Renormalons in Pole Mass of Quarks

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Outline

Introduction

- Quark masses are parameters of the Standard Model
  - cannot be measured directly
  - must be extracted from hadron masses
- Precise values m_b and m_c are needed for Higgs decays to $\bar{c}c$ and $\bar{b}b$
  - Goal: to calculate $\overline{\mathrm{MS}}$-renormalized masses of bottom and charm quarks
  - How: from heavy-light (or heavy-heavy) meson masses

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  - The perturbative relation between the pole mass and the $\overline{\text{MS}}$ mass is divergent $\rightarrow$ *renormalon* problem

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  - Borel resummation can be used to improve convergence
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First, let us start with a toy model

Toy Model: Divergence in Perturbation Theory

- QM exercise: find the ground-state energy of the schrödinger equation

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 + \lambda\, x^4\right)\psi(x) = E(\lambda)\,\psi(x)$$

- Answer: $E_0(\lambda) = \frac{1}{2} + \sum_{n=1}^{\infty} c_n \lambda^n$ where $c_1 = \frac{3}{4}$, $c_2 = -21/8, \cdots$

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  $c_n\sim(-1)^{n+1}\frac{\sqrt{6}}{\pi}\,\frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})}\,3^n$ as $n\to\infty$ [Bender and Wu, Phys.Rev. 184, 1231 (1969)]

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- Inserting $1=\frac{1}{n!}\int_0^\infty dt\,e^{-t}\,t^n$, the leading divergent part becomes

$$\begin{aligned}E_0(\lambda)\quad&\sim\quad\frac{1}{2}-\frac{\sqrt{6}}{\pi}\sum_{n=1}^{\infty}\int_0^\infty dt\,e^{-t}\,\frac{1}{n!}\,\frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})}\,(-3\lambda t)^n\\&\sim\quad\frac{1}{2}+\frac{\sqrt{6}}{\pi}\left(1-\int_0^\infty dt\,\frac{e^{-t}}{\sqrt{1+3\lambda t}}\right)\end{aligned}$$

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This is basically the method of Borel resummation

Borel Resummation and Borel Plane

- Now consider this series

$$f(x) = \sum_{n=0}^{\infty} r_n x^{n+1}$$

- Apply the same trick (for positive x)

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} r_n x^{n+1} \frac{1}{n!} \int_0^{\infty} dt\, t^n\, e^{-t} = \int_0^{\infty} dt\, e^{-t} \sum_{n=0}^{\infty} r_n x^{n+1} \frac{t^n}{n!} \\ &= \int_0^{\infty} ds\, e^{-s/x} \sum_{n=0}^{\infty} \frac{r_n}{n!} s^n \end{aligned}$$

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- Now we have a more convergent (less divergent) series

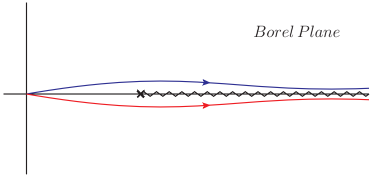
$$\begin{aligned} F(s) &= \sum_{n=0}^{\infty} \frac{r_n}{n!} s^n \\ f(x) &= \int_0^{\infty} ds\, e^{-s/x}\, F(s) \end{aligned}$$

Toy Model and Borel Plane

- The ground-state energy (the leading divergent part)

$$E_0(\lambda) \quad \sim \quad \frac{1}{2} + \frac{\sqrt{6}}{\pi}\left(1 - \int_0^\infty dt\, \frac{e^{-t}}{\sqrt{1+3\lambda t}}\right)$$

- Singularity on positive real axis when $\lambda < 0$
- Deform the contour of integration
- Which direction? ($\rightarrow$ Ambiguity)



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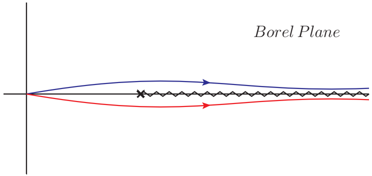

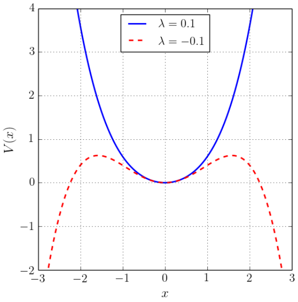

- Recall the potential $V(x) = \frac{1}{2}x^2 + \lambda\, x^4$
- For $\lambda > 0$, the ground-state is stable
- For $\lambda < 0$, the "ground-state" is not stable;
  inverse of life time $=$ imaginary part of energy
  $\propto$ contour integral around the branch cut

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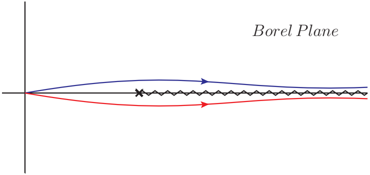

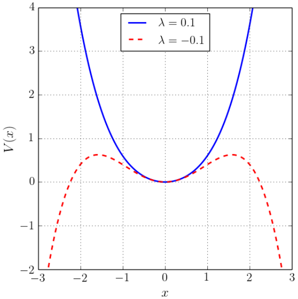

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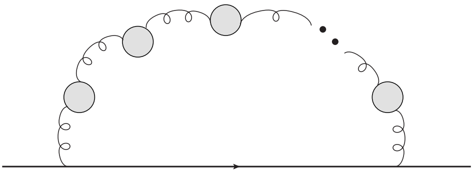
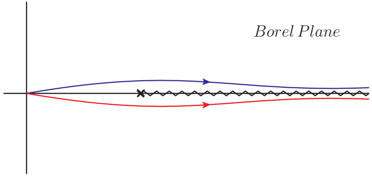
Singularities in Borel plane (R^+)
$\rightarrow$ Ambiguity in Borel sum
$\rightarrow$ Non-Pert. imaginary part of energy

Renormalons in Pole Mass: Feynman Diagrams

- Similar to the QM toy model, pert. calculations in QCD can be divergent
- For instance, in the self-energy of a quark:
  - Consider a bubble chain with n blobs in self-energy of quarks
  - The blob is the vacuum polarization of a gluon
  - Leads to a fixed-sign factorial growth in perturbative calculations
    $\Rightarrow$ Singularity on positive real axis of Borel plane
    $\Rightarrow$ Ambiguity in Borel sum of self-energy
    $\Rightarrow$ Ambiguity in Borel sum of pole mass





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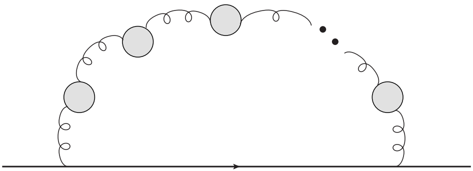

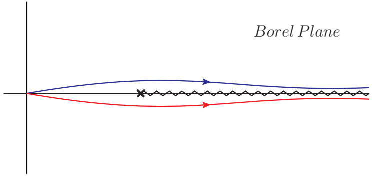

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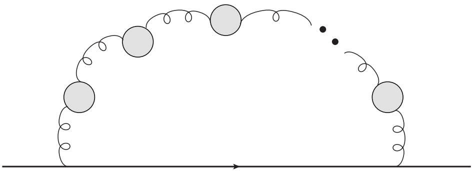

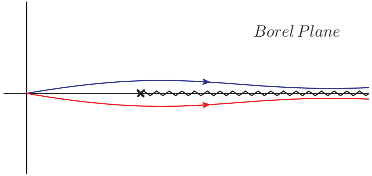

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- The pole mass of a quark in perturbation theory

$$m^{\text{pole}} = \overline{m}\left(1+\sum_{n=0}^{\infty} r_n\, \alpha_s^{n+1}(\overline{m})\right) \qquad \text{where} \quad \overline{m} = m^{\overline{\mathsf{MS}}}(\mu = \overline{m})$$

- r_n happens to grow as

$$r_n \sim \text{constant} \times (2\beta_0)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{s_1}{n+b} + \cdots\right)$$

  as $n \to \infty$, where $b = \beta_1/(2\beta_0^2)$, $s_1 = b^2 - \beta_2/(4\beta_0^3)$

- In Borel plane, this factorial growth leads to the so-called (leading) renormalon ambiguity

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- I use the following info to develop a new way to investigate renormalons

> The leading renormalon ambiguity in the pole mass is independent of $\overline{m}$

- The pole mass in terms of $\overline{m} = m^{\overline{\mathsf{MS}}}(\mu = \overline{m})$

$$m^{\text{pole}} \quad = \quad \overline{m}\,(1+y)\ , \qquad y = \sum_{n=0}^{\infty} r_n \alpha^{n+1}(\overline{m})$$

- Take derivative of m_{pole} with respect to $\overline{m}$

$$\begin{aligned} \frac{dm^{\text{pole}}}{d\overline{m}} &= 1 + y + 2\beta(\alpha) y' \\ \beta(\alpha) &= \overline{m}^2 \, \frac{d\alpha(\overline{m})}{d\overline{m}^2} = -\sum_{i=0}^{\infty} \beta_i \alpha^{2+i} \end{aligned}$$

- Some algebra:

$$\begin{aligned} \frac{dm^{\text{pole}}}{d\overline{m}} &= 1 + y + 2\beta(\alpha) y' \\ &= 1 + \sum_{n=0}^{\infty} r_n \alpha^{n+1} - 2 \left(\sum_{i=0}^{\infty} \beta_i \alpha^{2+i} \right) \left(\sum_{n=0}^{\infty} r_n (n+1) \alpha^n \right) \\ &= 1 + \sum_{n=0}^{\infty} \Big(r_n - 2\big(\beta_0 n r_{n-1} + \beta_1 (n-1) r_{n-2} + \cdots + \beta_{n-1} r_0\big) \Big) \, \alpha^{n+1} \end{aligned}$$

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- Recall

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\end{aligned}$$

- Construct a new power series

$$\frac{dm^{\text{pole}}}{d\overline{m}} = 1 + \sum_{n=0}^{\infty} r'_n \alpha^{n+1}$$

- Given the sequence r_n, we can map them to r'_n

$$r'_n = r_n - 2\big(\beta_0 n r_{n-1} + \beta_1 (n-1) r_{n-2} + \cdots + \beta_{n-1} r_0\big)$$

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- Recall

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- Thus, $\frac{dm^{\text{pole}}}{d\overline{m}}$ must be free of the leading renormalon
  $\Rightarrow$ The leading renormalon divergence of r_n cannot propagate to r'_k

Renormalon in Pole Mass: Recurrence Relation

- Note that the leading renormalon divergence of r_n cannot propagate to r'_k

$$r'_n = r_n - 2\big(\beta_0 n r_{n-1} + \beta_1 (n-1) r_{n-2} + \cdots + \beta_{n-1} r_0\big)$$

- This implies a recurrence relation

$$a_n = 2n\beta_0 a_{n-1} + 2(n-1)\beta_1 a_{n-2} + \cdots + 2\beta_{n-1} a_0\,, \quad n \geq 1$$

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  which has a solution that diverges as $n \to \infty$, but it cannot propagate to r'_k
- What is a_n?

$$\begin{aligned}
a_1 &= 2\beta_0 a_0 \\
a_2 &= (2\beta_0)^2 (2 + \frac{\beta_1}{2\beta_0^2}) a_0 \\
&\vdots \\
\Rightarrow \ a_n &= \text{constant} \times \ (2\beta_0)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{s_1}{n+b} + \cdots \right)
\end{aligned}$$

  where $b = \beta_1/(2\beta_0^2)$, $s_1 = b^2 - \beta_2/(4\beta_0^3)$.
- a_n is indeed proportional to the leading renormalon of the pole mass

$$r_n \sim C\, a_n \quad \text{as} \ \ n \to \infty$$

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- For example, try

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  to get the structure of the NL renormalon!
- Is it also possible to calculate the overall constants of the renormalons?

- Recall the relation between the pole mass and the $\overline{\text{MS}}$ mass $\overline{m}$

$$m^{\text{pole}} = \overline{m}\,(1+y)\;, \quad y = \sum_{n=0}^{\infty} r_n \alpha^{n+1}(\overline{m})$$

- And its derivative with respect to $\overline{m}$

$$\frac{dm^{\text{pole}}}{d\overline{m}} = 1 + y + 2\beta(\alpha)y'\,, \quad \frac{dm^{\text{pole}}}{d\overline{m}} = 1 + \sum_{n=0}^{\infty} r'_n \alpha^{n+1}$$

- This is a first order differential equation

$$\begin{aligned} y(\alpha) &= \int_{\alpha_{\text{base}}}^{\alpha} e^{-\int_{\alpha'}^{\alpha} \frac{d\alpha''}{2\beta(\alpha'')}} \left(\frac{dm_{\text{pole}}}{d\overline{m}} - 1\right) \frac{d\alpha'}{2\beta(\alpha')} \\ &= \int_{\alpha_{\text{base}}}^{\alpha} e^{-\int_{\alpha'}^{\alpha} \frac{d\alpha''}{2\beta(\alpha'')}} \left(\sum_{n=0}^{\infty} r'_n \alpha^{n+1}\right) \frac{d\alpha'}{2\beta(\alpha')} \end{aligned}$$

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- Note that $\sum_{n=0}^{\infty} r'_n \alpha^{n+1}$ has no information about the first renormalon
- Thus, the leading renormalon is generated because of the form of this integral
- We can investigate the leading renormalon by solving this integral
- Note that the solution to the differential equation can be also formally written as

$$y(\alpha) \quad = \quad \frac{1}{1+2\beta(\alpha)\frac{d}{d\alpha}} \sum_{k=0}^{\infty} r'_k \alpha^{k+1}$$

- For simplicity, perform the calculation in a scheme in which

$$\beta(\alpha) = -\frac{\beta_0 \alpha^2}{1 - \frac{\beta_1}{\beta_0}\alpha}$$

- In this scheme, the exact solution to the introduced recurrence relation is

$$a_n \quad = \quad 2n\beta_0 a_{n-1} + 2(n-1)\beta_1 a_{n-2} + \cdots + 2\beta_{n-1} a_0\,, \quad n \geq 1$$

  has a simple exact solution

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- Let us refer to a_n as "pure-renormalon" sequence
- The objective is now to calculate the initial value a_0 such that

$$r_n \sim a_n \quad \text{as} \quad n \to \infty$$

Renormalon in Pole Mass: Overall Normalization

- And its differential equation and its formal solution

$$\frac{dm^{\rm pole}}{d\overline{m}} = 1 + y + 2\beta(\alpha)y'\,, \quad \frac{dm^{\rm pole}}{d\overline{m}} = 1 + \sum_{n=0}^{\infty} r'_n \alpha^{n+1}$$

$$\Rightarrow \quad y(\alpha) = \frac{1}{1+2\beta(\alpha)\frac{d}{d\alpha}} \sum_{k=0}^{\infty} r'_k \alpha^{k+1} = \sum_{n=0}^{\infty} \left(-2\beta(\alpha)\frac{d}{d\alpha}\right)^n \sum_{k=0}^{\infty} r'_k \alpha^{k+1}$$

- After taking derivatives, summing up the terms and organizing in powers of α, we find that

$$y(\alpha) = \sum_{n=0}^{\infty} s_n\, \alpha^{n+1}$$

where the leading behavior of s_n is

$$s_n \sim N\,(2\beta_0)^n\, \frac{\Gamma(n+1+b)}{\Gamma(1+b)}$$

with

$$N = \sum_{k=0}^{\infty} r'_k\, \frac{\Gamma(1+b)}{\Gamma(2+k+b)}\, \frac{1+k}{(2\beta_0)^k}$$

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- We also find a heuristic condition that

$$s_n \sim N \, (2\beta_0)^n \, \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \qquad \text{if (at least)} \qquad |b| \lesssim n$$

- An important condition at vicinity of the conformal window of QCD where b is big

Overall Normalization: Large n_f Limit

- Renormalon singularities of pole mass are known at large n_f (number of flavors)
- Let us see if the derivative formula work at large n_f
- At leading order in this limit, one can keep only β_0 and drop all β_n for $n > 0$ and set $b = 0$, then

$$\begin{aligned} N\Big|_{(\text{large } n_f)} &= \sum_{k=0}^{\infty} r'_k \, \frac{1}{k!} \, \frac{1}{(2\beta_0)^k} = r_0 + \sum_{k=1}^{\infty} \left(r_k - 2\beta_0 \, k \, r_{k-1} \right) \frac{1}{k!} \, \frac{1}{(2\beta_0)^k} \\ &= \Big((1-2u) \, B[y](u) \Big)\Big|_{u=1/2} \\ &= \frac{4}{3\pi} \, e^{5/6} \end{aligned}$$

where $B[y](u)$ is the Borel transform of $y = \sum_{n=0}^{\infty} r_n \alpha^{n+1}$.

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- Now let us check numerically how fast a truncates series of N converges to its value

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  $\rightarrow$ This might be a remnant of a higher order renormalon
  It turns out that this is really the remnant of the NL renormalon

Overall Normalization: Finite n_f

- The relation between the pole mass and the $\overline{\text{MS}}$ mass is known up to order α_s^4
- Using that relation and the following truncated series, we find

$$N_{k_{\text{max}}} = \sum_{k=0}^{k_{\text{max}}} r'_k \frac{\Gamma(1+b)}{\Gamma(2+k+b)} \frac{1+k}{(2\beta_0)^k}$$

n_l \ k_{max}	0	1	2	3
0	0.299	0.501	0.577	0.592
1	0.299	0.494	0.566	0.576
2	0.301	0.487	0.554	0.558
3	0.304	0.483	0.539	0.535
4	0.310	0.480	0.522	0.505
5	0.319	0.482	0.498	0.463
6	0.335	0.489	0.461	0.396

- Considering the uncertainties, this table is consistent with previous calculations such as [C. Ayala, G. Cveti, and A. Pineda, JHEP 09, 045 (2014)] and [M. Beneke, P. Marquard, P. Nason, and M. Steinhauser, arXiv:1605.03609]

Overall Normalization: Conformal Window of QCD

- In QCD, the first two coefficients of the beta function, namely β_0 and β_1 are are scheme independent
- Both β_0 and β_1 are positive for small values of flavors
- There is a region in which β_0 is positive and β_1 is negative
  $\rightarrow$ indicates the presence of a non-trivial zero in the beta function in this region
- Is there any renormalon in this region?

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$$r_n \sim N\,(2\beta_0)^n\,\frac{\Gamma(n+1+b)}{\Gamma(1+b)} \quad \text{if (at least)} \quad |b| \lesssim n$$

- This implies that the factorial growth due to the leading renormalon is dominant only at very high orders in perturbation theory (if there are any renormalons at all) because $b = \beta_1/(2\beta_0^2)$ is very large in this region
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  and [C. Ayala, G. Cveti, and A. Pineda, JHEP 09, 045 (2014)]
- Also note that scheme conversions can be problematic in this region

Renormalon-Subtracted Mass

- The pole mass

$$m^{\rm pole} \quad = \quad \overline{m}\left(1+\sum_{n=0}^{\infty} r_n \alpha_s^{n+1}(\overline{m})\right)$$

where for large n

$$r_n \quad \sim \quad a_n = N\,(2\beta_0)^n\,\frac{\Gamma(n+1+b)}{\Gamma(1+b)}\,(1+\mathcal{O}(1/n))\;, \qquad b=\beta_1/(2\beta_0^2)$$

- Renormalon-subtracted mass: subtract the (leading) renormalon from the pole mass [A. Pineda hep-ph/0105008]

$$\begin{aligned}
m^{\rm RS} \quad &\equiv \quad m^{\rm pole} - \nu_f \sum_{n=0}^{\infty} a_n\,\alpha_s^{n+1}(\nu_f) \\
&= \quad \overline{m}\left(1+\sum_{n=0}^{\infty} r_n \alpha_s^{n+1}(\overline{m})\right) - \nu_f \sum_{n=0}^{\infty} a_n\,\alpha_s^{n+1}(\nu_f) \\
&= \quad \overline{m}\left(1+\sum_{n=0}^{\infty} r_n^{\rm RS}(\overline{m},\nu_f,\mu)\alpha_s^{n+1}(\mu)\right)
\end{aligned}$$

- $r_n^{\rm RS}$ is free of the (leading) renormalon $\Rightarrow$ improved convergence

HQET Description of Heavy-Light Mesons

- Expansion of the mass of a heavy-light pseudoscalar system in terms of the heavy quark mass m_Q

$$M_H = m_h + \bar{\Lambda} + \frac{\mu_\pi^2}{2m_h} - \frac{\mu_G^2(m_h)}{2m_h} + \cdots$$

- $\bar{\Lambda}$: energy of quark and gluons inside the system
- $\mu_\pi^2/2m_h$: kinetic energy of the heavy quark inside the system
- $\mu_G^2(m_h)/2m_h$: hyperfine energy due to heavy quark's spin (μ_G runs)
- m_h could be interpreted as the pole mass of the heavy quark (not a practical choice because of renormalons)
- A practical choice is to use the renormalon-subtracted mass $m_h \to m_h^{\mathsf{RS}}$

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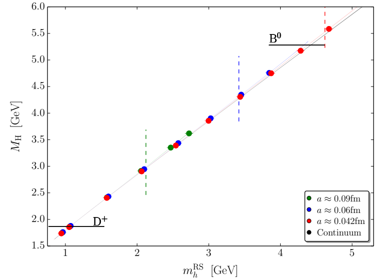
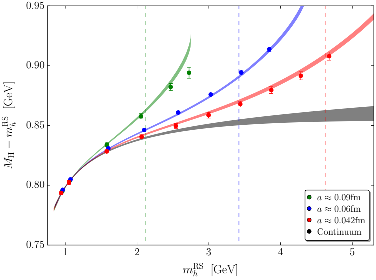
- A practical choice is to use the renormalon-subtracted mass $m_h \to m_h^{\mathsf{RS}}$
- Fit this function to lattice-QCD data to determine heavy quark masses in RS scheme, and in turn in $\overline{\mathsf{MS}}$ scheme

Extraction of Quark Masses from Lattice QCD

- Quark masses can be determined by means of Lattice-QCD simulations.
- A direct way to determine the quark masses is to match the tuned bare mass of a quark in lattice and the $\overline{\text{MS}}$ mass.
- Lattice simulations for the bottom quark mass are expensive and the matching uncertainty can be large.
- We employ HQET to construct a fit function for meson masses in terms of quark masses.
- We perform a combined correlated fit to the masses of mesons that contain a heavy-quark with mass from near charm to bottom.
- After continuum extrapolation, by fixing the meson mass to the mass of D and B mesons, we determine the charm and bottom quark masses.





Conclusion

- A toy model was used to explain the divergence in PT
- Renormalon problem in pole mass was discussed
- A novel method was introduced to investigate renormalons
- Renormalon-subtracted mass was discussed
- HQET description of heavy-light meson masses used in order to analysis lattice-QCD data and extract heavy quark masses
  - Statistical errors of lattice-QCD data are tiny: a challenge to good fits

Thank you for your attention!

Back-up Slides

- With change of variables $z = \frac{\beta_0}{\beta_1}\alpha^{-1}$, $x = \frac{\beta_1}{2\beta_0^2}$ and $d_k = r'_k\,(\beta_0/\beta_1)^{1+k}$, we need to solve

$$F(z,b) = \frac{1}{1+\frac{b^{-1}}{1-z^{-1}}\,\frac{d}{dz}}\,\sum_{k=0}^{\infty} d_k\, z^{-(k+1)} = \sum_{n=0}^{\infty}\Big(-\frac{b^{-1}}{1-z^{-1}}\,\frac{d}{dz}\Big)^n\,\sum_{k=0}^{\infty} d_k\, z^{-(k+1)}$$

- Calculations are simpler if we find a good basis instead of $z^{-1}, z^{-2}, \cdots$
- A useful basis can be constructed from this set of formal series

$$g_n(z;\nu) = \sum_{k=0}^{\infty}\frac{\Gamma^{(k)}(\nu+n+k)}{\Gamma(\nu)\Gamma(k+1)}\,z^{-(\nu+n+k)}$$

- They can be generated as

$$g_n(z;\nu) = \Big(\frac{-1}{1-z^{-1}}\,\frac{d}{dz}\Big)^n\,g_0(z;\nu)$$

- In this basis we have

$$\begin{aligned}
F(z,b) &= \frac{1}{1+\frac{b^{-1}}{1-z^{-1}}\,\frac{d}{dz}}\,g_0(z;\nu) = \sum_{n=0}^{\infty} g_n(z;\nu)\,b^{-n} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\Gamma^{(k)}(\nu+n+k)}{\Gamma(\nu)\Gamma(k+1)}\,b^{-n}\,z^{-(\nu+n+k)} \\
&= \sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{\Gamma^{(k)}(\nu+m)}{\Gamma(\nu)\Gamma(k+1)}\,b^{-(m-k)}\,z^{-(\nu+m)} \\
&= \sum_{m=0}^{\infty}\Big(\Gamma(\nu+m+b) - R_m(b;\nu+m)\Big)\,\frac{b^{-m}}{\Gamma(\nu)}\,z^{-(\nu+m)}
\end{aligned}$$

where

$$R_m(x;\nu+m) \equiv \int_0^x dt\,\frac{(x-t)^m}{\Gamma(m+1)}\Gamma^{(m+1)}(\nu+m+t)$$