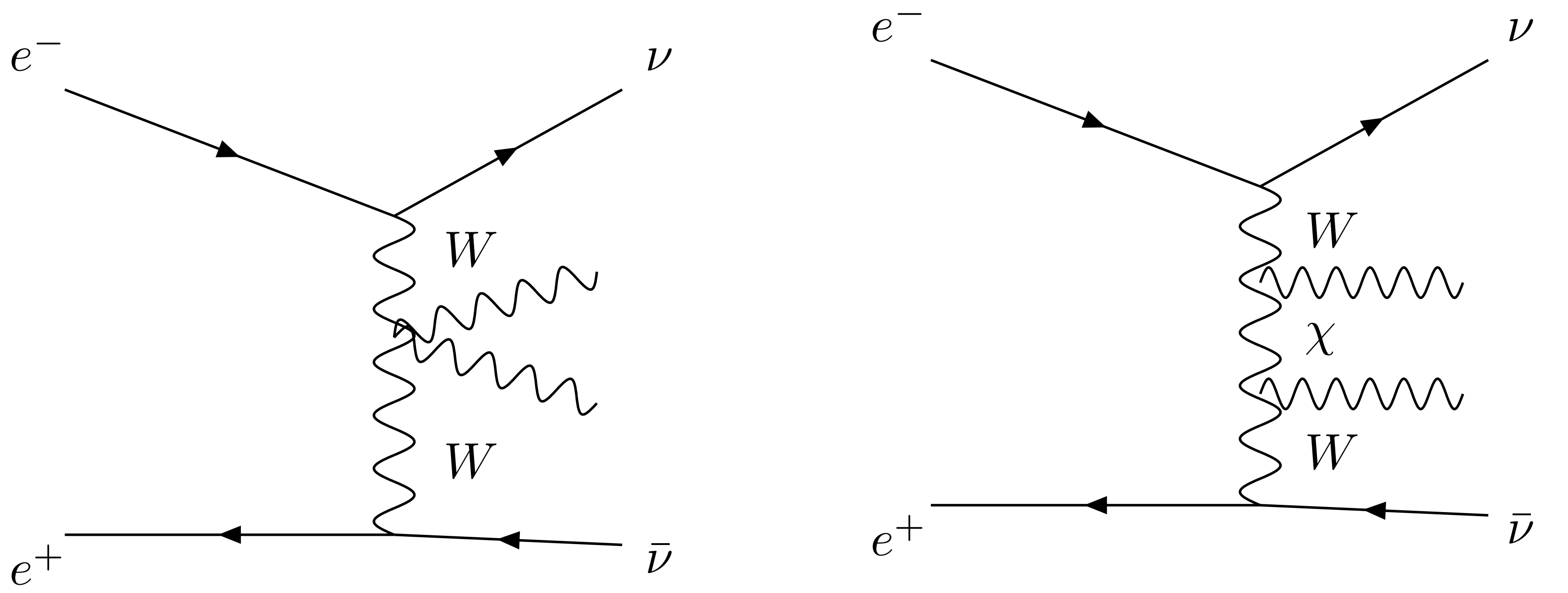


Figure 4: Four boson coupling and coupling for unphysical χ field.

This χ is needed for “pure QED”...

The formula for the complete spin amplitude (Z exchange only) can be easily re-ordered into consecutive contributions $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \ldots$, each gauge invariant:

$$\begin{aligned} \mathcal{M} =& \mathfrak{M}^{Z}_{2\{I\}} \binom{p\, k_1\, k_2}{\lambda \sigma_1 \sigma_2} \\ =& \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 + \mathcal{M}_5 + \mathcal{M}_6 + \mathcal{M}_7 \end{aligned} \tag{9}$$

We can now write the complete spin amplitude, of W interactions, as a sum of even more gauge invariant parts:

$$\begin{aligned} \mathcal{M}_{\mathcal{W}} =& \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 + \mathcal{M}_5 + \mathcal{M}_6 + \mathcal{M}_7 + \\ & \mathcal{M}_8 + \mathcal{M}_9 + \mathcal{M}_{10} + \mathcal{M}_{11}, \end{aligned} \tag{10}$$

Matrix Element: $q\bar{q} \to Jgg$ - part proportional to $T^A T^B$ fermion spinors dropped

$$I_{lr}^{(1,2)} = \left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{e}_1\not{k}_1}{2p\cdot k_1}\right)\not{J}\left(\frac{\not{k}_2\not{e}_2}{2q\cdot k_2} + \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2}\right)$$

$$I_{ll}^{(1,2)} = \frac{p\cdot k_2}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2}\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{e}_1\not{k}_1}{2p\cdot k_1}\right)\left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{\not{e}_2\not{k}_2}{2p\cdot k_2}\right)\not{J}$$

$$I_{rr}^{(1,2)} = \not{J}\,\frac{q\cdot k_1}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\left(\frac{q\cdot e_1}{q\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{k}_1\not{e}_1}{2q\cdot k_1}\right)\left(\frac{q\cdot e_2}{q\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{\not{k}_2\not{e}_2}{2q\cdot k_2}\right)$$

$$I_{e}^{(1,2)} = \not{J}\left(1 - \frac{p\cdot k_2}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2} - \frac{q\cdot k_1}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\right)\left(\frac{k_1\cdot e_2}{k_1\cdot k_2}\,\frac{k_2\cdot e_1}{k_1\cdot k_2} - \frac{e_1\cdot e_2}{k_1\cdot k_2}\right)$$

Remainder:

$$I_{p}^{(1,2)} = -\frac{1}{4}\,\frac{1}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2}\left(\frac{\not{e}_1\not{k}_1\not{e}_2\not{k}_2 - \not{e}_2\not{k}_2\not{e}_1\not{k}_1}{k_1\cdot k_2}\right)\not{J}$$

$$I_{q}^{(1,2)} = -\frac{1}{4}\,\not{J}\,\frac{1}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\left(\frac{\not{k}_1\not{e}_1\not{k}_2\not{e}_2 - \not{k}_2\not{e}_2\not{k}_1\not{e}_1}{k_1\cdot k_2}\right)$$

$\mathcal{M}$atrix $\mathcal{E}$lement: $q\bar{q} \to J gg$ - part proportional to $T^B T^A$ fermion spinors dropped

$$I^{(2,1)}_{lr} = \left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{\not{e}_2\not{k}_2}{2p\cdot k_2} \right) \not{J} \left(\frac{\not{k}_1\not{e}_1}{2q\cdot k_1} + \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{q\cdot e_1}{q\cdot k_1} \right)$$

$$I^{(2,1)}_{ll} = \frac{p\cdot k_1}{p\cdot k_2 + p\cdot k_1 - k_2\cdot k_1} \left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{\not{e}_2\not{k}_2}{2p\cdot k_2} \right) \left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{e}_1\not{k}_1}{2p\cdot k_1} \right) \not{J}$$

$$I^{(2,1)}_{rr} = \not{J} \frac{q\cdot k_2}{q\cdot k_2 + q\cdot k_1 - k_2\cdot k_1} \left(\frac{q\cdot e_2}{q\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{\not{k}_2\not{e}_2}{2q\cdot k_2} \right) \left(\frac{q\cdot e_1}{q\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{k}_1\not{e}_1}{2q\cdot k_1} \right)$$

$$I^{(2,1)}_{e} = \not{J} \left(1 - \frac{p\cdot k_1}{p\cdot k_2 + p\cdot k_1 - k_2\cdot k_1} - \frac{q\cdot k_2}{q\cdot k_2 + q\cdot k_1 - k_2\cdot k_1} \right) \left(\frac{k_2\cdot e_1}{k_2\cdot k_1} \, \frac{k_1\cdot e_2}{k_2\cdot k_1} - \frac{e_2\cdot e_1}{k_2\cdot k_1} \right)$$

$$I^{(2,1)}_{p} = -\frac{1}{4} \frac{1}{p\cdot k_2 + p\cdot k_1 - k_2\cdot k_1} \left(\frac{\not{e}_2\not{k}_2\not{e}_1\not{k}_1 - \not{e}_1\not{k}_1\not{e}_2\not{k}_2}{k_2\cdot k_1} \right) \not{J}$$

$$I^{(2,1)}_{q} = -\frac{1}{4} \not{J} \frac{1}{q\cdot k_2 + q\cdot k_1 - k_2\cdot k_1} \left(\frac{\not{k}_2\not{e}_2\not{k}_1\not{e}_1 - \not{k}_1\not{e}_1\not{k}_2\not{e}_2}{k_2\cdot k_1} \right)$$

For QCD we have separation too; 12 gauge invariant parts

- Terms like

$$\left(\frac{p \cdot e_1}{p \cdot k_1} - \frac{k_2 \cdot e_1}{k_2 \cdot k_1} - \frac{\not{e}_1 \not{k}_1}{2p \cdot k_1} \right) \qquad A$$

once integrated over part of phase space give Atarelli-Parisi kernel

- Terms

$$\frac{q \cdot k_1}{q \cdot k_1 + q \cdot k_2 - k_2 \cdot k_1} \qquad B$$

if combined with phase space Jacobians can be used to redefine fermionic fields from $v(q)$ to $v(q - k_2)$ for example. Term of such type appeared already in scalar QED (normalization of hadronic current).

- No applications for QCD developed ... But amplitudes properties predeced many of QCD phenomenology tools.

- I should now go after virtual corrections, but this would be too much for today.

Higher orders are installed in `KKMC` only.

1. That required work on spin amplitudes, it was not straightforward.
2. How to write fixed order amplitutes into parts, the ones which could have been obtained from from lower orders and the one which could have been used for higher order ones.
3. For $e^+e^- \to \tau^+\tau^-\gamma\gamma + \cdots$ that was already complicated, need to separation into initial state and final state amplitudes appeared.
4. Fortunately interferences were easy to introduce.
5. for $e^+e^- \to \bar{\nu}_e\nu_e\gamma\gamma + \cdots$ things became more complicated. Expansion around contact interaction was necessary to use and QED was not anymore pure. Care about charged higgs ghosts and their contributions was necessary.
6. separation into amplitude parts was essential for electroweak non-QED effects instalation.

- **Toward higher order 3 hard photon amplitudes** If previous bullet OK, that is feasible path. It must be training ground for NP.

- If my present **NP** attempt work that should not be of great problem.

- **Toward higher order 3 loop QED amplitudes** Seee Bennie Ward talks (e.g. 2410.09115) and Phys.Rev.D 99 (2019) 7, 076016

- **Toward two loop electroweak corrections.** Non QED Electroweak effects need to be separated out and implemented in a form of effective couplings of some sort. Matching with third order QED amplitudes and at least 4-5 explicit photons final states.

• Solution need to work with exclusive exponentiation where final states of multiple photons need to be present in predictions.

• At one loop level electroweak corrections were encapsulated into form factors multiplying couplings.

• That was working well with exclusive exponentiation, no breaking of gauge cancellations, because of the way how spin amplitudes for real emissions were divided into parts.

• That was challenge already at 1 loop level. To preserve all field theory constraints, analytic, dispersion relations ....

Finally solutions must be experimentalist friendly, they have many other things to worry. Some cuts may appear at random. For example of dead detector cells.

Interaction between sub-communities essential.