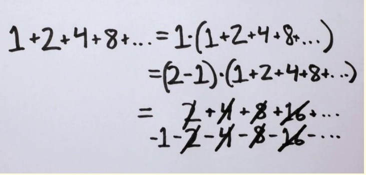
What are Renormalons?

Matthias Jamin
ICREA @ IFAE
Universitat Autònoma de Barcelona

Preamble
Asymptotic series
Borel transform
Vector correlator
Renormalons
OPE
Outlook

Preamble

- Physics aims at describing regularities of nature.
- The pertinent language is mathematics.
- The more words one knows,
  the better one can express oneself.
- No worries though, only complex analysis is required.



$$1+2+4+8+\dots = 1\cdot(1+2+4+8+\dots)$$
$$=(2-1)\cdot(1+2+4+8+\dots)$$
$$=\cancel{2}+\cancel{4}+\cancel{8}+\cancel{16}+\dots$$
$$-1-\cancel{2}-\cancel{4}-\cancel{8}-\cancel{16}-\dots$$

Asymptotic series

- Generally, perturbative expansions in QFT are divergent. (Dyson 1952)
- At best, the perturbative series is asymptotic.

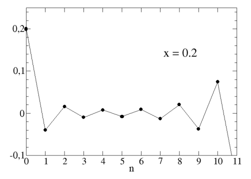
Consider the following integral:

$$F_1(x) \equiv \int\limits_0^\infty \frac{\mathrm{e}^{-t/x}}{(1+t)}\,\mathrm{d}t = \mathrm{e}^{1/x}\,\Gamma(0,1/x) \qquad (\mathrm{Re}\,x>0)$$

$$= x - x^2 + 2\,x^3 - 6\,x^4 + 24\,x^5 - + \ldots = \sum_{n=0}^{\infty} (-1)^n\, n!\, x^{n+1}\,.$$

With the incomplete Gamma-function:

$$\Gamma(n,z) \equiv \int\limits_z^\infty t^{n-1}\,\mathrm{e}^{-t}\mathrm{d}t$$



Consider another integral: $\displaystyle\int_0^\infty \frac{\mathrm{e}^{-t/x}}{(2-t)}\,\mathrm{d}t \qquad (\mathrm{Re}\,x > 0)$

As such, the integral is not well defined!

However: $$\lim_{\varepsilon\to 0^+} \frac{1}{(2-t\mp i\varepsilon)} = \mathscr{P}\left[\frac{1}{(2-t)}\right] \pm i\pi\,\delta(2-t)$$

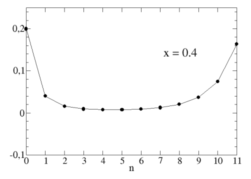
Then: $$F_2(x) \equiv \mathscr{P}\!\int_0^\infty \frac{\mathrm{e}^{-t/x}}{(2-t)}\,\mathrm{d}t = \mathrm{e}^{-2/x}\,\mathrm{Ei}(2/x)$$

$$= \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{4} + \frac{3x^4}{8} + \frac{3x^5}{4} + \ldots = \sum_{n=0}^\infty n!\left(\frac{x}{2}\right)^{n+1}.$$

Regulating the divergence

entails an (exp small) ambiguity!

2nd series terms $1/2^{n+1}$ suppressed.



Borel transform

The Borel transform of a function $F(x)$ is defined by:

$$F(x) \equiv \int_0^\infty e^{-t/x}\, \mathscr{B}[F](t)\, dt$$

Therefore:

$$\mathscr{B}[F_1](t) = \frac{1}{(1+t)} = 1 - t + t^2 - t^3 + - \ldots = \sum_{n=0}^{\infty} (-1)^n t^n$$

$$\mathscr{B}[F_2](t) = \frac{1}{(2-t)} = \frac{1}{2} + \frac{t}{4} + \frac{t^2}{8} + \frac{t^3}{16} + \ldots = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{t}{2}\right)^n$$

- The Borel transform produces a convergent series.
- Individual terms are suppressed by $n!$.
- Dominant contribution with pole closest to $t = 0$.

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Vector correlator

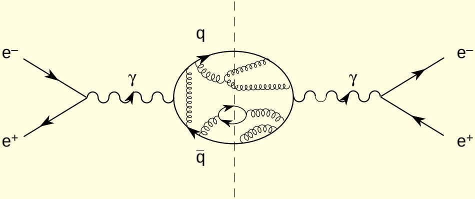
(Central QCD object.)

The two-point correlation function $\Pi(s)$ $(s \equiv p^2)$ is defined by

$$\Pi_{\mu\nu}(p) \;\equiv\; i\int \mathrm{d}^4x\, \mathrm{e}^{ipx}\,\langle\Omega|\,T\{j_\mu(x)j_\nu(0)\}|\Omega\rangle$$
$$=\; (p_\mu p_\nu - g_{\mu\nu}p^2)\,\Pi(p^2)\,,$$

with $j_\mu(x) = \bar{q}(x)\gamma_\mu q(x)$ being the $\bar{q}q$ vector current.

Then: $R_q(s) \;\equiv\; \dfrac{\sigma(e^+e^- \to \gamma^*(p) \to \bar{q}q)}{\sigma(e^+e^- \to \mu^+\mu^-)} \;=\; 12\pi\,\mathrm{Im}\,\Pi(s+i0)\,.$



Besides the physical observable $R_q(s)$, we define:

$$D(s) \equiv -s \frac{\mathrm{d}}{\mathrm{d}s} \Pi(s)$$

Both quantities are related by:

$$R_q(s) = 6\pi i \int\limits_{s-i0}^{s+i0} \frac{D(s')}{s'} \, \mathrm{d}s'$$

Perturbative expansion of $D(s)$ in terms of $a_Q \equiv \alpha_s(Q)/\pi$:

$$(Q^2 \equiv -q^2 = -s)$$

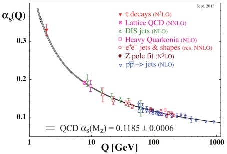
$$4\pi^2 D(s) = 1 + a_Q + c_2 \, a_Q^2 + c_3 \, a_Q^3 + \ldots \equiv 1 + \widehat{D}(a_Q)$$

Investigate the Borel transform of $\widehat{D}(a_Q)$:

$$\widehat{D}(a_Q) = \frac{2\pi}{\beta_1} \int\limits_0^\infty \mathrm{e}^{-\frac{2u}{\beta_1 a_Q}} \, \mathscr{B}[\widehat{D}](u) \, \mathrm{d}u$$

Scaling of the QCD coupling: ($\overline{\mathrm{MS}}$-scheme)

$$-Q\,\frac{\mathrm{d}a_Q}{\mathrm{d}Q} \;\equiv\; \beta(a_Q) \;=\; \beta_1 a_Q^2 + \beta_2 a_Q^3 + \ldots$$



However, a_Q is a dimensionless quantity. Hence:

$$a_Q \;=\; f(Q/\Lambda) \;=\; \frac{1}{\beta_1 \ln(Q/\Lambda)} + \ldots$$

Renormalons

('t Hooft 1977)

Scaling arguments (RG) show that $\mathscr{B}[\widehat{D}](u)$ is a sum of:

UV renormalon poles:

$$\frac{d_k^{\mathrm{UV}}}{(p+u)^{\gamma_k}}\Big[1+\mathscr{O}(p+u)\Big]$$

$$p \;=\; 1,\,2,\,3,\,\ldots$$

IR renormalon poles:

$$\frac{d_i^{\mathrm{IR}}}{(p-u)^{\gamma_i}}\Big[1+\mathscr{O}(p-u)\Big]$$

$$p \;=\; 2,\,3,\,4,\,\ldots$$

- Large-order behaviour dominated by UV poles at $u = -1$. (Yield sign-alternating series.)
- This is not seen in the known coefficients c_2, c_3, c_4. ($\Rightarrow$ At low orders IR renormalons dominant.)
- γ_i, γ_k and sub-leading terms are perturbatively calculable.
- Residues $d_i^{\rm IR}$ and $d_k^{\rm UV}$ are non-perturbative objects.
- What happens with the ambiguity of IR renormalons?

Operator Product Expansion

(Wilson 1969)

$$D(a_Q) \;=\; D^{(0)}(a_Q) + \sum_{d=4,6,\ldots}^{\infty} \underset{\substack{\Uparrow \\ \text{Wilson} \\ \text{coefficients}}}{C^{(d)}(a_Q)} \, \underset{\substack{\Uparrow \\ \text{local} \\ \text{operators}}}{\frac{\langle\Omega|O_d(0)|\Omega\rangle}{Q^d}}$$

- In massless QCD, Λ is the only inherent scale.

- Non-perturbative condensates must behave as $\langle\Omega|O_d(0)|\Omega\rangle \sim \Lambda^d$.

- Power corrections scale like: $\left(\frac{\Lambda}{Q}\right)^d = \mathrm{e}^{-\frac{d}{\beta_1 a_Q}}\,\big[1+\ldots\big]$

- Operators of dimension d correspond to IR renormalon poles at location $p = d/2$.

- The lowest-dimensional operator is the gluon condensate $\langle\Omega|G_{\mu\nu}G^{\mu\nu}|\Omega\rangle \sim \Lambda^4 \;\Leftrightarrow\;$ pole at $u = 2$.

- No gauge-invariant operator with dimension 2!

- The ambiguities between the perturbative series and operator definitions have to cancel.

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- Matching known perturbative results with Borel models.
- Aim at obtaining more information on residues.
- Interesting new mathematical line: resurgence theory.

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Thank You! And...

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Enjoy!