Axionic Exercises

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1

Exercise: QCD potential
Prove that the effective potential for $\theta_{\rm QCD}$

$$e^{-\int d^4x V[\theta]} = \int \mathcal{D}A_{a\mu} e^{-S_E[A_{a\mu}] - i\theta \int d^4x_E \frac{\alpha_s}{8\pi} G^a_{\mu\nu}\widetilde{G}^{\mu\nu}_a} \tag{1}$$

has its absolute minimum at $\theta_{\rm QCD} = 0$, $V[0] \leq V[\theta]$ using the triangular inequality.

Exercise: Axion mass and mixing
At low energies, below QCD confinement and absorbing $\theta_{\rm SM}$ in the axion field,

$$V \sim -m_u v^3 \cos(\theta_0 + \theta_3) - m_d v^3 \cos(\theta_0 - \theta_3) - \Lambda^4 \cos(2\theta_0 + \theta_\phi) \tag{2}$$

Integrating out $\eta' = \eta^0 + \beta\phi = 0$ ($\beta = f/2f_\phi$) the system becomes 2x2. Find the linear combinations of mass eigenstates that diagonalise the mass matrix in the $\beta \to 0$ limit, and their masses.

Exercise: Compute the axion to photon coupling for hadronic axions

Even if axions would not couple to photons directly, they would inherit a coupling from their mixing with η^0 and π_3. These anomalous couplings follow from the divergence of the $\mathrm{U}(1)_A$ and third generator of $\mathrm{SU}(2)_A$

$$\mathcal{L} \ni [6\left(\frac{2}{3}\right)^2 + 6\left(\frac{1}{3}\right)^2]\frac{\eta^0}{f}\frac{\alpha}{8\pi}F_{\mu\nu}\widetilde{F}^{\mu\nu} + [6\left(\frac{2}{3}\right)^2 - 6\left(\frac{1}{3}\right)^2]\frac{\pi_3}{f}\frac{\alpha}{8\pi}F_{\mu\nu}\widetilde{F}^{\mu\nu} \tag{3}$$

$$= \frac{10}{3}\frac{\eta^0}{f}\frac{\alpha}{8\pi}F_{\mu\nu}\widetilde{F}^{\mu\nu} + 2\frac{\pi_3}{f}\frac{\alpha}{8\pi}F_{\mu\nu}\widetilde{F}^{\mu\nu} \tag{4}$$

Now use the axion mixings computed in the previous exercise to compute the axion-2-photon coupling

Exercise: Additional axion potential

Consider the KSVZ model and add a term of the short

$$\mathcal{L} \ni c\frac{\Phi^n}{M^{n+4}} + \text{h.c.} \tag{5}$$

with c a generally complex constant and M another energy scale $M > f_a$. After SSB compute the contribution to the axion potential (assuming it is small).

Adding it to the meson-axion potential, find the minimum of the potential. Show that $2\theta_0 + \theta_\phi - \theta_{\rm SM}$ is in general non-zero.

Global symmetries are violated by gravity effects (black hole's no-hair theorem). Some authors thus suggest that the violation of the PQ symmetry (axion shift-symmetry) is violated by the above type of operators where M is the Planck scale $M_P = 1/\sqrt{G_N} = 1.22 \times 10^{19}$ GeV. Choose $c = |c|e^{i\delta}$ and discuss the NEDM constraint on $|c|$ as a function of δ and n (Use the d_n formula with $\theta = \theta_0 + \frac{1}{2}\theta_a - \theta_{\rm SM}$, although it is not entirely correct).

Exercise: Photon axion mixing in a B-field

Write the equations of motion of the EM field and the axion in the presence of a strong external B-field. Linearise them to find mass mixing $\propto B$ between photons and the axion. Starting at position $x = 0$ with a purely EM wave, compute the axion wave at a certain distance L. Assume the B-field is transverse to the direction of propagation.

Exercise: Axion Dark matter abundance

In the pre-inflation PQ breaking scenario, the axion field becomes homogeneous in our local Universe due to inflation. We assume that the Universe was reheated at a very high temperature T, but not enough to restore the PQ symmetry thermally. In this case, the axion field in our local Universe can be taken to start with homogeneous initial conditions $\theta(x, t \sim 0) = \theta_I$. The equation of motion for the axion field in the expanding Universe is

$$\ddot{a} + 3H\dot{a} + \frac{\partial V(a)}{\partial a} = 0. \tag{6}$$

and the energy density

$$\rho = \frac{1}{2}(\dot{a})^2 + V(a). \tag{7}$$

The potential energy of the axion field at temperatures above the confinement phase transition $T_c \sim 150$ MeV is not the one described in the lectures to show how the strong CP problem is solved in today's relatively cold conditions. As a model we can use (I use $\theta = a/f_a$ absorbing $\theta_{\rm SM}$ inside)

$$V(a) = \chi(T)(1 - \cos\theta) \quad , \quad \chi \sim (75.5\mathrm{MeV})^4 \left\{ \begin{array}{cc} 1 & T < T_c \\ (T_c/T)^n & T > T_c \end{array} \right. \tag{8}$$

where $n \sim 7 - 8$ can be keep implicitly. If we restrict to $\theta_I \lesssim 1$ we can use $V(a) = \chi\theta^2/2$ and use a linearised equation of motion $\ddot{\theta} + 3H\dot{\theta} + m_a^2(T)\theta = 0$ with $m_a^2(T) = \chi(T)/f_a^2$.

Calculate the axion energy density *today* as a function of $m_a = m_a(t_0)$ or f_a.

Help!: Assume radiation domination when the axion field starts to oscillate, (the expansion rate is $H = \dot{R}/R = 1/2t$, $R(t)$ scale factor of the Universe expansion, t is time) and calculate the approximate time when oscillations commence t_1, $m_a(t_1)t_1 \sim 1$. (Use $H^2 = (8\pi^3/90M_{\rm Planck}^2)g_*T^4$ with $g_* \sim$ constant to get $T(t)$). Before that time the EOM is approximated by $\ddot{\theta} + 3H\dot{\theta} = 0$ which solves as $\theta = \theta_I$ $\dot{\theta} = 0$, which we can use as initial conditions. Solve the $t > t_1$ evolution of the axion field in the WKB approximation by assuming a solution

$$\theta(t) = q(R(t))e^{i\int^t m_a(t')dt'} \quad \text{with} \quad \dot{q} \ll m_a \tag{9}$$

(the solution is the real part $\mathrm{Re}[\theta]$). Show that the solution satisfies the conservation of the so-called comoving number of axions $m_a\theta^2R^3$. Show that the previous expression is the density of axions n_a (energy density/mass of an axion) times a comoving volume $n_aR^3 = (\rho/m_a)R^3$ (in the WKB or adiabatic approximation). Due to axion number conservation, we can obtain the density now $\rho(t_0) = m_a(t_0)n_a = m_a(t_0)n_a(t_1)(R_1/R_0)^3$. To estimate the dilution factor $(R_1/R_0)^3 = (R(t_1)/R(t_0))^3$ one can use the conservation of comoving entropy $g_S(T_1)T_1^3R_1^3 = g_S(T_0)T_0^3R_0^3$. Compute the axion energy density today in keV/cm^3 or in units of the critical density $\Omega_a = \rho_a/\rho_c$, $3H^2(t_0)M_{\rm Planck}^2/8\pi$. The degrees of freedom $g_S(T_0) \sim 3.9$ and $g_S(T_1) \sim g_*(T_1)$ can be kept explicitly. Todays's temperature is $T_0 = 2.725$ K.