On the non-linear scale of cosmological perturbation theory

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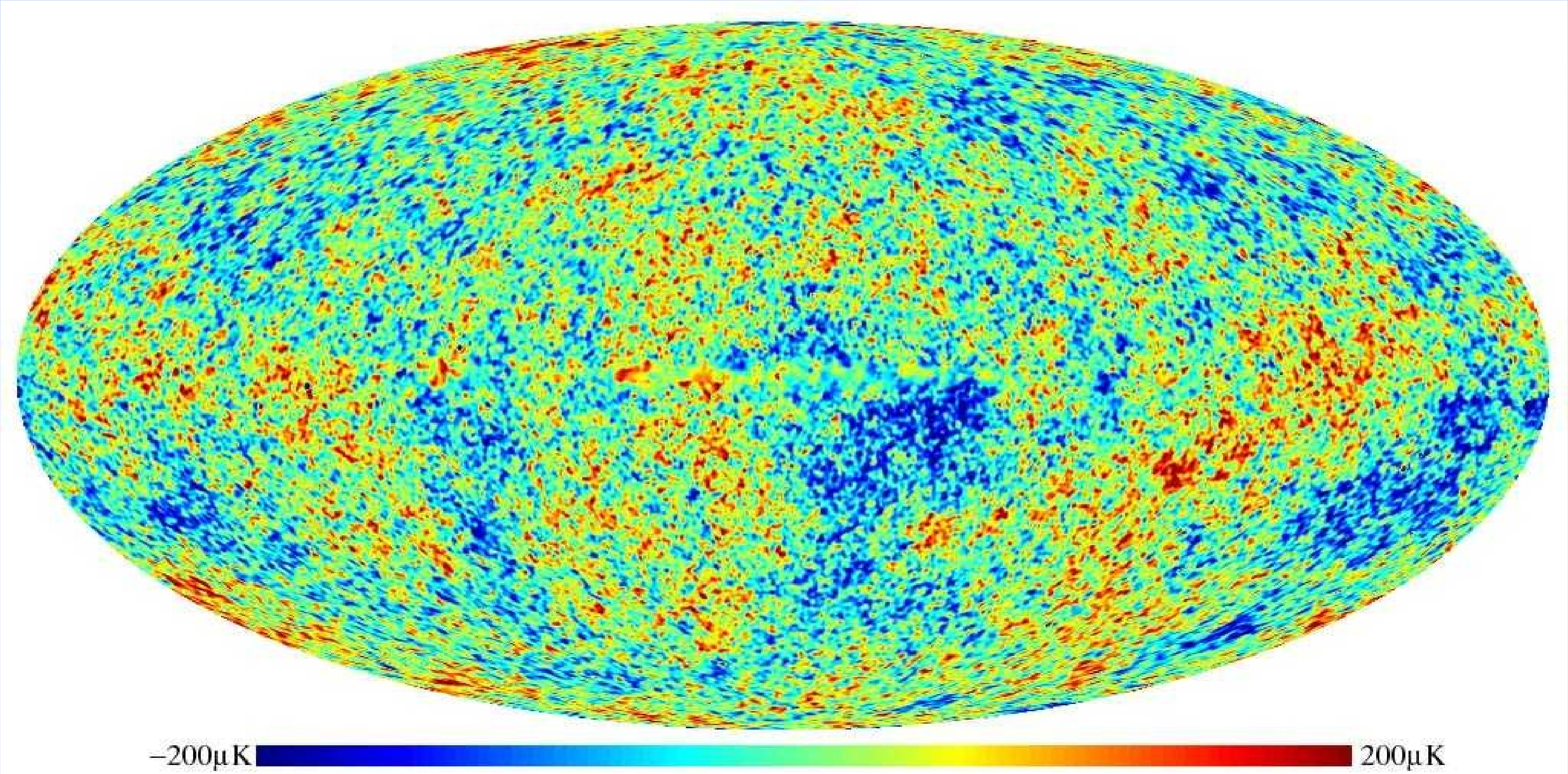
DESY

in collaboration with D. Blas and M. Garny

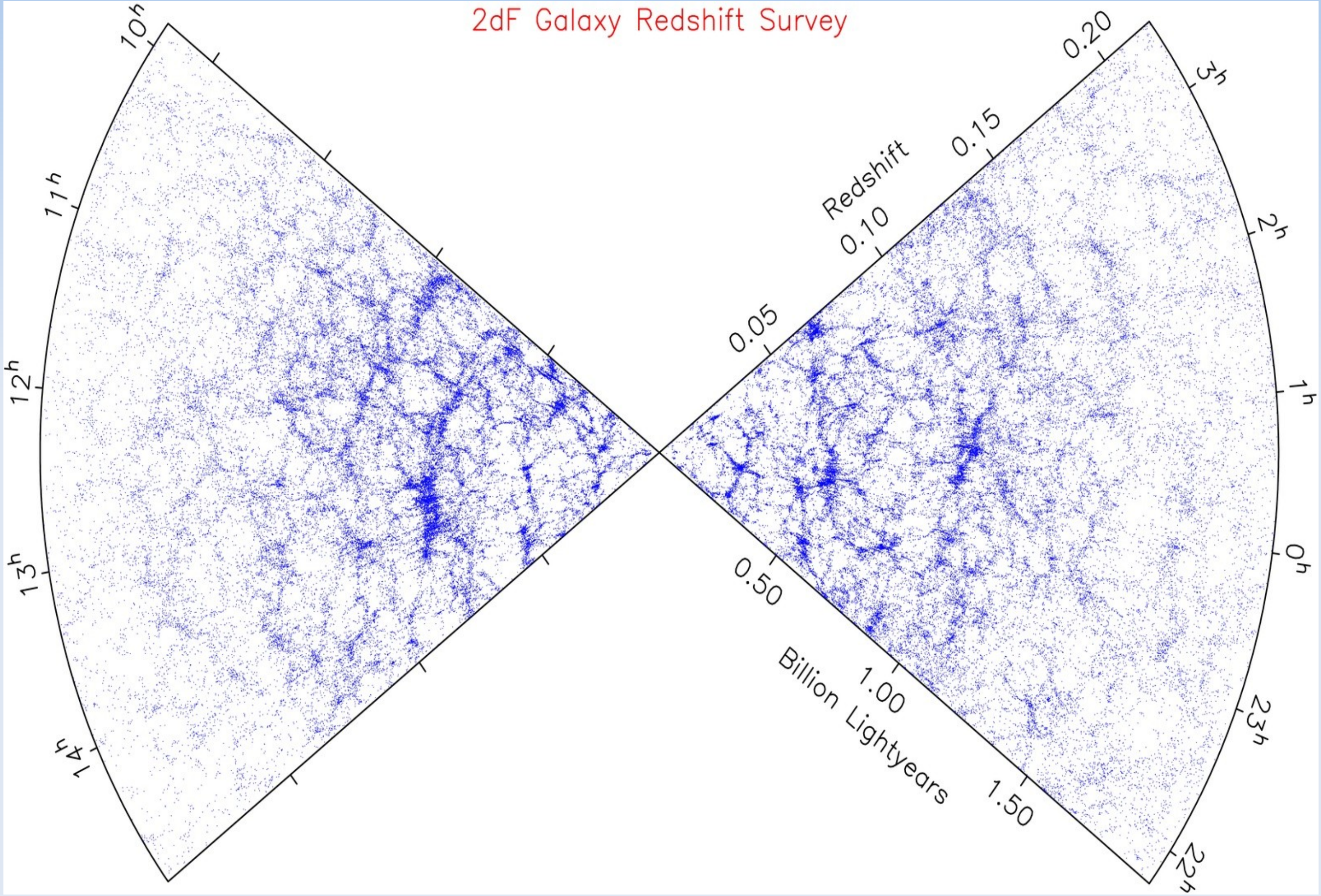
Barcelona, March 20, 2014

How to get from here ...



Initial conditions of structure formation are very well established by CMB measurements

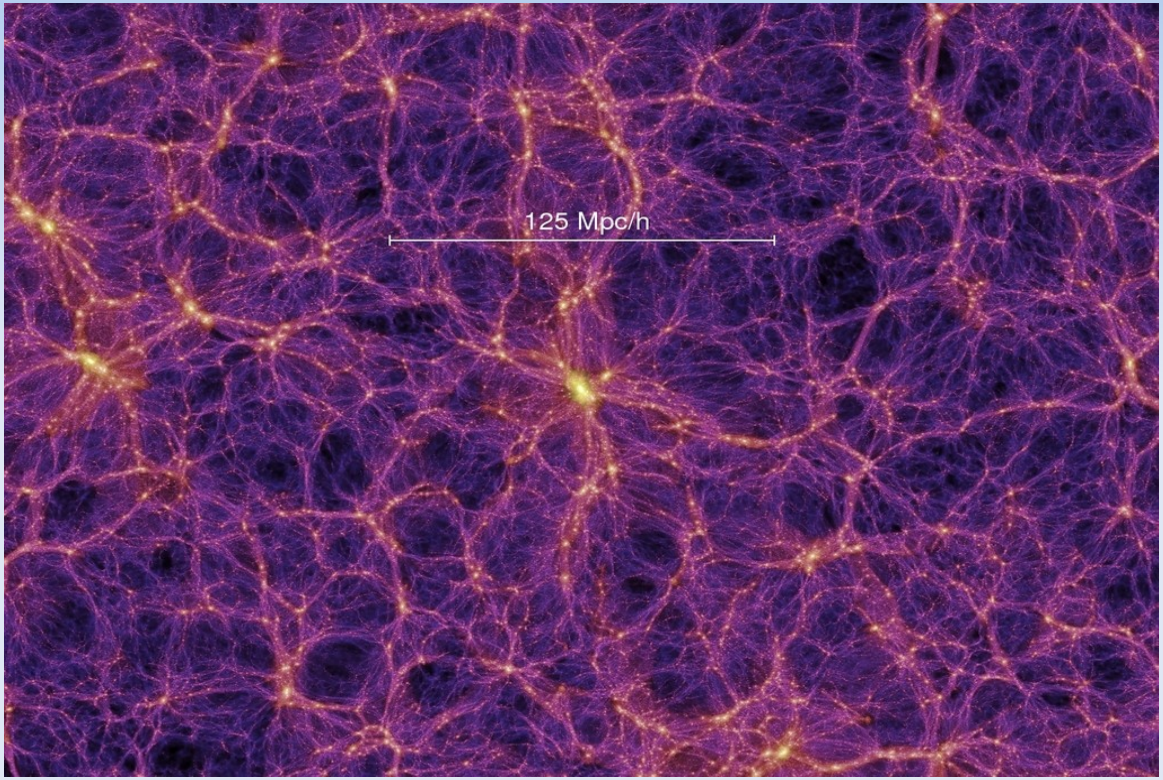
... to here ...



Galaxy surveys determine the matter distribution on large scales.

It is sensitive to different red-shifts and probes the real expansion history and not just a model.

.. without this.



The most reliable predictions for the matter distribution come from n-body simulations. But they are very demanding.

Outline

Introduction

Standard perturbation theory (SPT)

Resummation

Conclusions

Observables

$$\rho(x,\tau)$$ matter density

$$\delta(x,\tau) = \rho(x,\tau)/\bar{\rho}(\tau) - 1$$ matter density contrast

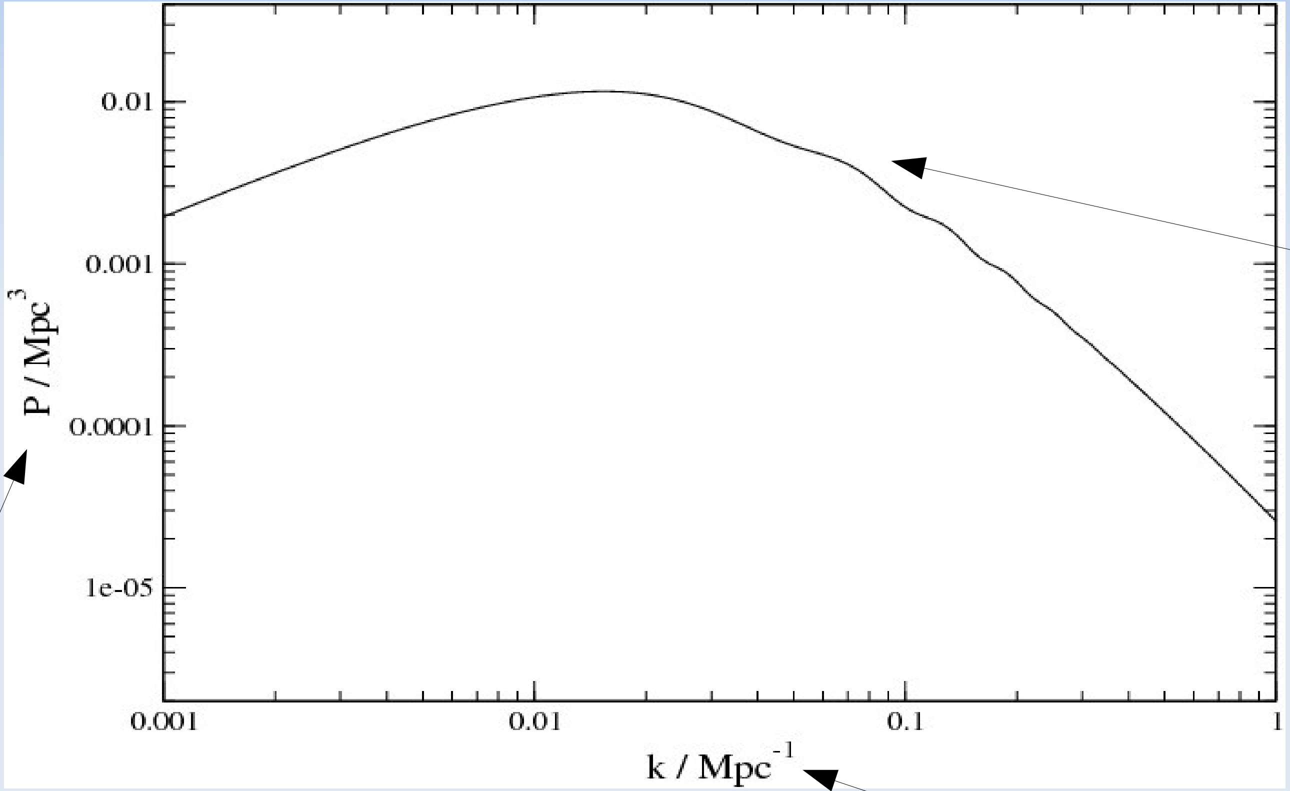
$$\xi(x-y,\tau) \equiv \langle \delta(x,\tau)\delta(y,\tau) \rangle$$ correlation function

$$P(k,\tau)\,\delta_K(k+k') \equiv \langle \delta(k,\tau)\delta(k',\tau) \rangle$$ equal-time power spectrum

There are many observables, but we are mainly interested in the equal-time power spectrum

The power spectrum at early times

The initial dark matter powe spectrum is predicted from CMB physics.



Z = 100

Baryonic acoustic oscillations (BAO)

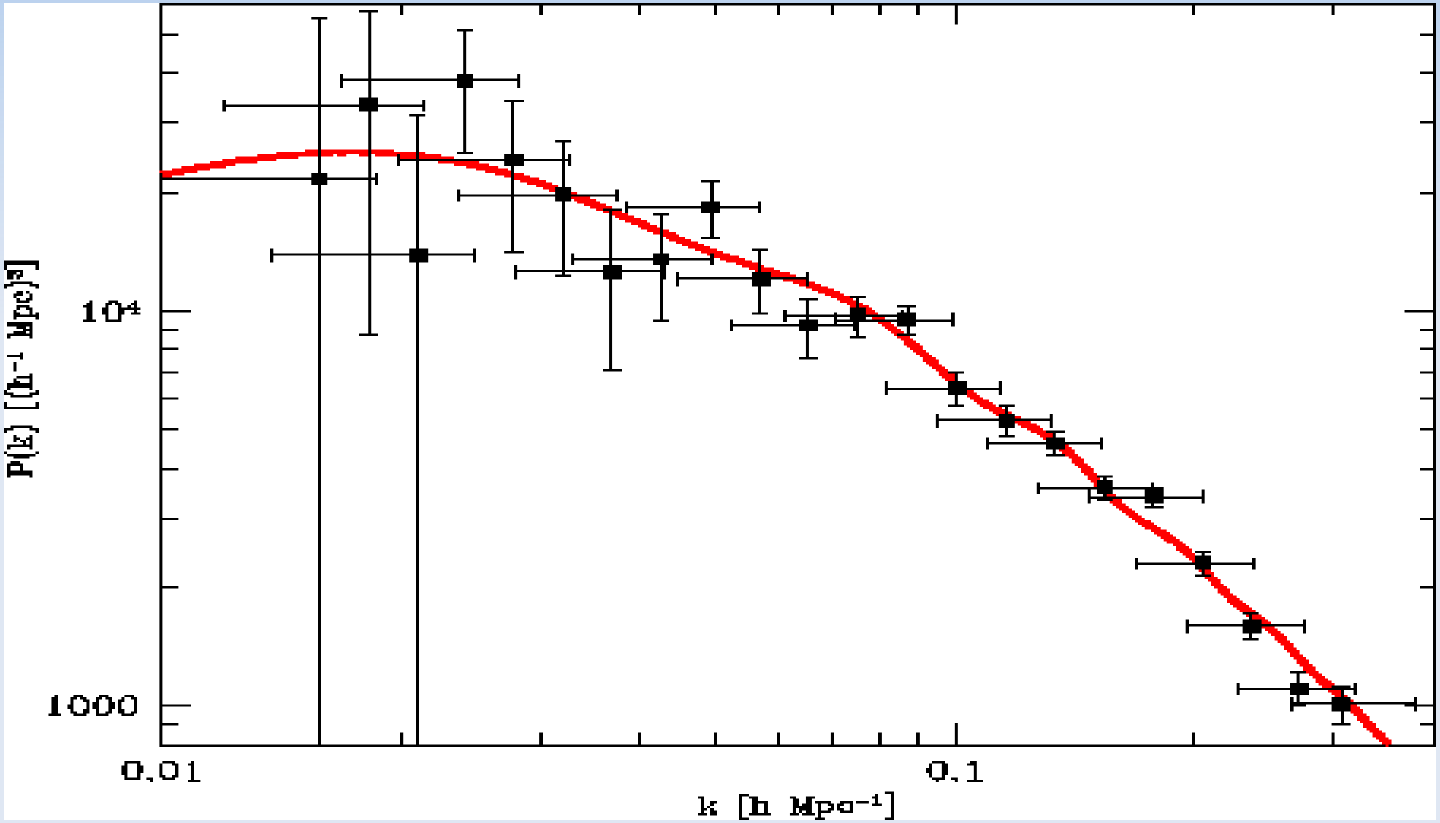
scales roughly as $a(\tau)^2 \sim (1+z)^{-2}$

comoving momentum

Initial conditions are uniquely specified by the power spectrum if the fluctuations are Gaussian.

The power spectrum measured today

Sloan Digital Sky Survey (2006)



This measurement should improve tremendously with Euclid and other LSS measurements. The BAO can then be used to probe the late expansion history of the Universe.

Outline

Standard perturbation theory (SPT)

Feynman rules of large scale structure

Dark matter as a fluid

Starting point are the hydrodynamic fluid equations in an expanding universe

continuity:

$$\frac{\partial \delta}{\partial \tau} + \nabla \cdot (1+\delta)\vec{v} = 0$$

Euler:

$$\frac{\partial \vec{v}}{\partial \tau} + H\vec{v} + \vec{v} \cdot \nabla \vec{v} = -\nabla \Phi$$

$\vec{v}$ fluid velocity

H Hubble parameter

Φ grav. potential

$\delta = \rho/\bar{\rho} - 1$ density contrast

in combination with the Poisson equation

$$\nabla^2 \Phi = 4\pi G a^3 \bar{\rho}\, \delta(x,\tau)$$

(assumptions: matter is collisionless, pressureless, single-streaming)

Linear solutions: growth

The linearized system can be phrased as

$$\frac{\partial \Psi}{\partial \eta} + \Omega \Psi = 0$$

with

$$\Psi = \begin{pmatrix} \delta \\ \frac{\nabla \vec{v}}{H} \end{pmatrix} \qquad \eta = \log a(\tau)$$

where (for Einstein-de Sitter)

$$\Omega = \begin{pmatrix} 0 & -1 \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

Newton force → $-\frac{3}{2}$

expansion → $\frac{1}{2}$

Linear solutions: growth

$$\Psi^L(k,\eta) \equiv g(\eta,\eta_0)\Psi(k,\eta_0)$$

The corresponding Green's function is

$$g(\eta,\eta_0) = \frac{e^{(\eta-\eta_0)}}{5}\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} + \frac{e^{-3(\eta-\eta_0)/2}}{5}\begin{pmatrix} 2 & -2 \\ -3 & 3 \end{pmatrix}$$

For the power spectrum

$$P_{ab}(k,\tau)\,\delta_K(k+k') \equiv \langle \Psi_a(k,\tau)\Psi_b(k',\tau)\rangle$$

growing mode →

$$P_{L,ab}(\tau,k) \simeq a(\tau)^2\,P_0(k)\,\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Fluid equations in Fourier space

Assuming no vorticity ($\nabla \times \vec{v} = 0$) the equations read in Fourier space

$$\partial_\eta \Psi_a(k,\eta) + \Omega_{ab}\Psi_b(k,\eta) = \int dq\, \gamma_{abc}(q, k-q)\, \Psi_b(q,\eta)\Psi_c(k-q,\eta)\,,$$

where

$$\gamma_{121} = \alpha(k_1,k_2)/2, \quad \gamma_{112} = \alpha(k_2,k_1)/2, \quad \gamma_{222} = \beta(k_1,k_2),$$

and

$$\alpha(k_1,k_2) \equiv \frac{(k_1+k_2)\cdot k_1}{k_1^2}$$

$$\beta(k_1,k_2) \equiv \frac{(k_1+k_2)^2 k_1\cdot k_2}{2k_1^2 k_2^2}$$

SPT in integrated form

The solution to this equation can formally be written

$$\Psi_a(\eta) = g_{ab}(\eta, \eta_0)\Psi_b(\eta_0)$$

$$+ \int_{\eta_0}^{\eta} d\bar{\eta} \ \int dq \ g_{ab}(\eta, \bar{\eta}) \ \gamma_{bcd} \ \Psi_c(\bar{\eta})\Psi_d(\bar{\eta})$$

SPT in integrated form

The solution to this equation can formally be written

$$\Psi_a(\eta) = g_{ab}(\eta, \eta_0)\Psi_b(\eta_0)$$

$$+ \int_{\eta_0}^{\eta} d\bar{\eta} \int dq \; g_{ab}(\eta, \bar{\eta}) \; \gamma_{bcd} \; \Psi_c(\bar{\eta})\Psi_d(\bar{\eta})$$

And iteratively solved

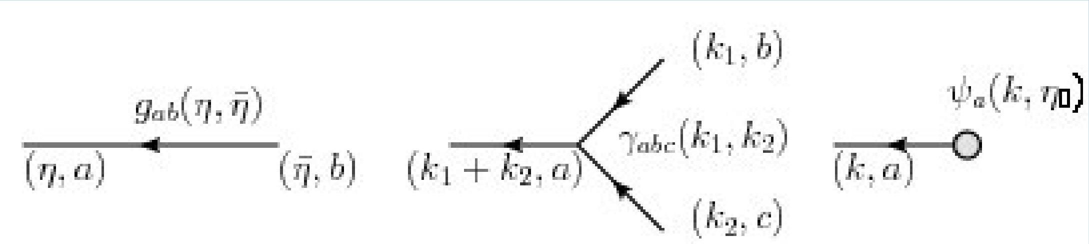
$$\Psi_a(\eta) = \Psi_a^L(\eta_0)$$

$$+ \int_{\eta_0}^{\eta} d\bar{\eta} \int dq \; g_{ab}(\eta, \bar{\eta}) \; \gamma_{bcd} \; \Psi_c^L(\bar{\eta})\Psi_d^L(\bar{\eta})$$

$$+O(\Psi^L)^3$$

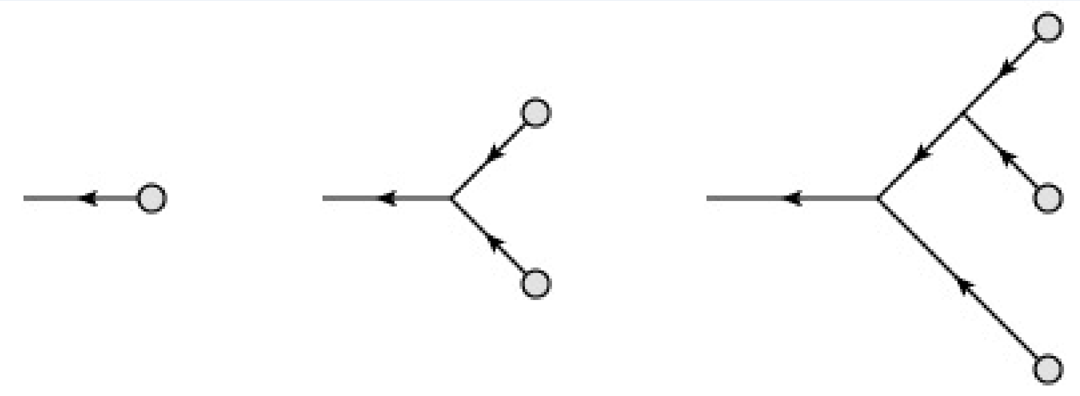
Diagrammatic representation

Linear propagator, interaction and initial density can be represented as



such that the expansion of the density perturbation at late times reads

$$\Psi_a(k,\eta) =$$



The arrows denote the flow of time due to the classical causal structure

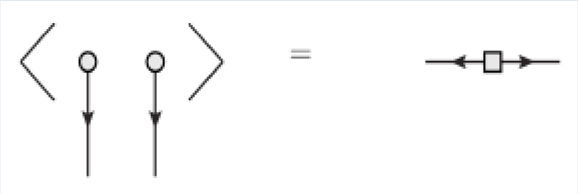
Wick's theorem and initial conditions

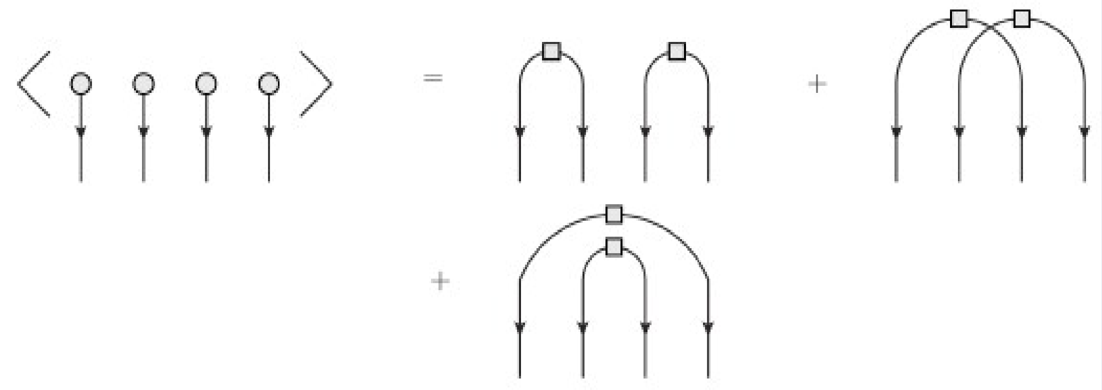
Gaussian fluctuations obey certain contraction rules that are analogeous to Wick's theorem in QFT

$$P_0(k)\,\delta_K(k+k') \equiv \langle \delta_0(k)\delta_0(k' \rangle$$

$$\begin{aligned}\langle \delta_0(k_1)\delta_0(k_2)\delta_0(k_3)\delta_0(k_4)\rangle = {} & \langle \delta_0(k_1)\delta_0(k_2)\rangle \, \langle \delta_0(k_3)\delta_0(k_4)\rangle \\ & + \langle \delta_0(k_1)\delta_0(k_3)\rangle \, \langle \delta_0(k_2)\delta_0(k_4)\rangle \\ & + \langle \delta_0(k_1)\delta_0(k_4)\rangle \, \langle \delta_0(k_2)\delta_0(k_3)\rangle\end{aligned}$$

Or diagrammatically



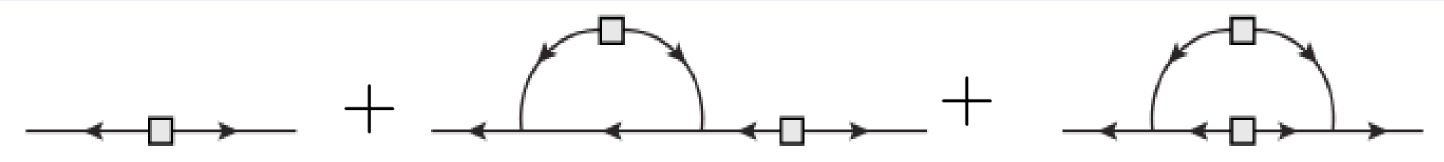


Expansion of the power spectrum

This gives for the power spectrum the expansion

$$P(k,\tau)\,\delta(k+k') \equiv \langle \delta(k,\tau)\delta(k',\tau)\rangle$$

$$P(k,\eta) =$$

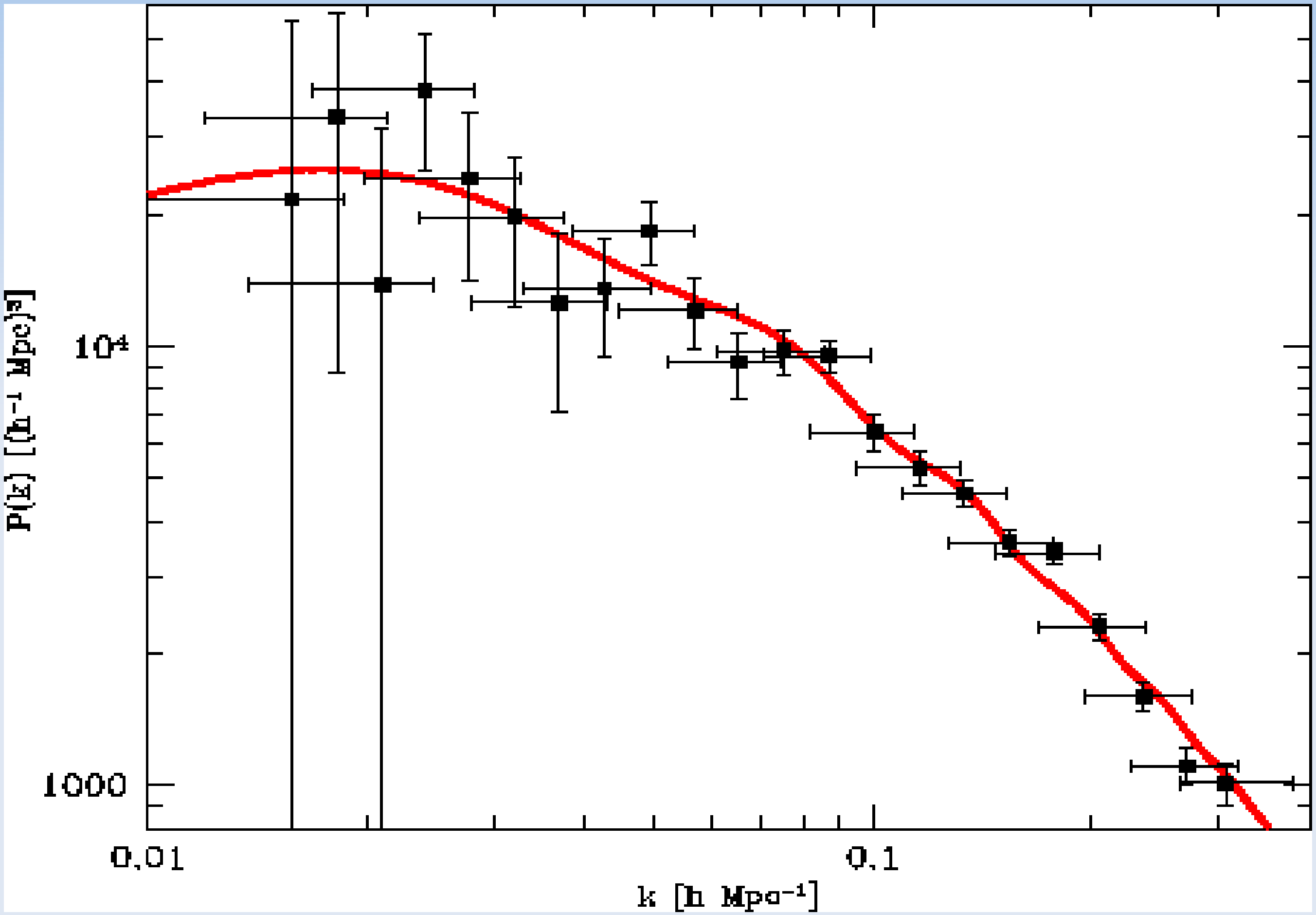


$$+ \quad O(P_L^3)$$

In this expansion the number of linear power spectra is the number of loops plus one. The linear result has no loops.

Is this a good expansion scheme?

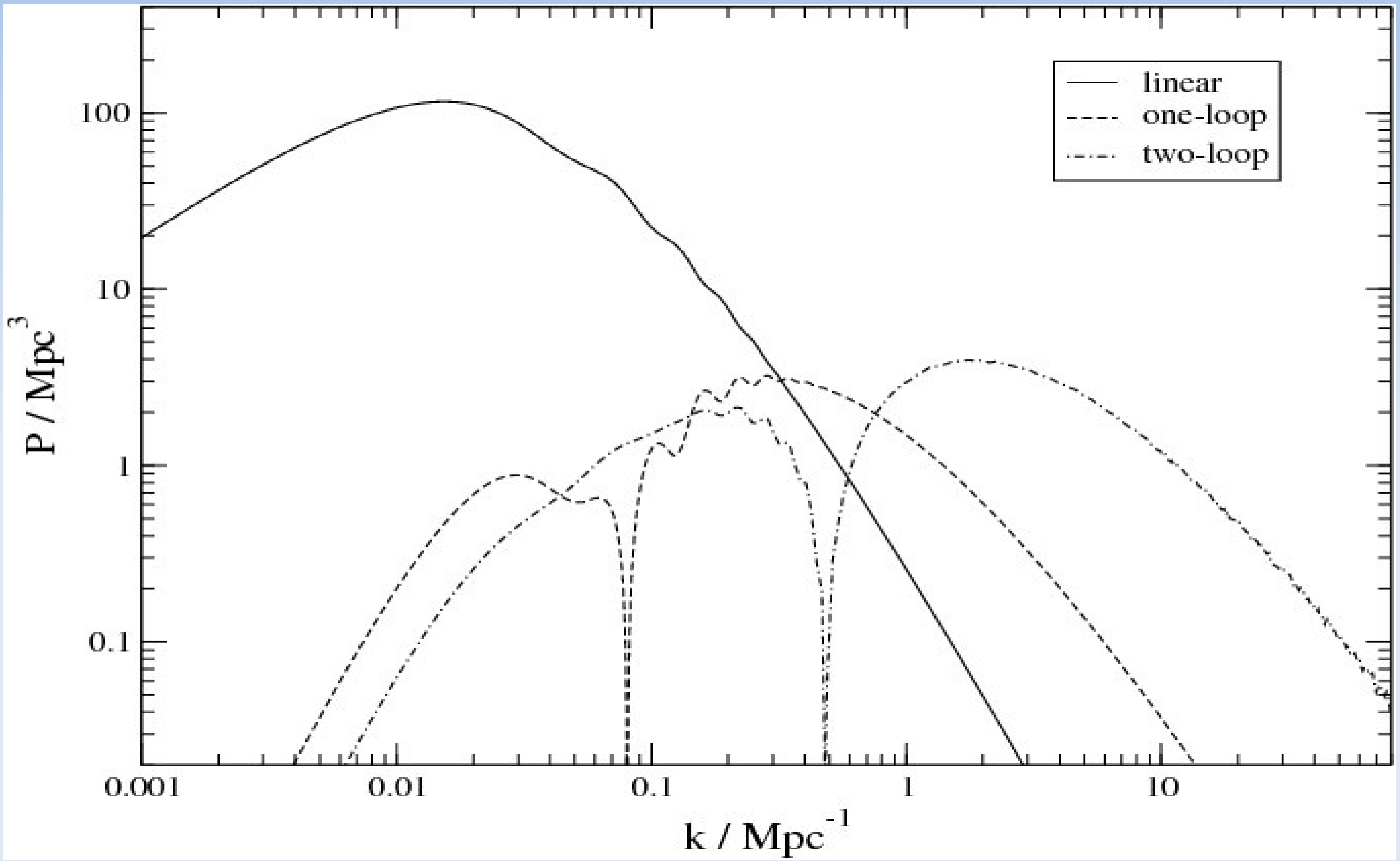
Expansion parameter in SPT?



$$\epsilon = P_L(k_0)(k_0/2\pi)^3 \simeq 10^{-3} \times (a/a_*)^2$$

The convergence is better at larger redshift (smaller scale factor a)

Power spectrum in SPT

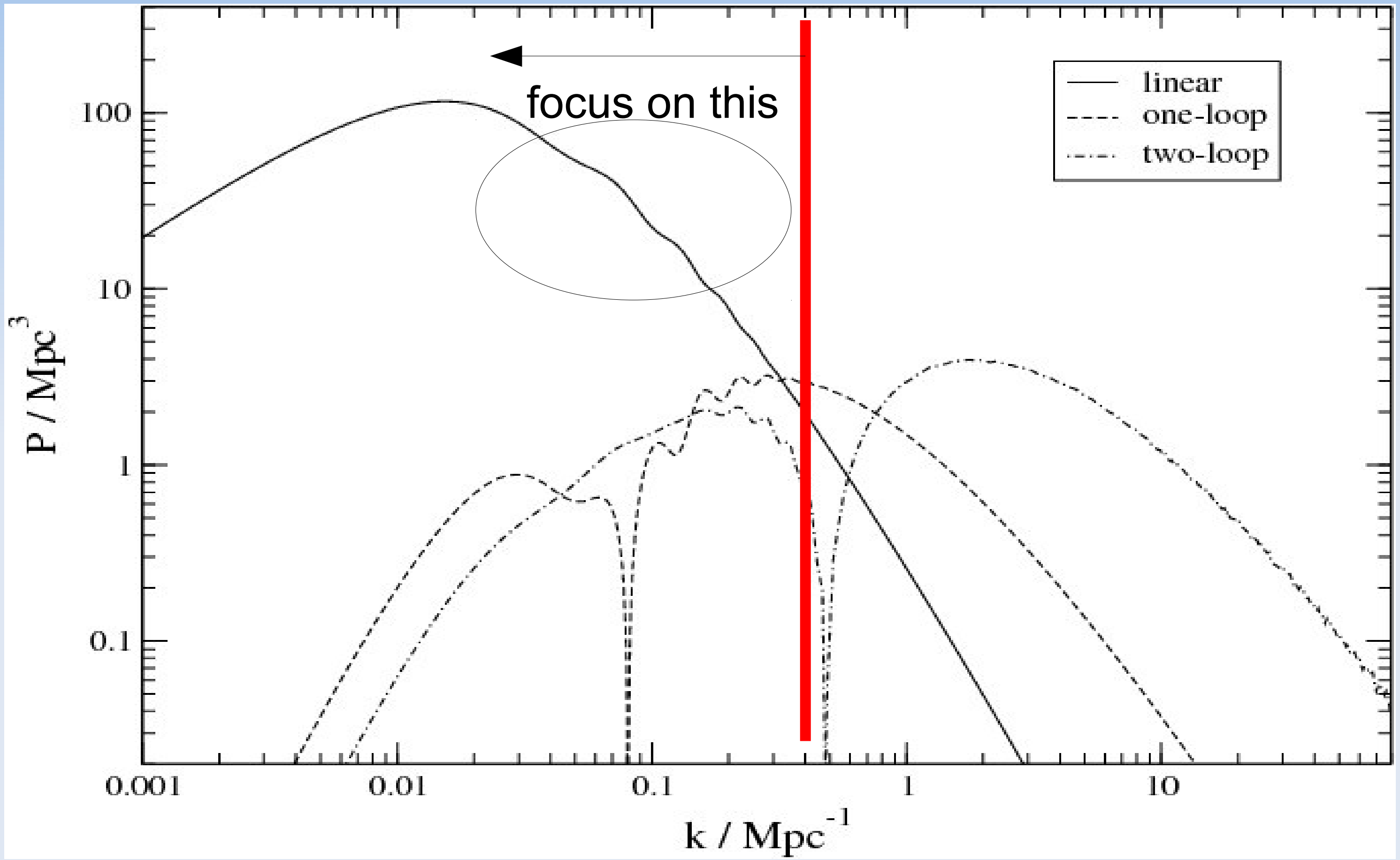


Z = 0

The convergence is for large momentum very bad for z = 0 but improves for larger redshift, since the n-loop scales with

$a(\tau)^{2n}$ relative to the linear result.

Power spectrum in SPT



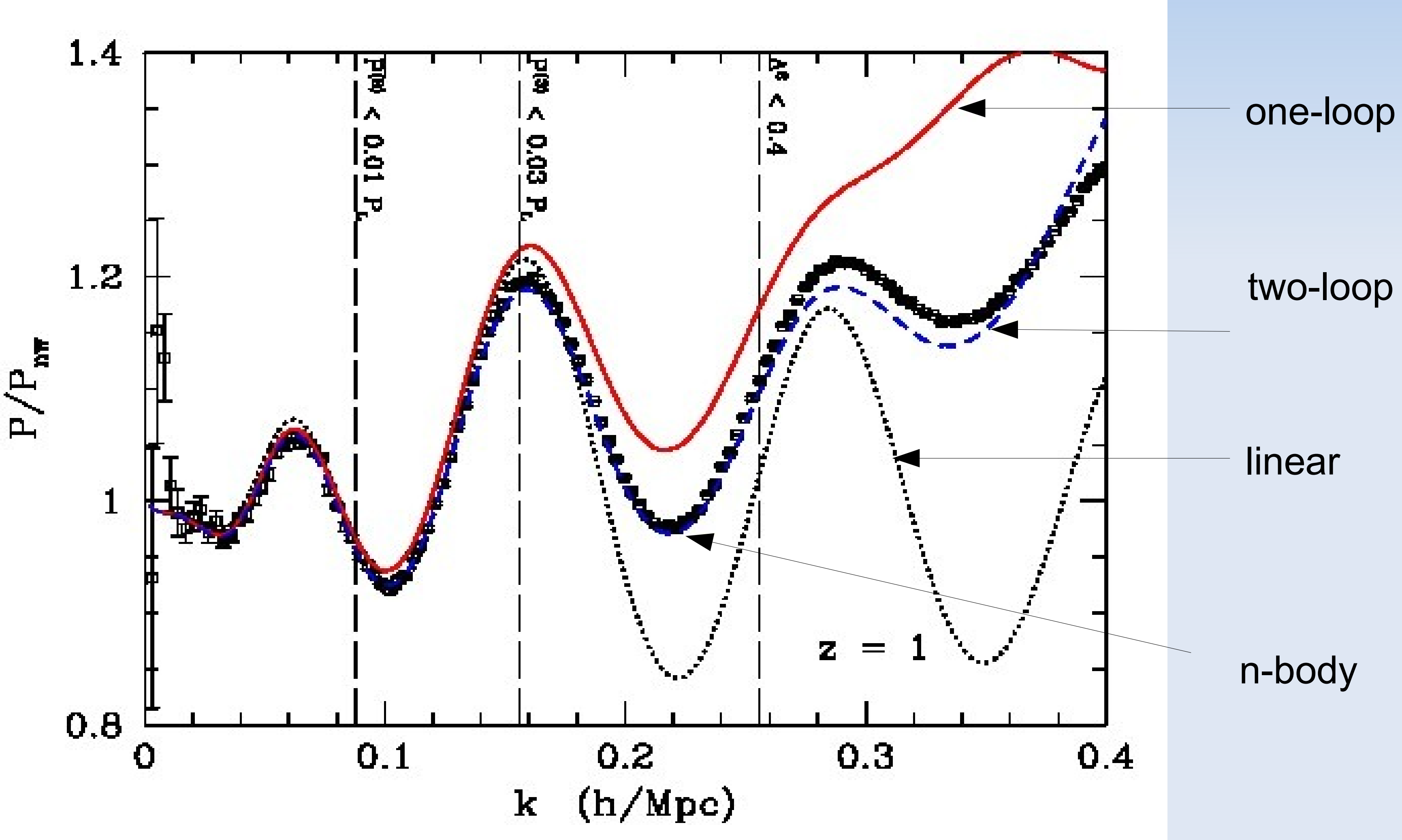
Z = 0

The convergence is for large momentum very bad for z=0 but improves for larger redshift, since the n-loop scales with

$a(\tau)^{2n}$ relative to the linear result.

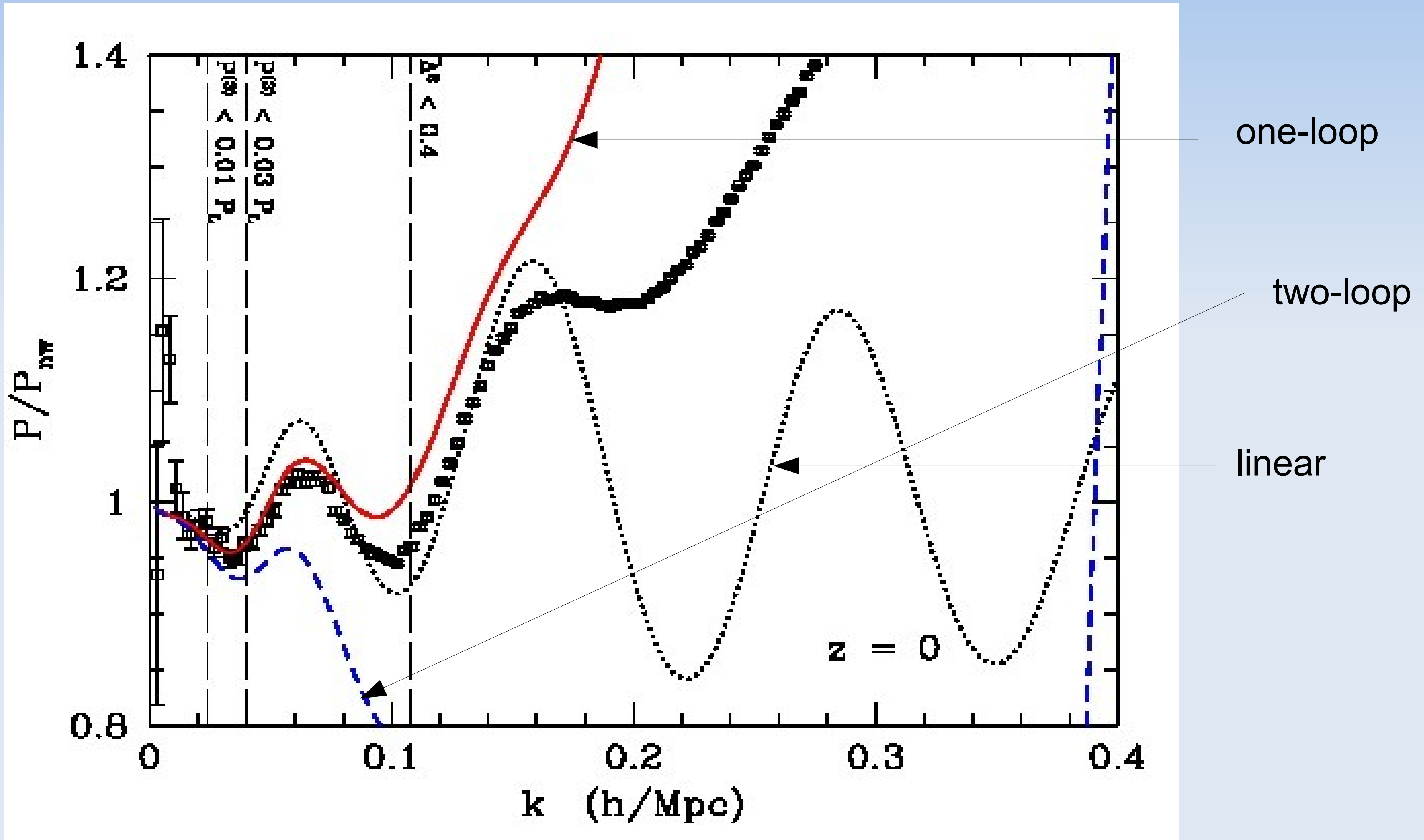
Deviations z=1

Results are often normalized to the smooth Eisenstein-Hu spectrum (without BAO)



[Carlson, White & Padmanabhan '09]

Deviations z=0



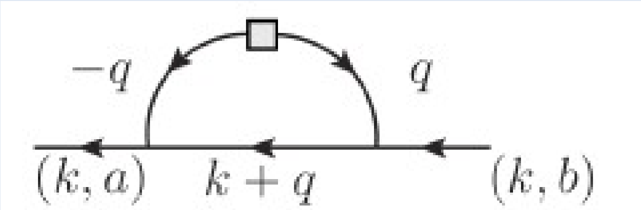
[Carlson, White & Padmanabhan '09]

A problem

For a large hierarchy in the momenta (q<<k), the vertex behaves like

$$\gamma_{abc}(q,k) \to \frac{q\cdot k}{q^2}\delta_{ab}\delta_{c2}$$

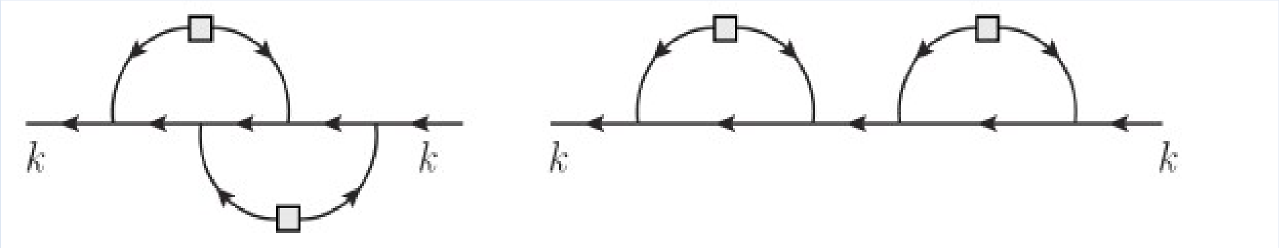
what leads for soft loops to



$$\simeq -\int d^3q \left(\frac{k\cdot q}{q^2}\right)^2 P(q) \equiv -\frac{1}{2}k^2\sigma_d^2$$

Even though the variance is typically small,
this leads potentially to bad convergence for large external momenta

$$\sigma_d^2 k_0^2 \simeq 10^{-2}$$



$$\simeq k^4\sigma_d^4$$

Outline

Resummation

pictures only

Many resummation schemes

- Renormalized perturbation theory
  [Crocce and Scoccimarro '05 + '07]
- Simple renormalization group PT [McDonald '06]
- Large-N path-integral methods [Valageas '06]
- Renormalization group techniques
  [Matarrese and Pietroni '07]
- Closure theory [Taruya and Hiramatsu '07]
- Lagrangian resummation [Matsubara '07]
- Time-RG theory [Pietroni '08]
- Multi-point propagators [Bernardeau et al '08]
- Eikonal approximation [Bernardeau et al '11]

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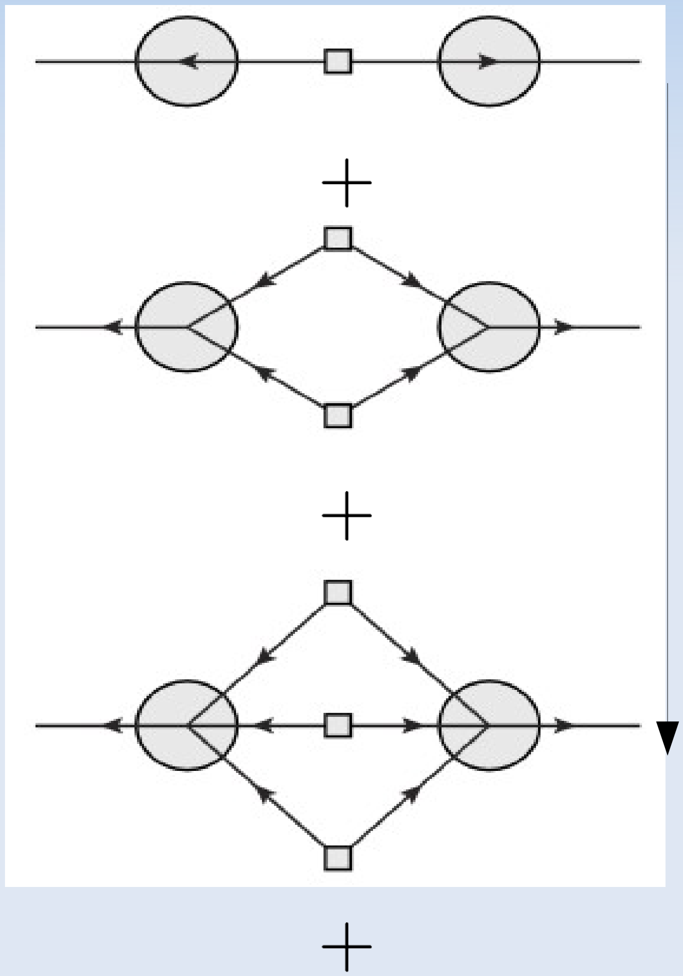
Renormalized PT

In RPT different contributions from SPT are collected in different order (Gamma – expansion)

$$P(k,\eta) =$$

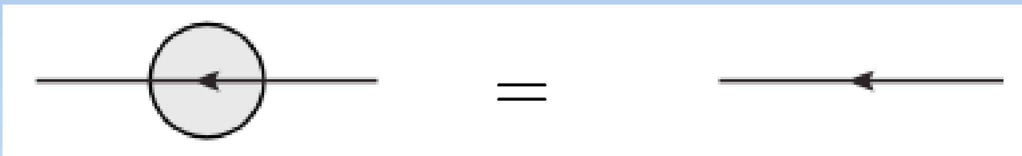


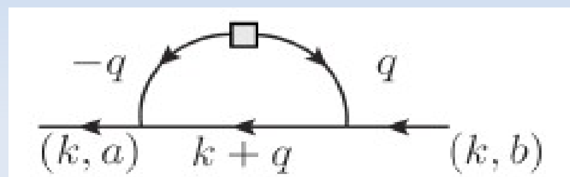
truncate

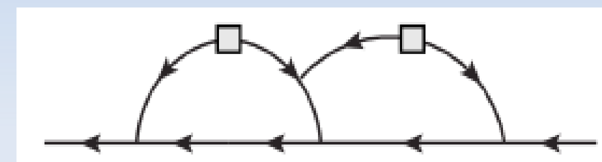
All contributions positive which avoids cancellations!

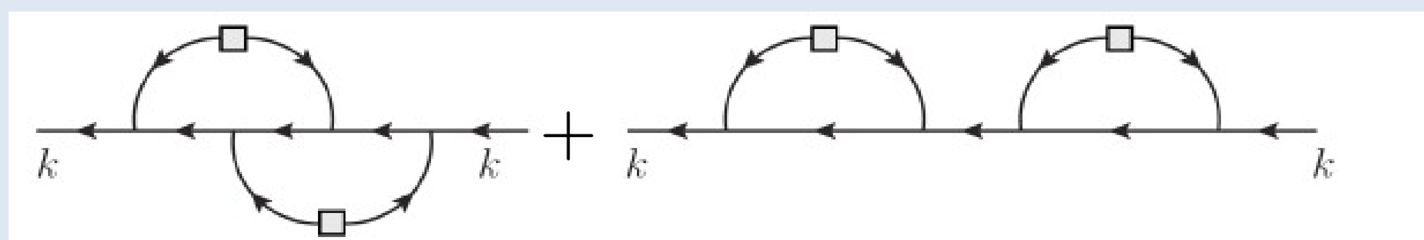
[Crocce and Scoccimarro '05 + '07]

Full propagator



\+ 

\+ 

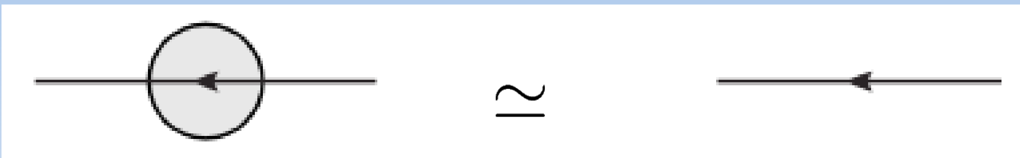
\+ 

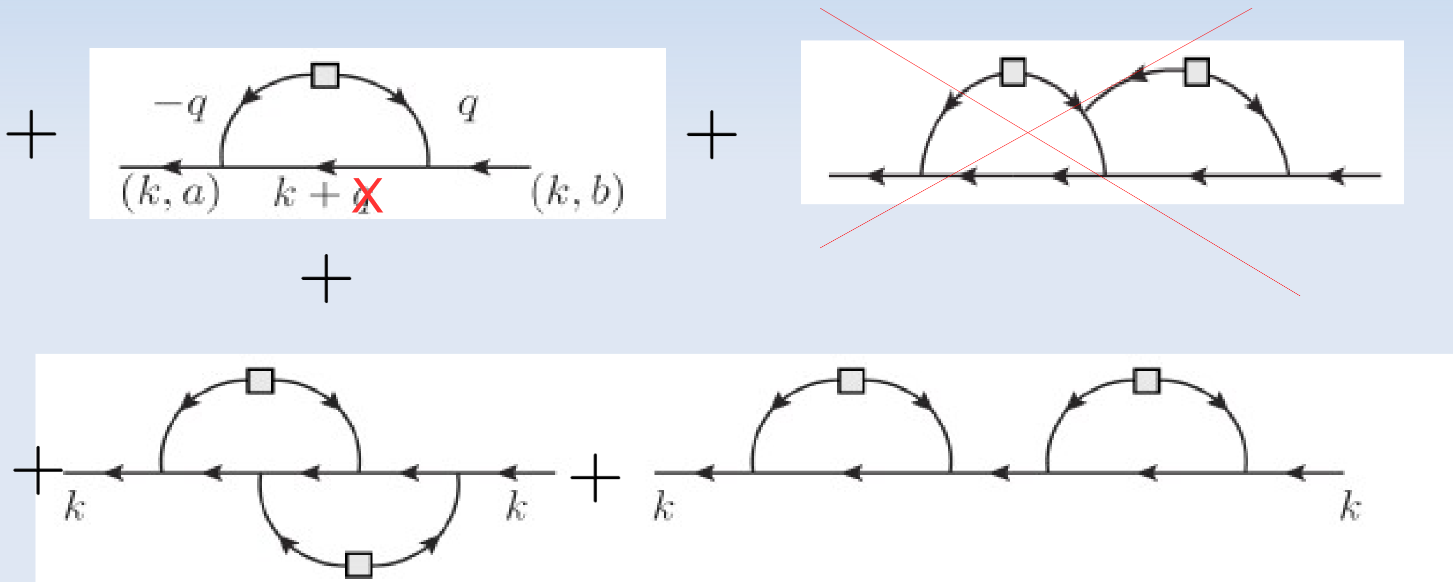
And formally

$$G(k, \eta, \eta_0)\, \delta(k - k') = \left\langle \frac{d\Psi(k, \eta)}{d\Psi(k', \eta_0)} \right\rangle$$

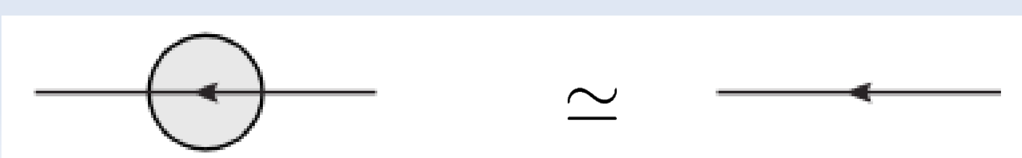
Leading soft corrections

[Crocce and Scoccimarro '05 + '07]



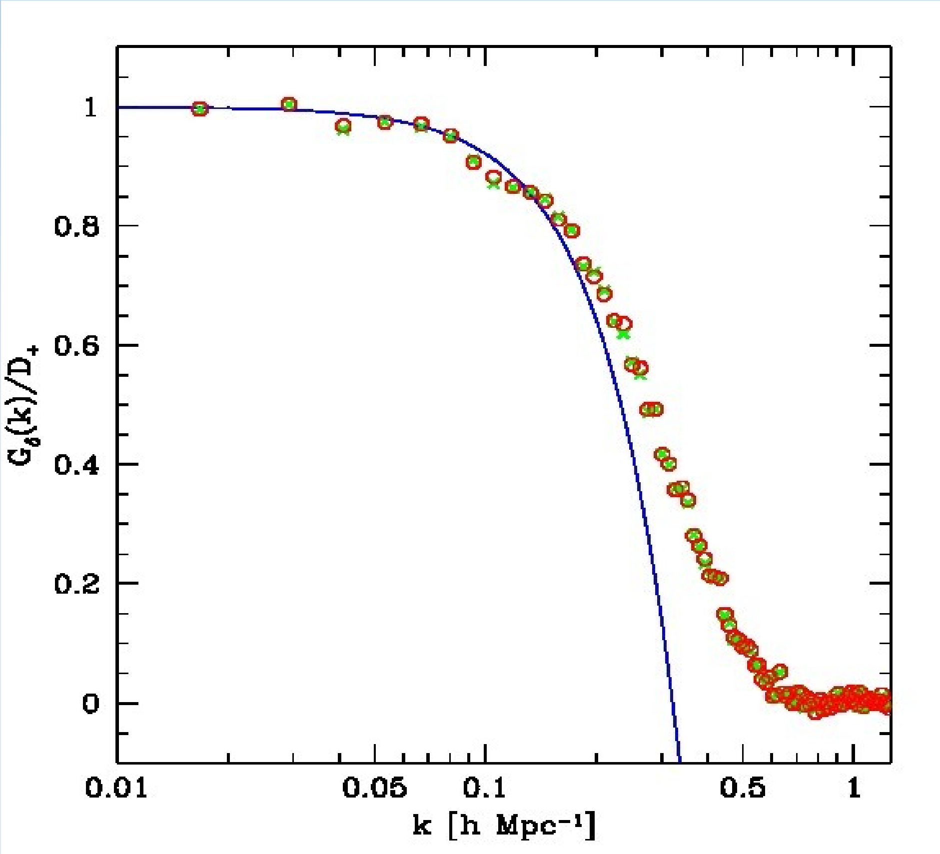


In the limit of hard external momentum, all higher loop contributions can be resummed leading to

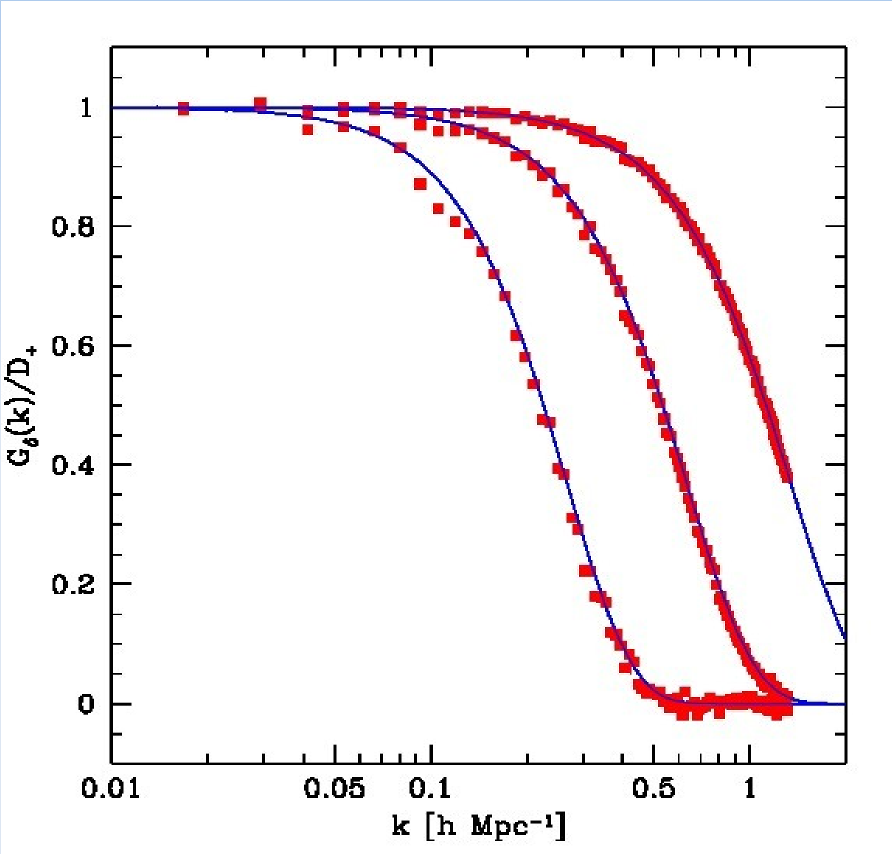


$$\times \quad e^{-\frac{1}{2}k^2\sigma_d^2}$$

Full propagator results



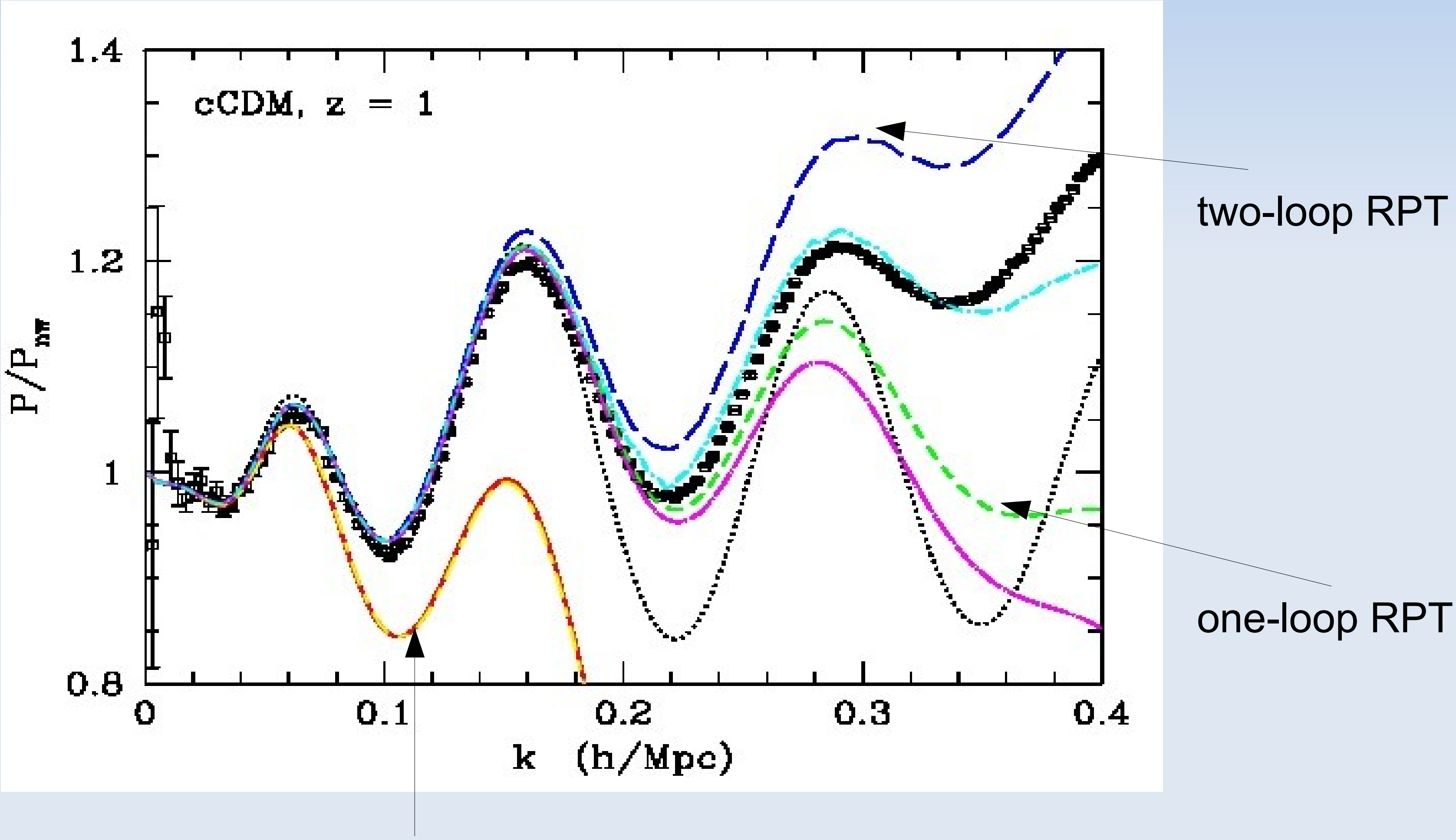
SPT (1-loop)



RPT

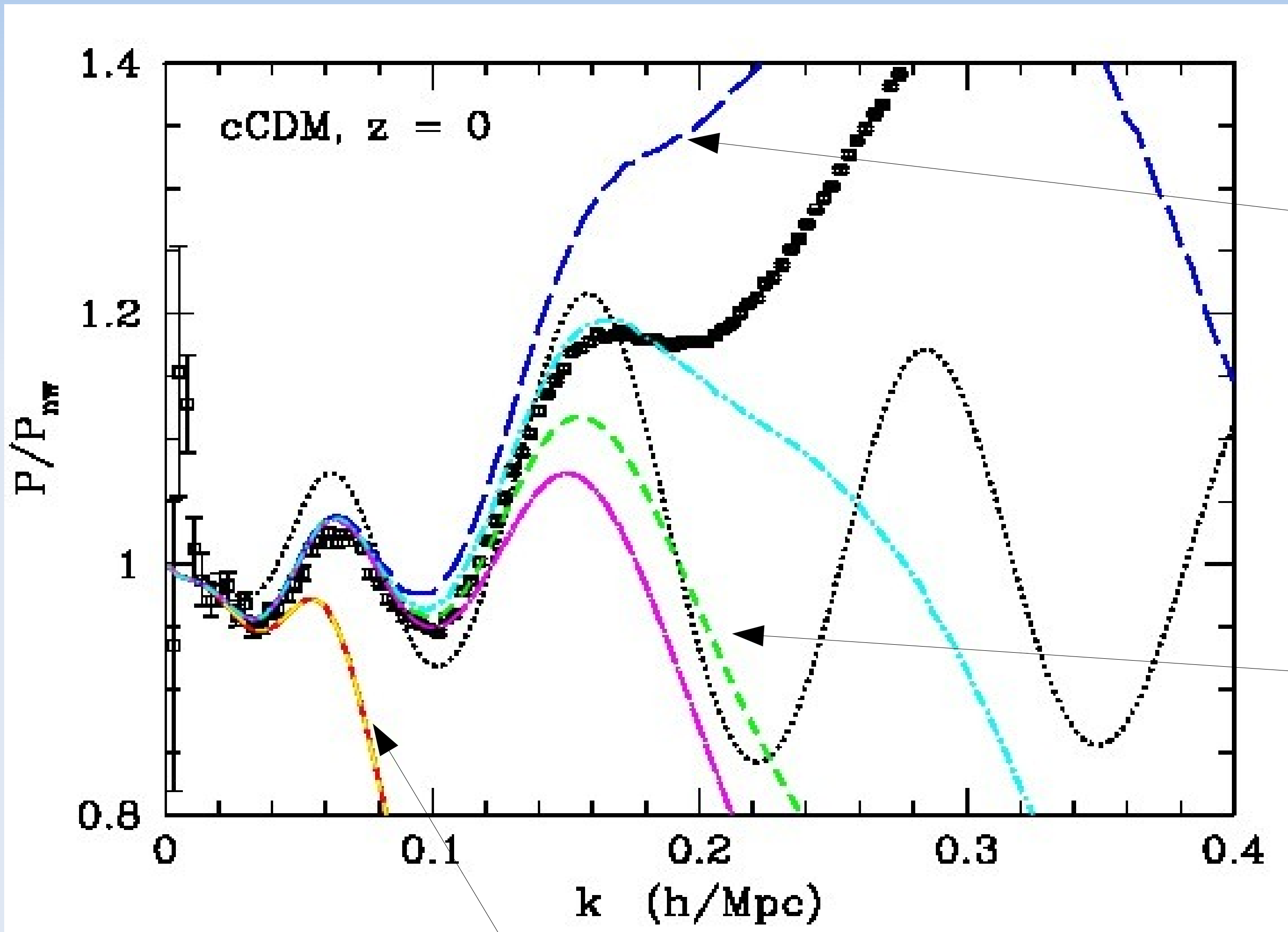
RPT power spectrum

$$P_{RPT,tree} = P_L\, e^{-k^2\sigma_d^2}$$



[Carlson, White & Padmanabhan '09]

RPT power spectrum



two-loop RPT

one-loop RPT

tree RPT

[Carlson, White & Padmanabhan '09]

RPT conclusions

- RPT reproduces very well the full propagator, which is a correlator at unequal times. In this case soft corrections seem important.
- For the power spectrum, the improvement by RPT is debatable
- No a priori justification for
- power spectrum is by construction positive – all contributions in the Gamma-expansion positive

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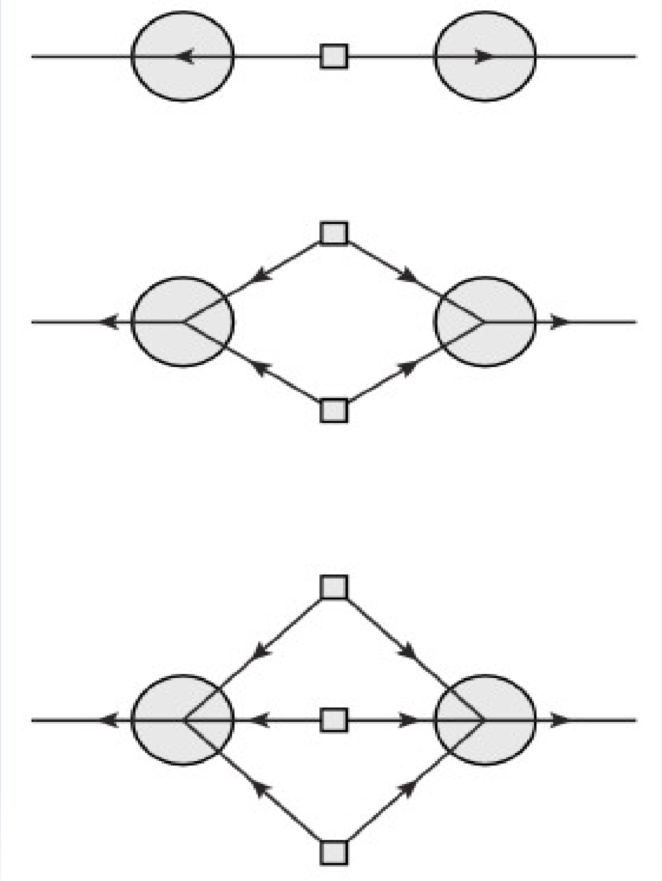

Eikonal approximation

The eikonal approximation implements the soft limit on the level of the equations of motion

$$\partial_\eta \Psi_a(k,\eta) + \Omega_{ab}\Psi_b(k,\eta) = \int \gamma_{abc}(k_1,k_2)\Psi_b(k_1,\eta)\Psi_c(k_2,\eta)\,,$$

[Bernardeau et al '11]

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leads to

$$q \ll k$$

$$\int dq\, \frac{k\cdot q}{q^2} \Psi_a(k,\eta)\Psi_2(q,\eta)$$

$$\Psi_a(k,\eta) = g_{ab}(\eta,\eta_0)\,\xi(\eta,\eta_0,k)\,\Psi_b(k,\eta_0)$$

$$\xi(\eta,\eta_0,k) = \exp\left[\int d\bar\eta \int dq\, \frac{k\cdot q}{q^2}\Psi_2(q,\bar\eta)\right]$$

[Bernardeau et al '11]

Cumulants

These exponential expressions can be exactly evaluated assuming that the soft modes are independent from the hard ones.

For example one finds

$$\left\langle e^{X} \right\rangle = e^{\sum \frac{1}{n!} c_n}$$

This involves the cumulants.

$$c_n = \langle X^n \rangle_c$$

In diagrammatic language these are the connected diagramms. This technique is quite similar to the effective action used in QFT.

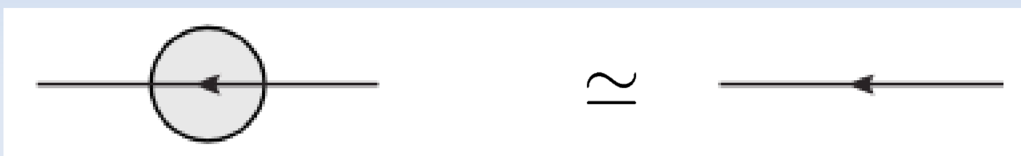
[Bernardeau et al '11]

Eikonal results

[Bernardeau et al '11]

In the eikonal approximation, one reproduces the (in leading order) the propagator including soft corrections

$$G(k,\eta,\eta_0) = g(\eta,\eta_0)\ \langle \xi(k,\eta,\eta_0) \rangle$$

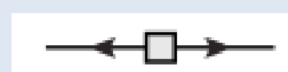


$$\times \quad e^{-\frac{1}{2}k^2\sigma_d^2}$$

However, the effect of the soft modes cancels in the power spectrum and one recovers the linear result

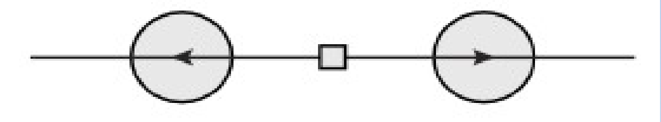
$$P(k,\eta) = P_L(k,\eta)\ \langle \xi(k)\xi(-k) \rangle$$

$$P(k,\eta) =$$

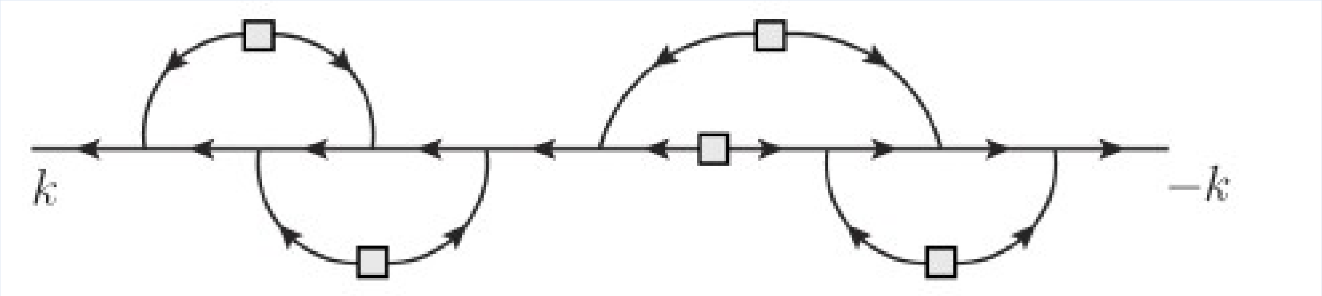


How does this fit with RPT?
Is the eikonal approximation missing some dynamics in the power spectrum?

More resummations

$P_{RPT} \simeq$ 

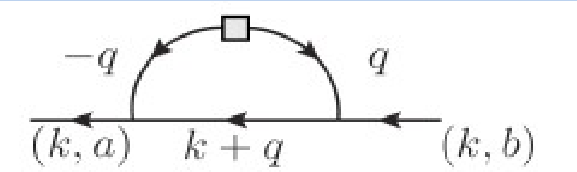
In fact RPT is missing something



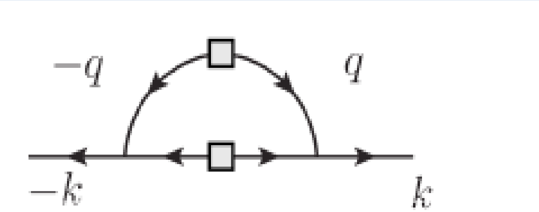
These diagrams can also be resummed if all loops are soft. In RPT they appear in the high multi-point contributions.

[Blas, Garny & Konstandin '13]

Enhancement through soft modes



$$\simeq -\frac{1}{2}\int d^3q \left(\frac{k\cdot q}{q^2}\right)^2 P(q) \equiv -\frac{1}{2}k^2\sigma_d^2$$



$$\simeq \int d^3q \left(\frac{k\cdot q}{q^2}\right)^2 P(q) \equiv k^2\sigma_d^2$$

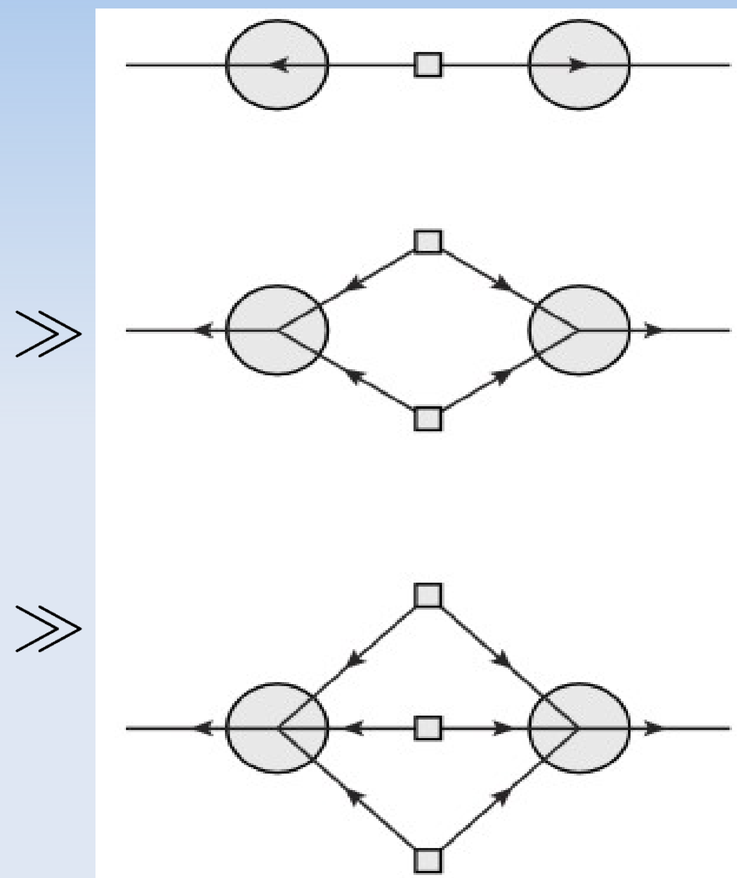
The relative factor 2 comes from the ordering, the sign from the hard momenta.

$$\sum_{m,n,l} \frac{1}{m!n!l!}(-\sigma_d^2 k^2/2)^{m+n}(\sigma_d^2 k^2)^l = 1$$

There is an intricate cancellation of soft effects at fixed loop order.

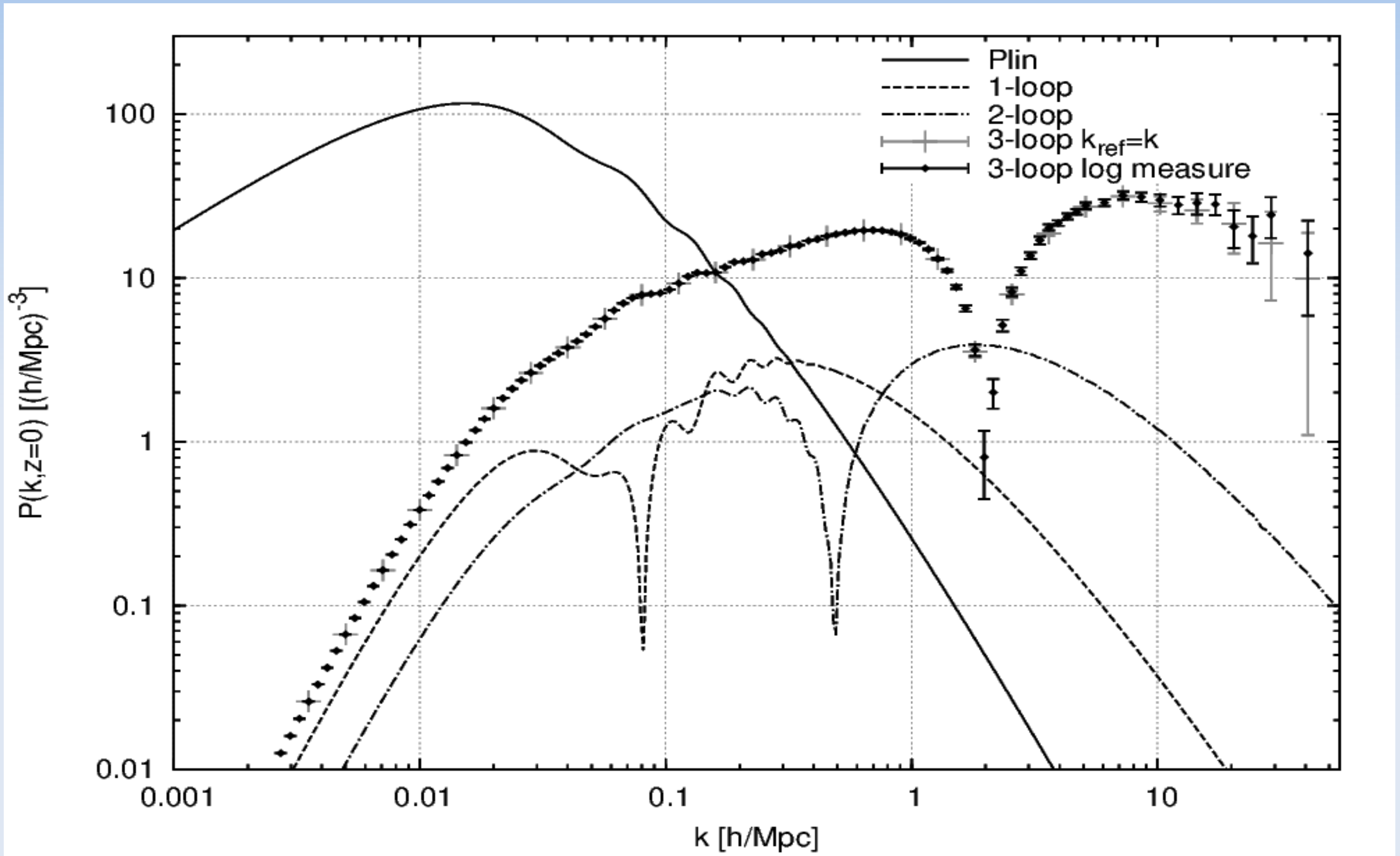
How does it fit with RPT?

- For large momenta, the contributions from higher multipoint correlations dominate.
- This cancellation does not only appear in the leading soft corrections. In fact, corrections from soft loops should not be relevant in equal time correlators (Galilean invariance – GR equivalence principle).
  [Peloso & Pietroni '13]
- The observed cancellation can be made explicit before the integrals are performed. This is important for the numerical stablilty.
  [Blas, Garny & Konstandin '13]
  [Senatore et al '13]



So does SPT converge after all?

Actually not ...



Z = 0

Three loop results in SPT indicate that PT does not even converge for small momenta at z=0.

But the result has some features of an asymptotic series.

[Blas, Garny & Konstandin '13]

Asymptotic behavior

For realistic initial conditions the asymptotic behavior of SPT can be inferred. For large k one finds terms of the form (l < n)

$$P_{n-loop} \ni \left([k\partial_k]^l P^L(k) \right) \; \sigma_l^{2n}(k)$$

And for small k

$$P_{n-loop}(k) \propto k^2 P^L(k) \int_0^\infty dq P^L(q) \, \sigma_l^{2n-2}(q)$$

Both expressions involve

$$\sigma_l^2(\Lambda, \eta) = \int^\Lambda d^3q \, P^L(q, \eta)$$

What diverges logarithmically and is larger than unity for z=0.

[Blas, Garny & Konstandin '13]

Pade approximant

The Pade approximant represents well complex functions with a branch cut or poles as a rational function

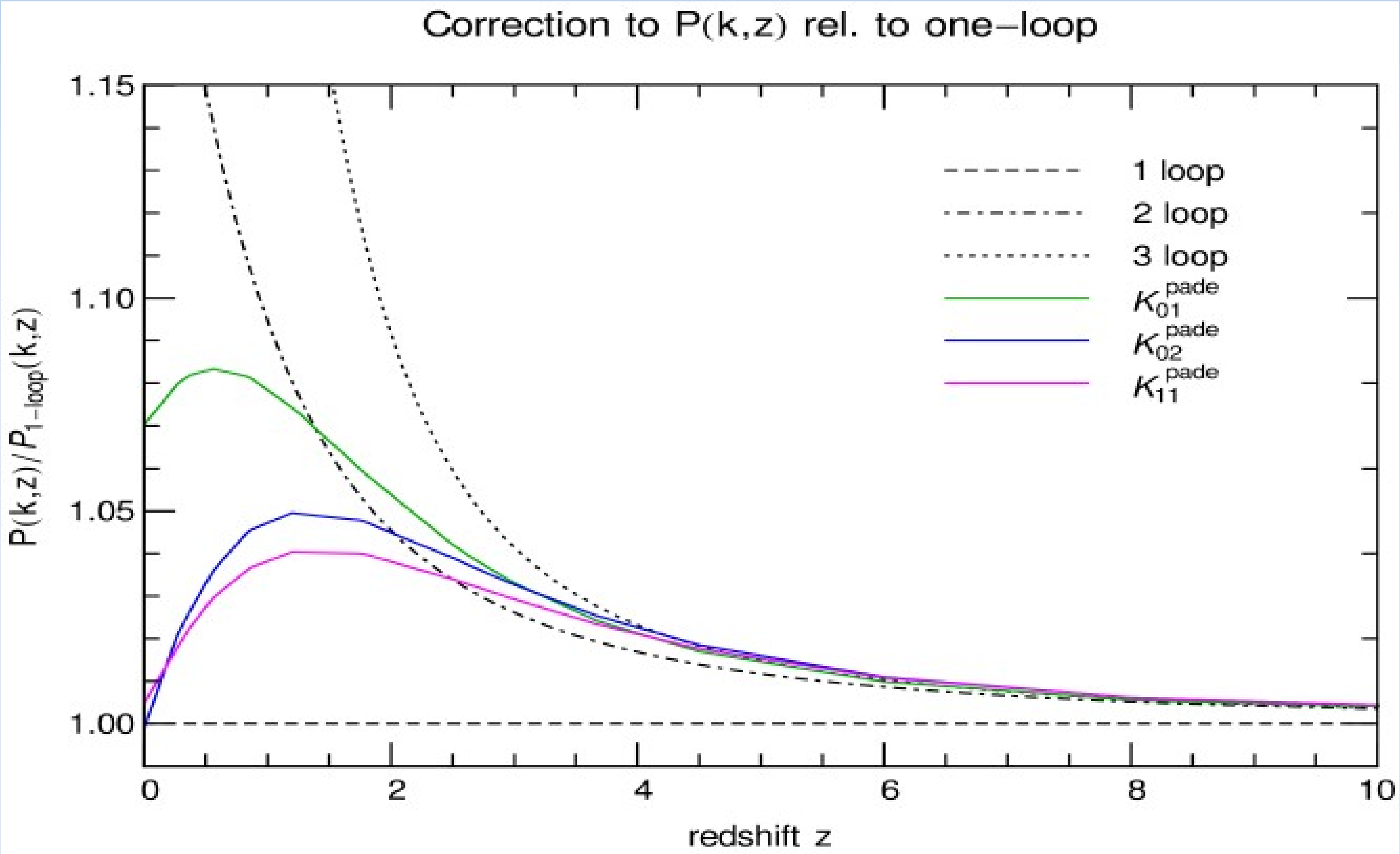
$$f(x) \simeq \frac{a_0 + a_1\, x + a_2 x^2 + \cdots}{1 + b_1\, x + b_2 x^2 + \cdots}$$

and can be matched to the Taylor expansion.

Here we use the Pade approximant of the integrand with $x = \sigma_l^2$

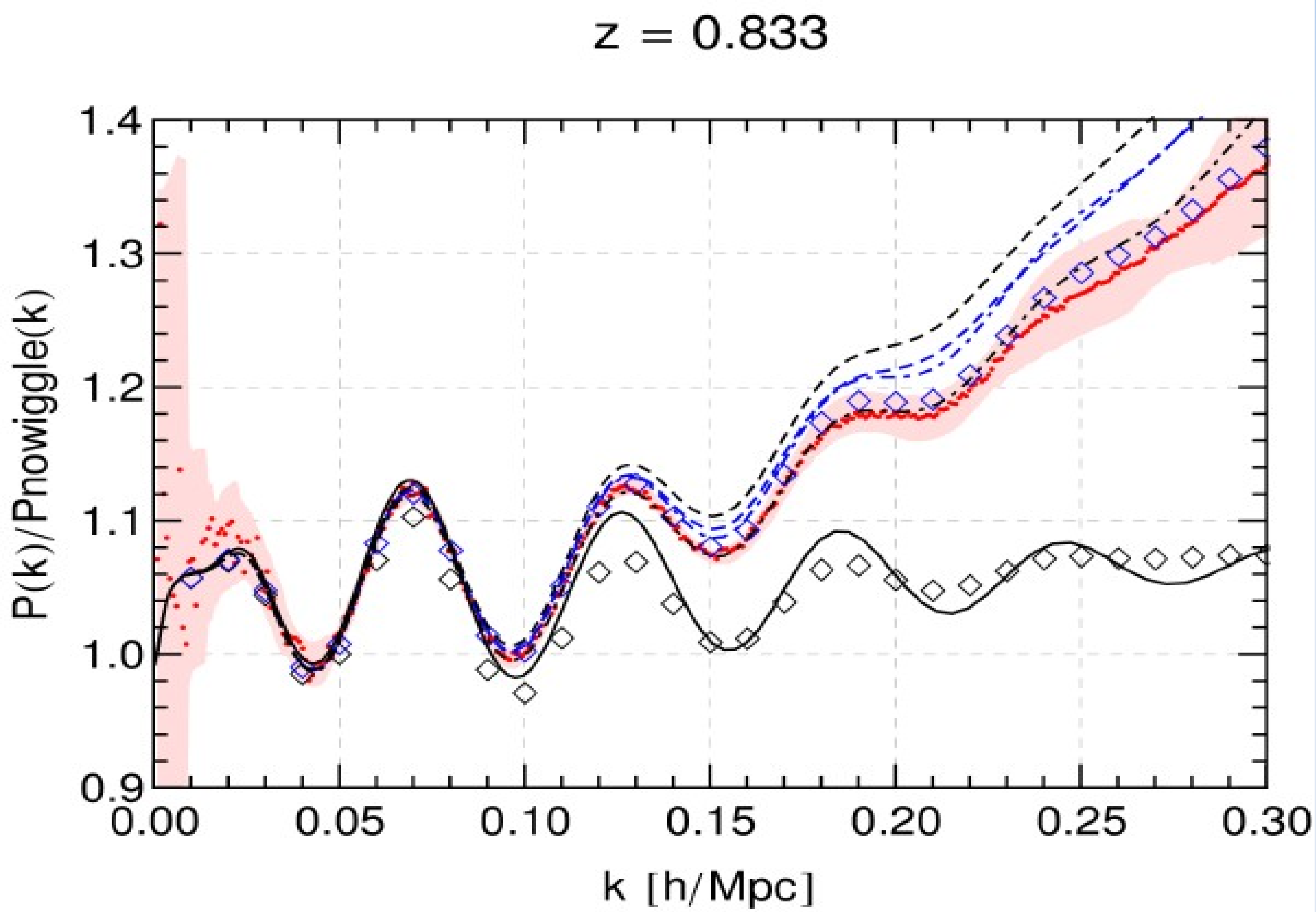
Resummed leading monomial

$$P(k,z) \simeq P_L(k) + c(z)\,k^2\,P_L(k)$$

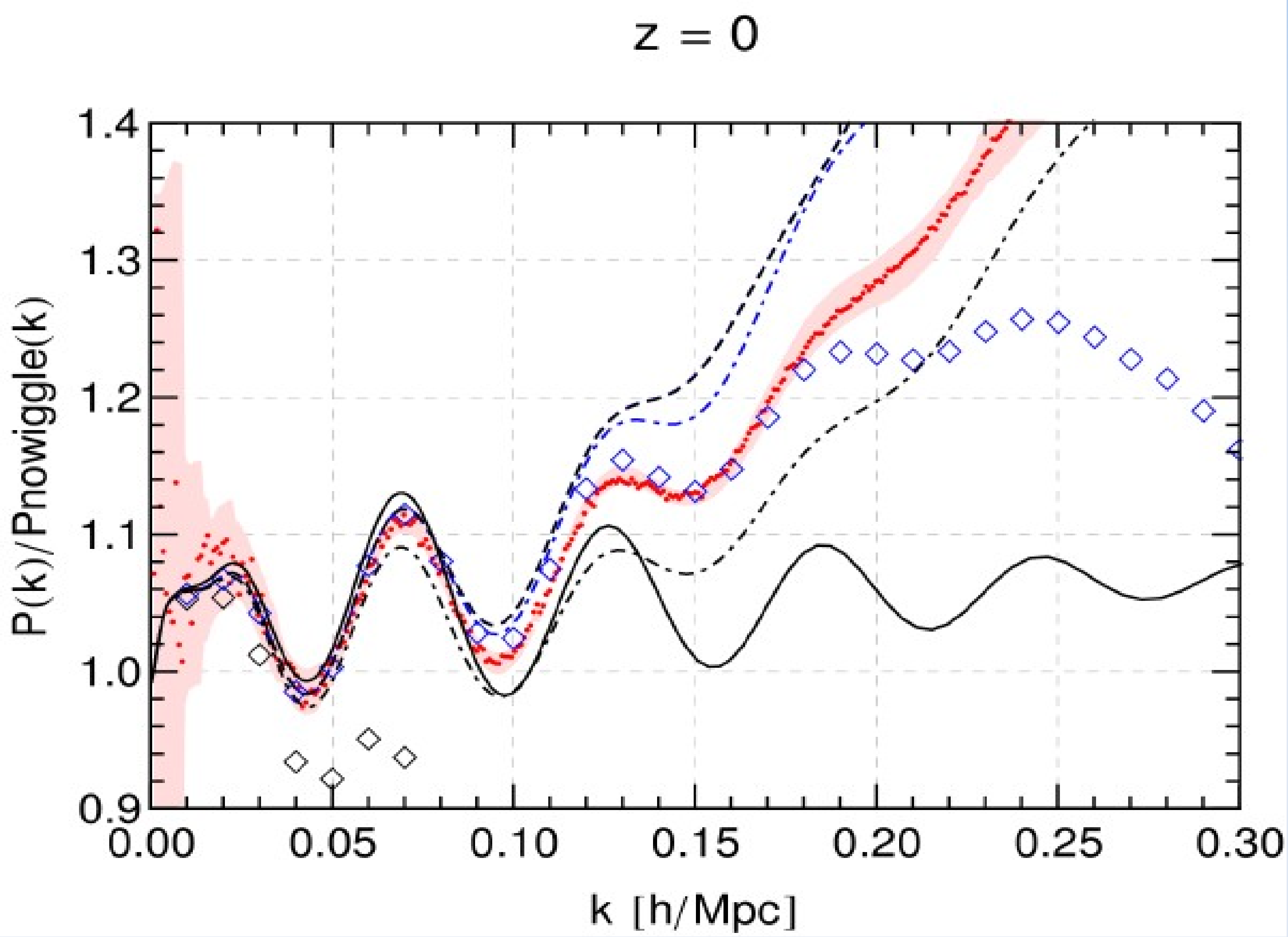


$$a(z) = \frac{1}{1+z}$$

Pade power spectrum z = 0.833



Pade power spectrum z = 0



Conclusions

Most resummation schemes focus on resumming soft effects related to the scale

$$\sigma_d^2 \equiv \int \frac{d^3q}{q^2} P_L(q,\eta)$$

However, in equal-time correlators there is no enhancement related to the scale.

The failure in the convergence of SPT at late times is related to another quantity that is large, namely

$$\sigma_l^2(\Lambda,\eta) = \int^{\Lambda} d^3q \, P_L(q,\eta)$$

Even for small momenta, SPT converges at best asymptotically. Most resummation schemes share this property.

A Pade resummed SPT result looks promising in this regime.
[Carlson, White & Padmanabhan '09]

Asymptotic behavior

For large k one can extract the leading log.
For small k one can extract the leading power law.