VNiVERSiDAD
D SALAMANCA

Quantum Field Theory and Standard Model

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Universidad de Salamanca

Taller de Altas Energías 2017, Benasque

CENTRO DE CIENCIAS
DE BENASQUE
PEDRO PASCUAL

An “ideal” list of topics

- ✱ The very basics of QFT
- ✱ Loops, divergences and renormalization
- ✱ What to ask from a “healthy” QFT
    - ☑ Lorentz invariance
    - ☑ Locality
    - ☑ Unitarity
    - ☑ Renormalizability?
- ✱ Symmetries and their breaking
- ✱ Gauge invariance
- ✱ Massive gauge fields
- ✱ Building the standard model

Schedule

* **Lectures**: Monday to Thursday (first week), from 9:00 to 10:00.

* **Tutorials**:

    - Monday 4th, 15:30 to 16:30
    - Tuesday 5th, 16:30 to 17:30
    - Thursday 7th, 18:00 to 19:00

Tutor: Ramon Miravitllas.

A sample of textbooks

- L. Álvarez-Gaumé & M.A. Vázquez-Mozo, "An Invitation to Quantum Field Theory", Springer 2012.
- M.E. Peskin & D.V. Schroeder, "An Introduction to Quantum Field Theory", Perseus Books 1995.
- C. Quigg, "Gauge theories of the strong, weak, and electromagnetic interactions" (2nd edition), Princeton University Press 2013.
- M. Schwartz, "Quantum field theory and the standard model", Cambridge 2014.

A note about conventions:

✱ We use the "**mostly minus**" metric (a.k.a. West Coast metric):

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

✱ Unless otherwise said, **natural units** are used throughout:

$$\hbar = c = 1$$

✱ We use **Heaviside-Lorentz** electromagnetic units:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

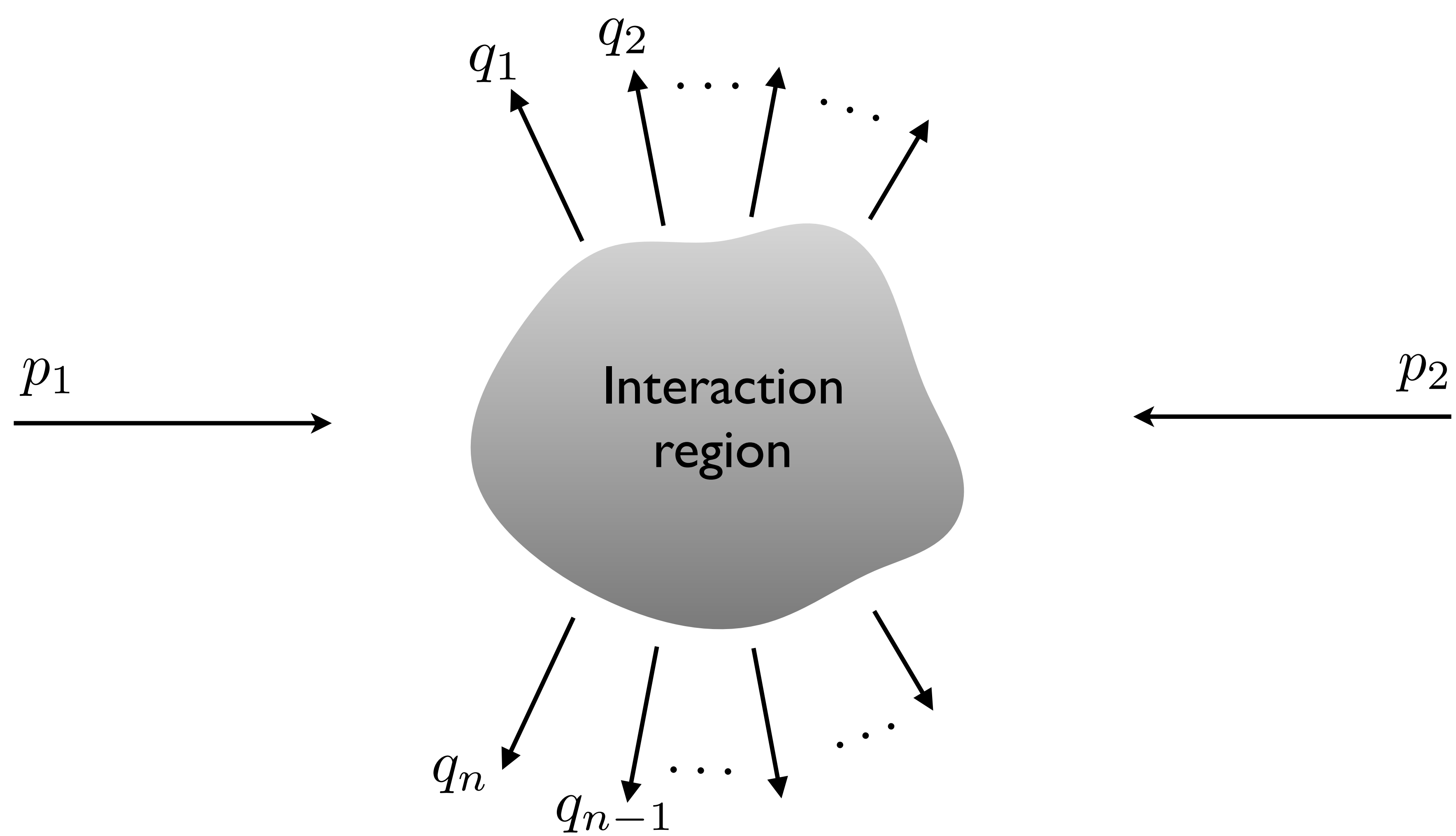
$$\boldsymbol{\nabla} \times \mathbf{B} = \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t}$$

$$\alpha = \frac{e^2}{4\pi}$$ (fine structure constant)

Part I

From elementary particles to quantum fields

Elementary particles are studied through **scattering experiments**, typically



Quantum mechanics, even relativistic, is **not enough** to describe these **high energy** experiments…

Let us consider the **relativistic** quantum evolution of a **localized, single-particle wave packet**:

positive frequency

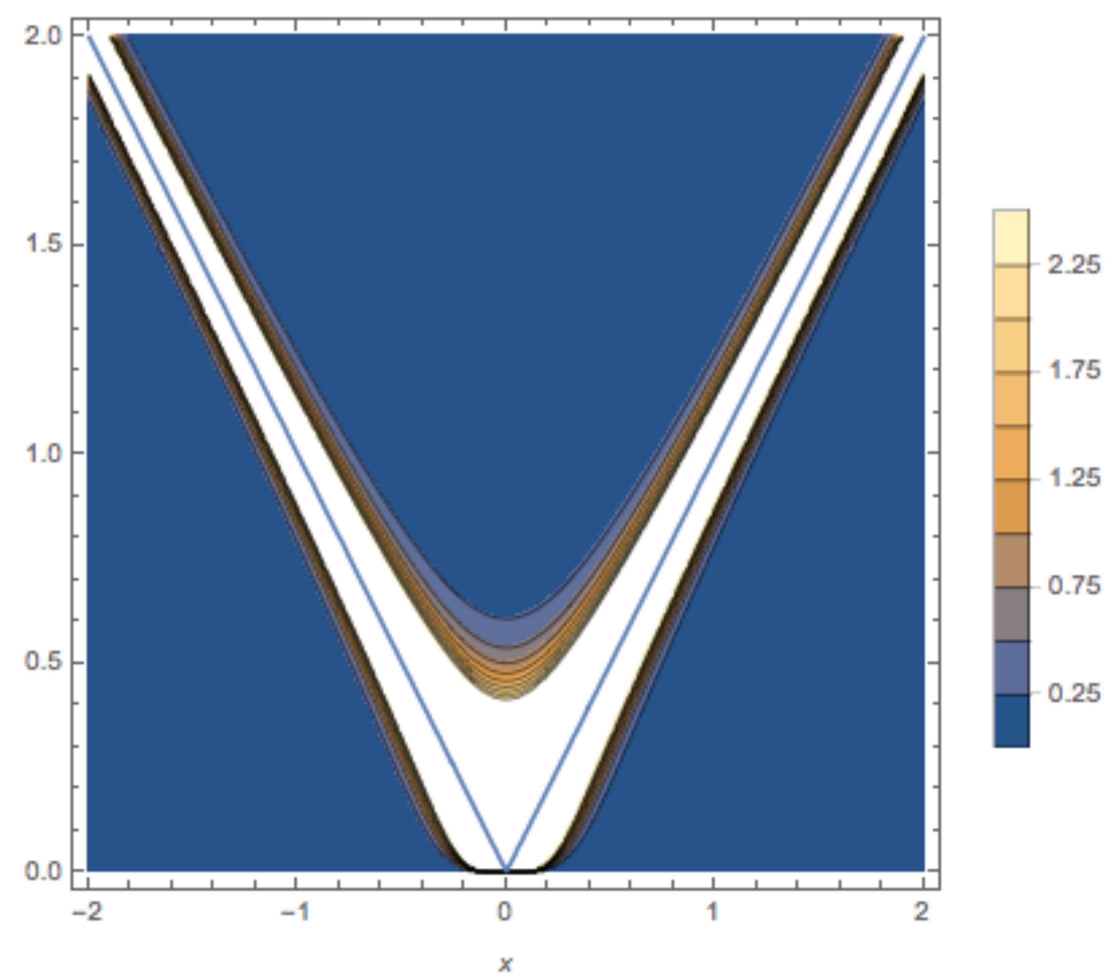
$$\psi(t,\mathbf{x}) = e^{-it\overbrace{\sqrt{-\nabla^2+m^2}}^{H=\sqrt{p^2+m^2}}}\delta^{(3)}(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{x}-it\sqrt{k^2+m^2}}$$

$$= -\frac{i}{4\pi^2|\mathbf{x}|}\int_{-\infty}^{\infty} kdk\, e^{ik|\mathbf{x}|-it\sqrt{k^2+m^2}} = \frac{1}{2\pi^2|\mathbf{x}|}\int_0^{\infty} kdk\,\sin(k|\mathbf{x}|)e^{-it\sqrt{k^2+m^2}}$$

The integral can be regularized by $t \to t - i\epsilon$, to give

$$\psi(t,\mathbf{x}) = -\frac{i}{2\pi^2}\frac{m^2 t}{t^2-\mathbf{x}^2}K_2\left(im\sqrt{t^2-\mathbf{x}^2}\right)$$

The probability $|\psi(t,\mathbf{x})|^2$ **spills** outside the light-cone



Causality is violated!

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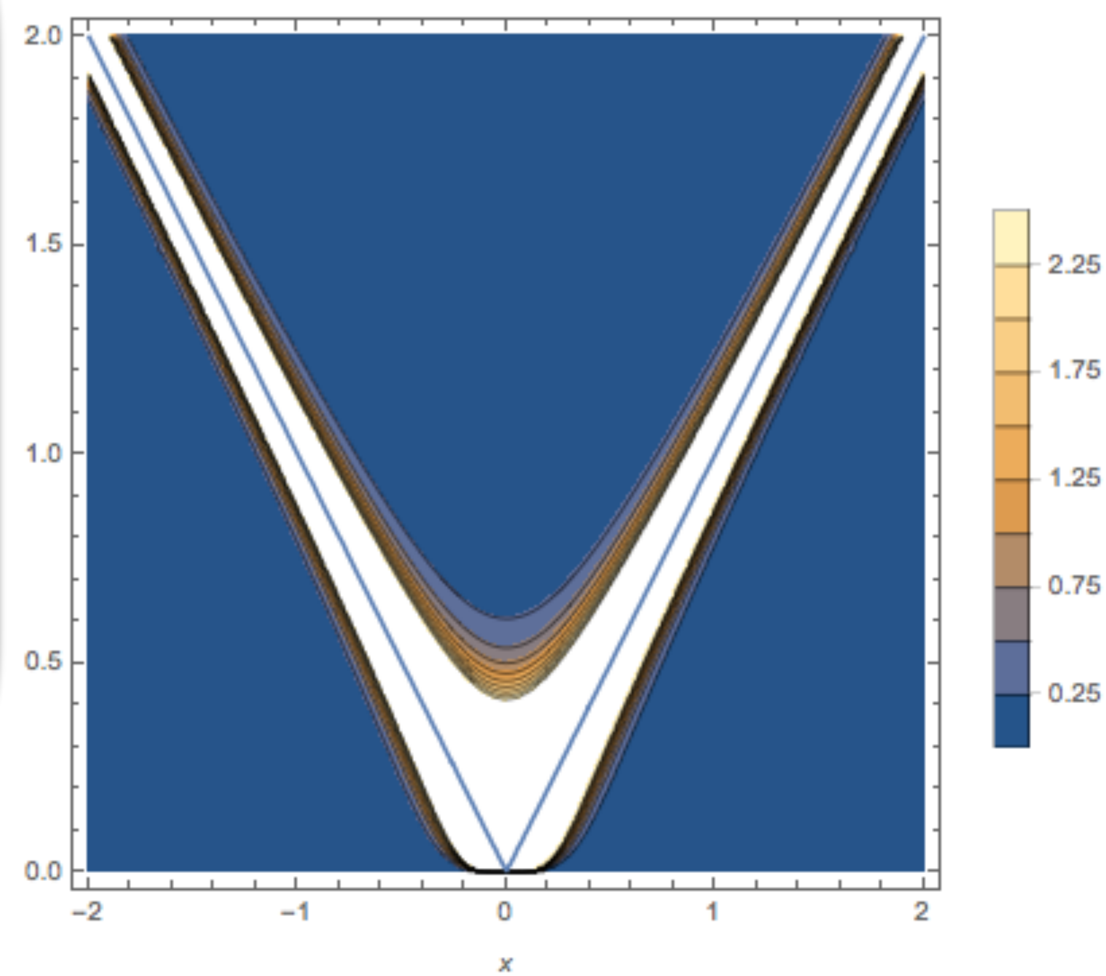
What we are computing is the **propagator** of a relativistic particle:

$$\psi(t,\mathbf{x}) = \langle \mathbf{x}|e^{-itH}|\mathbf{0}\rangle \equiv G(t,\mathbf{x};0,\mathbf{0})$$

Relativistic quantum mechanics propagates states **outside** the light cone

$$G(t',\mathbf{x}';t,\mathbf{x}) \neq 0 \quad \text{when} \quad (t'-t)^2-(\mathbf{x}'-\mathbf{x})^2<0$$



Causality is violated!

But when $t^2 - \mathbf{x}^2 < 0$ there are **frames** in which $(t, \mathbf{x})$ **happens before** $(0, \mathbf{0})$. Thus,

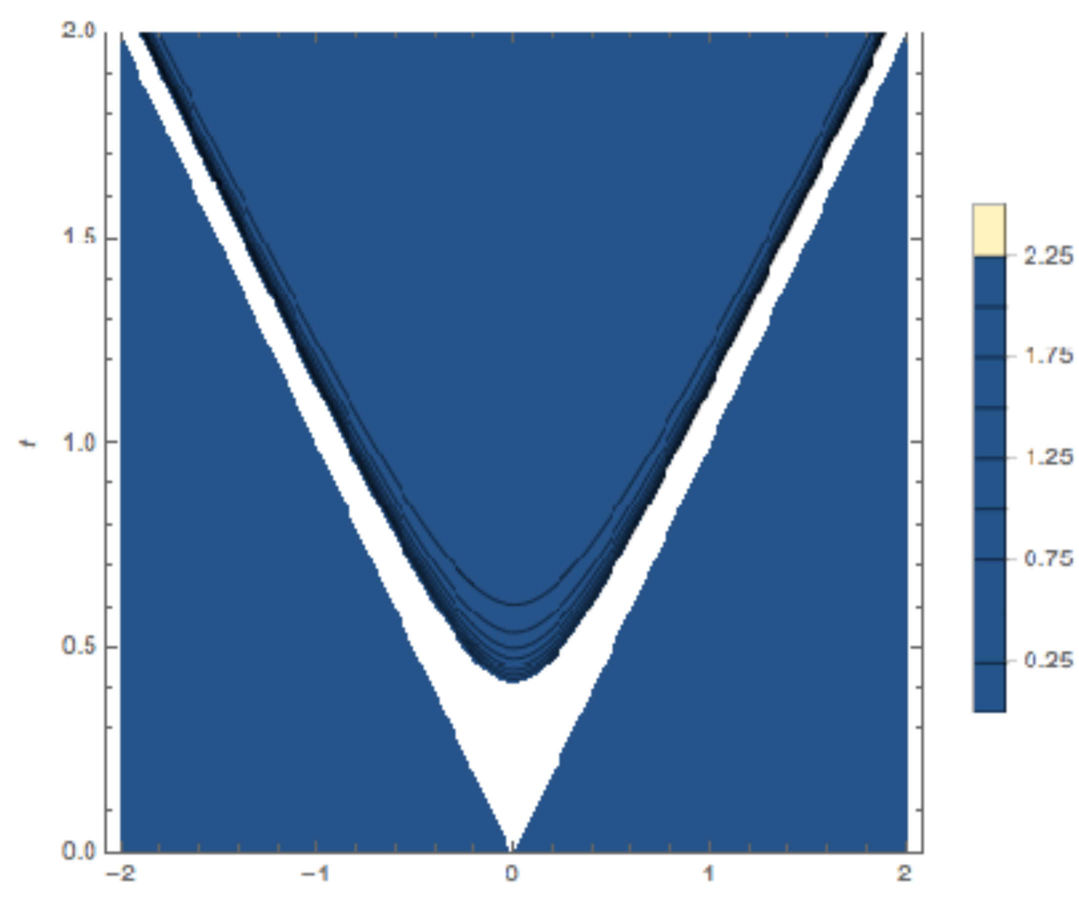
$$\psi(t,\mathbf{x}) = \begin{cases} \langle \mathbf{x}|e^{-itH}|\mathbf{0}\rangle \quad \text{when} \quad t^2 - \mathbf{x}^2 > 0 \\ \\ \langle \mathbf{x}|e^{-itH}|\mathbf{0}\rangle + \langle \mathbf{0}|e^{itH}|\mathbf{x}\rangle = 2\mathrm{Re}\,\langle \mathbf{x}|e^{-itH}|\mathbf{0}\rangle \quad \text{when} \quad t^2 - \mathbf{x}^2 < 0 \end{cases}$$

But since we have computed

$$\langle \mathbf{x}|e^{-itH}|\mathbf{0}\rangle = -\frac{i}{2\pi^2}\frac{m^2 t}{t^2 - \mathbf{x}^2}K_2\left(im\sqrt{t^2 - \mathbf{x}^2}\right)$$

the result is

$$\psi(t,\mathbf{x}) = -\frac{i}{2\pi^2}\frac{m^2 t}{t^2 - \mathbf{x}^2}K_2\left(im\sqrt{t^2 - \mathbf{x}^2}\right)\theta(t^2 - \mathbf{x}^2)$$



Causality is restored!

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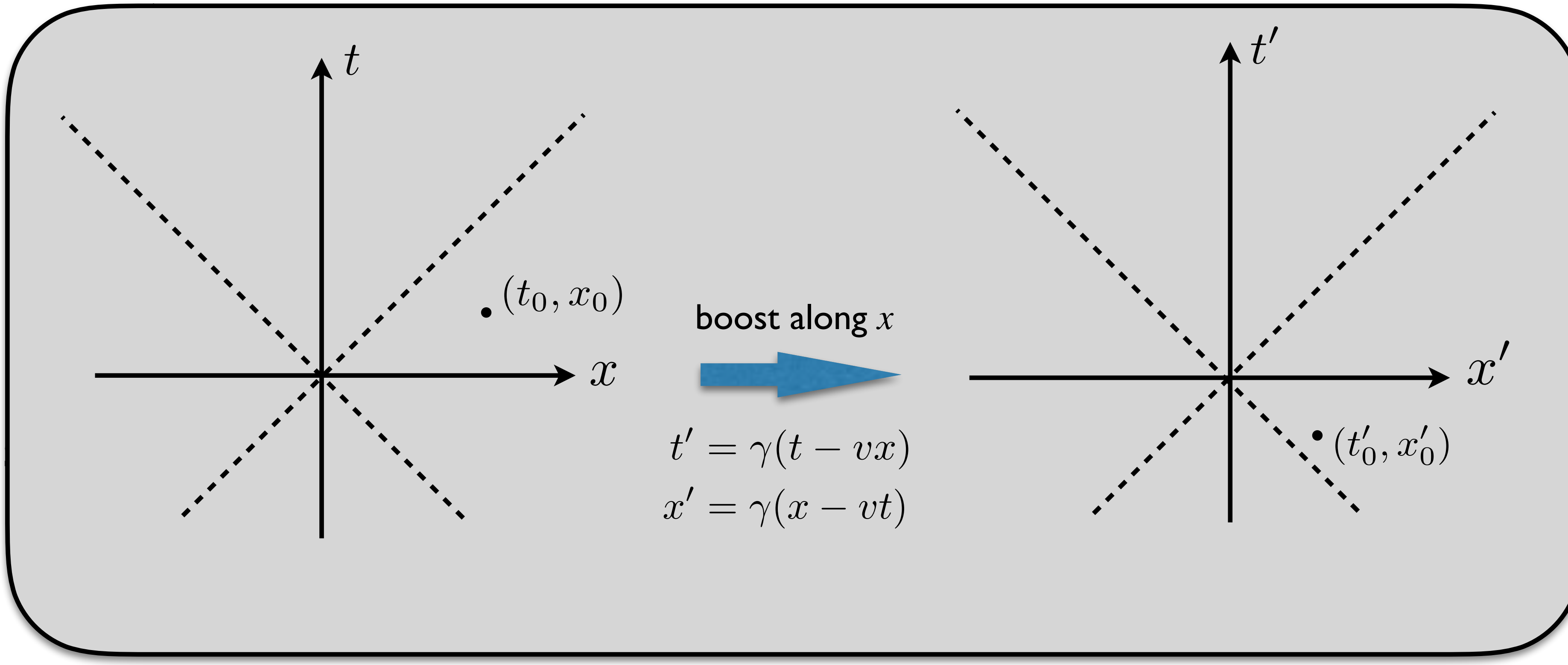

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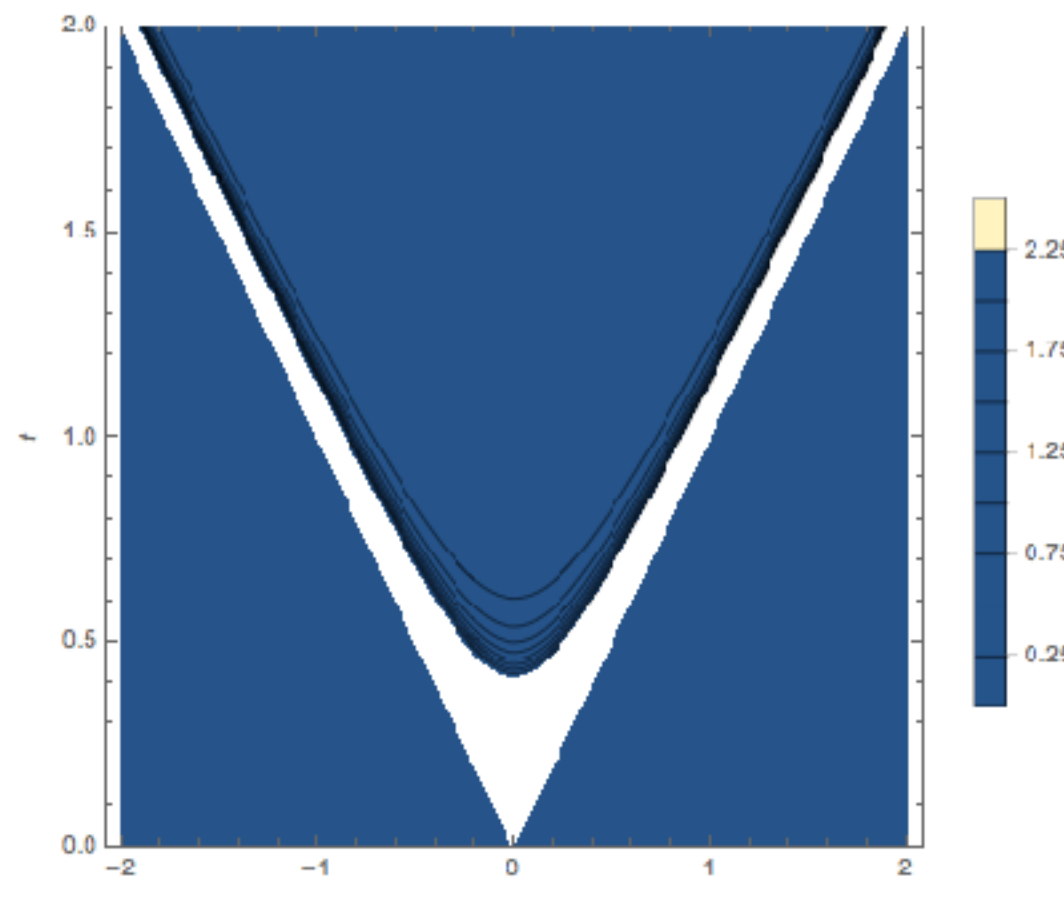

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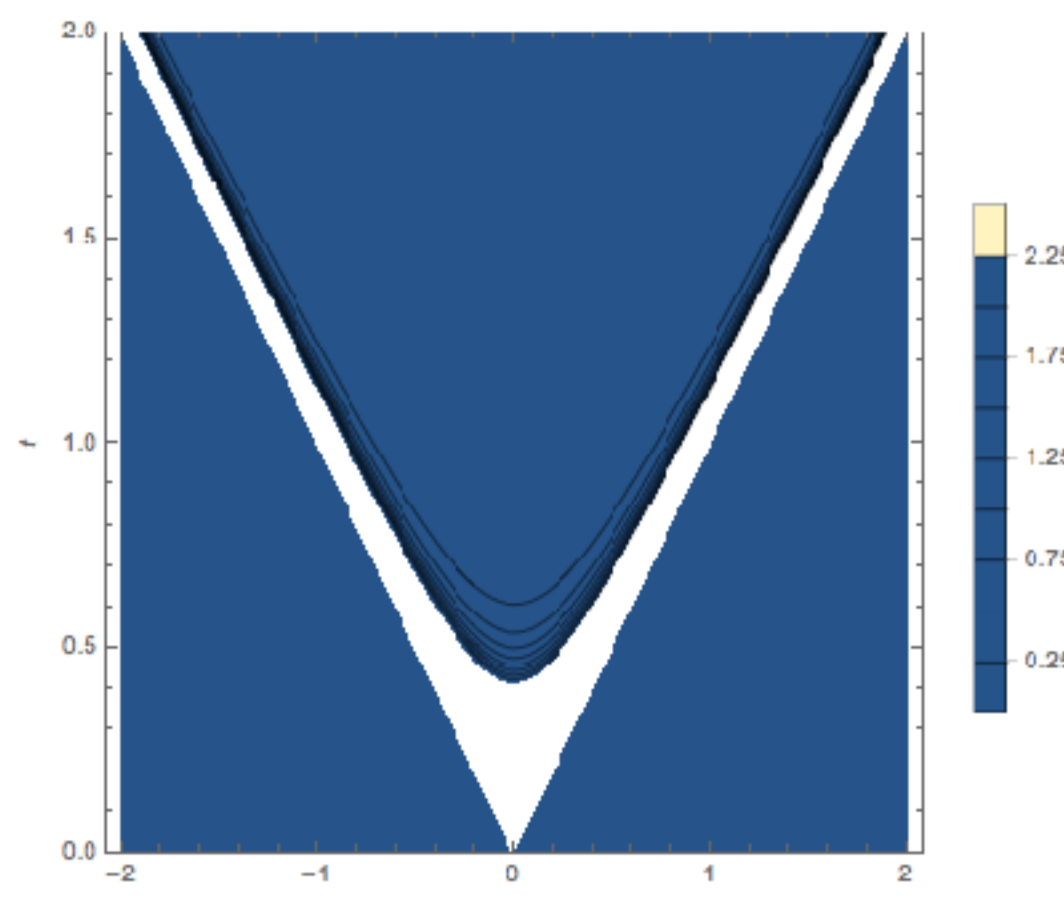

Causality is restored!

We have to allow **particles travelling backward in time**!!

Their wave functions are

$$\psi(t,\mathbf{x})_{\Downarrow} = \langle \mathbf{0}|e^{itH}|\mathbf{x}\rangle = \langle \mathbf{x}|e^{-itH}|\mathbf{0}\rangle^{*} = \psi(t,\mathbf{x})^{*}_{\Uparrow}$$

Ernst Stückelberg
(1905-1984)

Richard Feynman
(1981-1988)

Thus, under any **global** $\mathrm{U}(1)$ **symmetry**

$$\psi(t,\mathbf{x})_{\Uparrow} \longrightarrow e^{iq\theta}\psi(t,\mathbf{x})_{\Uparrow} \qquad\qquad \psi(t,\mathbf{x})_{\Downarrow} \longrightarrow e^{-iq\theta}\psi(t,\mathbf{x})_{\Downarrow}$$

these particles have **oposite charges**, $q_{\Downarrow} = -q_{\Uparrow}$ (but the same mass! $H_{\Uparrow,\Downarrow} = \sqrt{-\nabla^2 + m^2}$)

To **restore causality** we are forced to introduce **antiparticles**!!

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States moving backward in time can be reinterpreted as **negative frequency states** with **reversed** momentum, propagating **forward** in time:

$$\psi(t,\mathbf{x})_{\Downarrow} = \int \frac{d^3k}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}+it\sqrt{k^2+m^2}}$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{i(-\mathbf{k})\cdot\mathbf{x}-it(-\sqrt{k^2+m^2})}$$

negative frequency

$$\to e^{-iq\theta}\psi(t,\mathbf{x})_{\Downarrow}$$

mass! $H_{\Uparrow,\Downarrow} = \sqrt{-\nabla^2+m^2}$)

To **restore causality** we are forced to introduce **antiparticles**!!

Switching on **interactions**, charge conservation allows the creation of **particle-antiparticle pairs**, provided **enough energy** is available.

For example, localizing particle below their **Compton wavelength**

$$\Delta x \sim \frac{1}{m} \xrightarrow{\Delta x \Delta p \sim 1} \Delta p \sim m \longrightarrow \Delta E \sim m$$

and due to energy **quantum fluctuations** the creation of particle-antiparticle pairs **cannot be prevented**.

$$\Downarrow$$

We have to **give up** the **single-particle description**!

Relativistic quantum mechanics is a **dead end** for high energy particle physics…

To handle many particles, **second quantization** seems the best approach, introducing **creation-annihilation operators** for particles with **on-shell momentum** p

$$a(p),\ a(p)^\dagger \quad \xrightarrow[\omega_{\mathbf{p}}=\sqrt{\mathbf{p}^2+m^2}]{p^2=m^2} \quad \begin{array}{l} [a(p),a(p')^\dagger]=(2\pi)^3(2\omega_{\mathbf{p}})\delta^{(3)}(\mathbf{p}-\mathbf{p}') \\ [a(p),a(p')]=[a(p)^\dagger,a(p')^\dagger]=0 \end{array}$$

Lorentz invariant (exercise)

(Multi-)particle states are obtained from the **Poincaré-invariant vacuum** $|0\rangle$

$$|p\rangle = a(p)^\dagger|0\rangle \quad \longrightarrow \quad \langle p|p'\rangle=(2\pi)^3(2\omega_{\mathbf{p}})\delta^{(3)}(\mathbf{p}-\mathbf{p}')$$

Lorentz invariant (exercise)

$$|f\rangle=\int\left[\prod_{i=1}^{n}\frac{d^3p_i}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}_i}}\right]f(\mathbf{p}_1,\ldots,\mathbf{p}_n)a(p_1)^\dagger\ldots a(p_n)^\dagger|0\rangle$$

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Lorentz invariant (exercise)

$$\mathscr{U}(\Lambda)|0\rangle=e^{-ia\cdot P}|0\rangle=|0\rangle$$

$$\mathscr{U}(\Lambda)a(p)\mathscr{U}(\Lambda)^\dagger=a(\Lambda p)$$

$$\mathscr{U}(\Lambda)|p\rangle=|\Lambda p\rangle$$

where $\mathscr{U}(\Lambda)\in\mathrm{SO}(1,3)$

$$_1,\ldots,\mathbf{p}_n)a(p_1)^\dagger\ldots a(p_n)^\dagger|0\rangle$$

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Free fields are **linear combinations** of creation-annihilation operators. E.g., for a free **Hermitian** scalar field

$$\phi(x) = \phi(x)^\dagger \qquad \phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}\Big[f(x,\mathbf{p})a(p) + f(x,\mathbf{p})^* a(p)^\dagger\Big]$$

Imposing the **equations of motion**,

$$(\Box + m^2)\phi(x) = 0 \qquad f(x,\mathbf{p}) = e^{-i\omega_{\mathbf{p}}t + i\mathbf{p}\cdot\mathbf{x}}$$

The free quantum field satisfies:

* **Equal-time** canonical commutation relations

$$[\phi(t,\mathbf{x}),\dot{\phi}(t,\mathbf{x}')] = i\delta^{(3)}(\mathbf{x}-\mathbf{x}'), \qquad [\phi(t,\mathbf{x}),\phi(t,\mathbf{x}')] = [\dot{\phi}(t,\mathbf{x}),\dot{\phi}(t,\mathbf{x}')] = 0$$

* **Microcausality**

$$[\phi(x),\phi(x')] = 0 \qquad \text{when} \qquad (x-x')^2 < 0$$

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negative frequency

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The **many-particle** Fock states **diagonalize** the free field **Hamiltonian**

$$H = \frac{1}{2}\int d^3x \left[\dot{\phi}^2 + (\boldsymbol{\nabla}\phi)^2 + m^2\phi^2\right] \xrightarrow{\text{(exercise)}} H = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\left[a(p)^\dagger a(p) + (2\pi)^3\omega_{\mathbf{p}}\delta^{(3)}(\mathbf{0})\right]$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}\left[\omega_{\mathbf{p}} a(p)^\dagger a(p)\right] + E_0$$

Subtracting the (divergent) zero-point energy E_0

$$[a(p), a(p')^\dagger] = (2\pi)^3(2\omega_{\mathbf{p}})\delta^{(3)}(\mathbf{p}-\mathbf{p}')$$

$$H|p\rangle = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3} a(k)^\dagger a(k) a(p)^\dagger|0\rangle = \omega_{\mathbf{p}}|p\rangle$$

$$H|p_1,\ldots,p_n\rangle \equiv H a(p_1)^\dagger \ldots a(p_n)^\dagger|0\rangle = \left(\sum_{i=1}^{n}\omega_{\mathbf{p}_i}\right)|p_1,\ldots,p_n\rangle$$

$$\Downarrow$$

Particles are the **low-lying excitations** of **quantum fields**

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$$=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}\left[\omega_{\mathbf{p}}a(p)^\dagger a(p)\right]+E_0$$

$$\boxed{E_0=\langle 0|H|0\rangle=\sum_{\mathbf{p}}\frac{1}{2}\omega_{\mathbf{p}}}$$

Subtracting the (divergent) zero-point energy E_0

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Particles are the **low-lying excitations** of **quantum fields**

A **particle** is characterized by a number of **"Casimirs"**:

✱Poincaré group: $\begin{cases} P_\mu P^\mu = m^2 & \longrightarrow \text{Mass} \\ W_\mu W^\mu = -m^2 s(s+1) & \longrightarrow \text{Spin} \end{cases}$

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} J_{\nu\alpha} P_\beta$$

vector de Pauli-Lubański

✱Internal symmetry groups: $\begin{cases} \text{electric charge} \\ \vdots \end{cases}$

To do particle physics, we have to **choose** the appropriate **interpolating field:**

✱It **transforms correctly** (i.e., the right value for the "Casimirs")

✱It **creates** the corresponding particle out of the **vacuum**:

$$\langle 0|\phi(x)|p\rangle \neq 0$$

The x-dependence is **fixed** by the Poincaré invariance of the vacuum

$$\langle 0|\phi(x)|p\rangle = \langle 0|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}|p\rangle = \langle 0|\phi(0)|p\rangle e^{-ip\cdot x}$$

The fields can be **canonically normalized**, such that:

* **Scalar field**: $\langle 0|\phi(0)|p\rangle = 1$

* **Dirac field**: $\begin{cases} \langle 0|\psi_\alpha(0)|p,\sigma;0\rangle = u_\alpha^{(\sigma)}(p) \\ \langle 0|\overline{\psi}_\alpha(0)|0;p,\sigma\rangle = \overline{v}_\alpha^{(\sigma)}(p) \end{cases}$

* **Photon field**: $\langle 0|A_\mu(0)|p,\lambda\rangle = \varepsilon_\mu^{(\lambda)}(p)$

Any properly normalized interpolating field **does the job**, provided it satisfies **microcausality**

$$[\phi(x),\phi(x')] = 0 \qquad \text{when} \qquad (x-x')^2 < 0$$

$\longrightarrow$ Borchers classes

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To describe scalar particles, instead of $\phi(x)$ we can also use

$$\Phi(x) = -\frac{1}{m^2}\Box\phi(x)$$

$$\langle 0|\Phi(x)|p\rangle = -\frac{1}{m^2}\Box\langle 0|\phi(x)|p\rangle = -\frac{1}{m^2}\Box e^{-ip\cdot x} = e^{-ip\cdot x}$$

and

$$[\Phi(x),\Phi(x')] = \frac{1}{m^4}\Box_x\Box_{x'}[\phi(x),\phi(x')] = 0$$

for $(x-x')^2 < 0$

$\sigma)(p)$

$\sigma)(p)$

, provided it satisfies

< 0

Borchers classes

Still, to study particle physics we need to **introduce interactions**…

In interacting field theories, **particles** still emerge as **weakly coupled excitations:**

dimensionless
effective couplings

$$\mathscr{L}(\phi,\partial\phi)=\mathscr{L}(\phi,\partial\phi)_{\rm free}+\sum_i g_i\mathscr{O}_i(\phi,\partial\phi)$$

nonquadratic terms

where g_i is **"small"** and

ground state of
the **full** theory

$$\langle\Omega|\phi(x)|p\rangle=\phi(\mathbf{p})e^{-ip\cdot x}$$

one-particle
wave function

Thus:

* Particles are identified by **quantizing** the **free theory**.

* Interactions are treated in **perturbation theory**.

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Examples

✱**QED**: electrons+photons

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(\gamma^\mu\partial_\mu - m)\psi - qA_\mu\overline{\psi}\gamma^\mu\psi$$

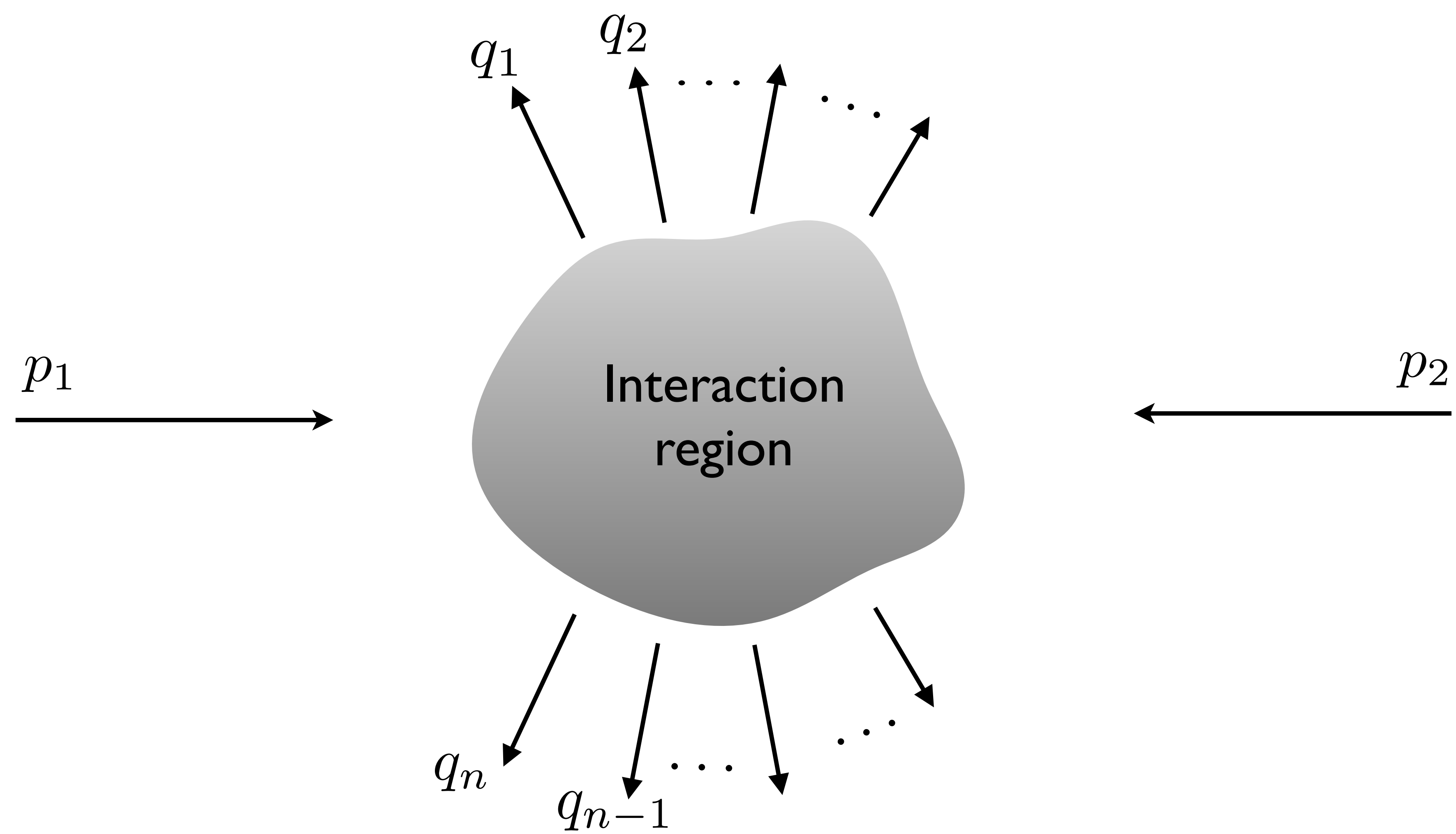
✱**QCD** (**high energies**): quarks+gluons

$$\mathscr{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \sum_{f=1}^{6}\overline{Q}_f(\gamma^\mu\partial_\mu - m_f)Q_f - gA^a_\mu\sum_{f=1}^{6}\overline{Q}_f T^a\gamma^\mu Q_f$$

✱**QCD** (**low energies**): pions (+nucleons)

$$\mathscr{L} = \frac{1}{2}\mathrm{tr}\left(\partial_\mu\boldsymbol{\pi}\partial^\mu\boldsymbol{\pi}\right) - \frac{1}{3f_\pi^2}\mathrm{tr}\Big(\partial_\mu\boldsymbol{\pi}[\boldsymbol{\pi},[\boldsymbol{\pi},\partial^\mu\boldsymbol{\pi}]]\Big) + \ldots$$

✱ Interactions are treated in **perturbation theory**.



A **scattering** experiment is characterized by its initial (*in*) and final (*out*) multiparticle state:

$$|p_1, p_2\rangle_{\rm in} \qquad\qquad |q_1, q_2, \ldots, q_{n-1}, q_n\rangle_{\rm out}$$

Both are **Heisenberg-picture** (i.e., time-independent) **states** in a very complicated **interacting theory**.

Our aim is to **compute** the **probability amplitude**:

$$S(i \longrightarrow f) = {}_{\rm out}\langle q_1, \ldots, q_n | p_1, p_2\rangle_{\rm in}$$

$$|p_1, p_2\rangle_{\rm in} \qquad\qquad |q_1, q_2, \ldots, q_{n-1}, q_n\rangle_{\rm out}$$

These states can also be seen as belonging to the **free, multiparticle Fock space**

$$|p_1, p_2\rangle, |q_1, q_2, \ldots, q_n\rangle \in \mathscr{F} \equiv \bigoplus_{n=0}^{\infty} \mathscr{H}_1 \otimes \overset{(n)}{\cdots} \otimes \mathscr{H}_1$$

The scattering experiment is then described by the ***S*-matrix operator**

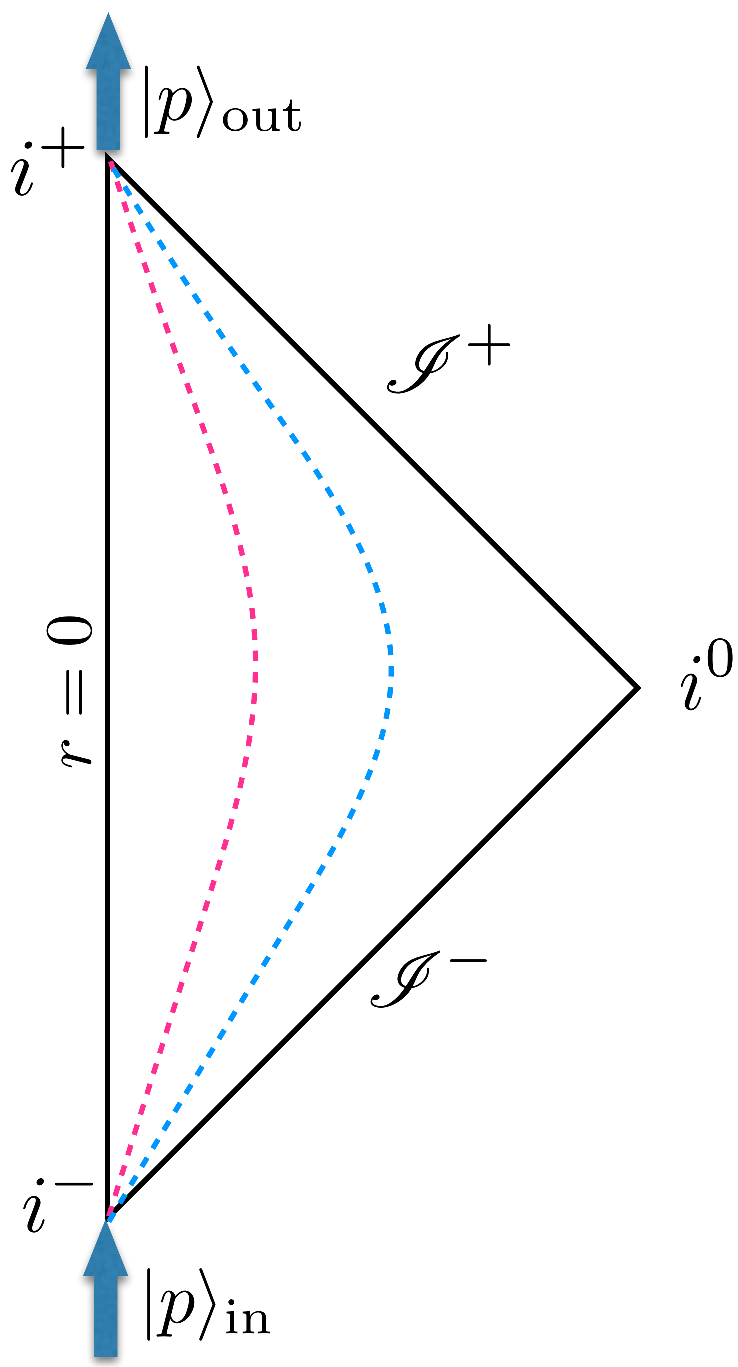
$$S : \mathscr{F} \longrightarrow \mathscr{F}$$

$$S(i \longrightarrow f) = {}_{\rm out}\langle q_1, \ldots, q_n | p_1, p_2\rangle_{\rm in} \equiv \langle q_1, \ldots, q_n | S | p_1, p_2\rangle$$

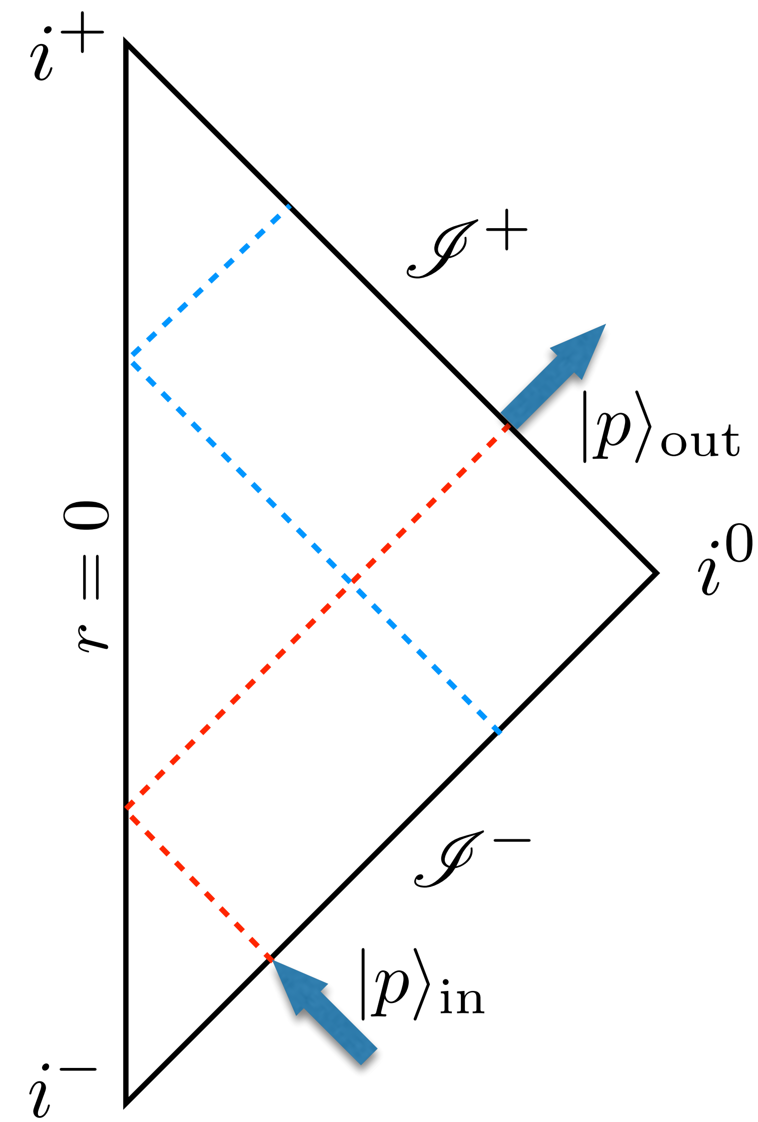
The ***S*-matrix** operator satisfies a number of **properties**:

* **Unitarity**: $S^\dagger = S^{-1}$
* **Lorentz invariance**: $\mathscr{U}(\Lambda) S \mathscr{U}(\Lambda)^\dagger = S$ with $\Lambda \in \mathrm{SO}(1{,}3)$
* $\langle q_1, \ldots, q_n | S | p_1, p_2\rangle$ is **analytic** in the external momenta.

The ***S*-matrix** is a kind of **holographic** quantity in **Minkowski** space-time: *in*- and *out*-states **live** on its **boundary**.



massive states



massless states

The computation of the S-matrix in terms of the interacting field theory is done using the **Lehmann-Symanzik-Zimmermann (LSZ) reduction formula**.

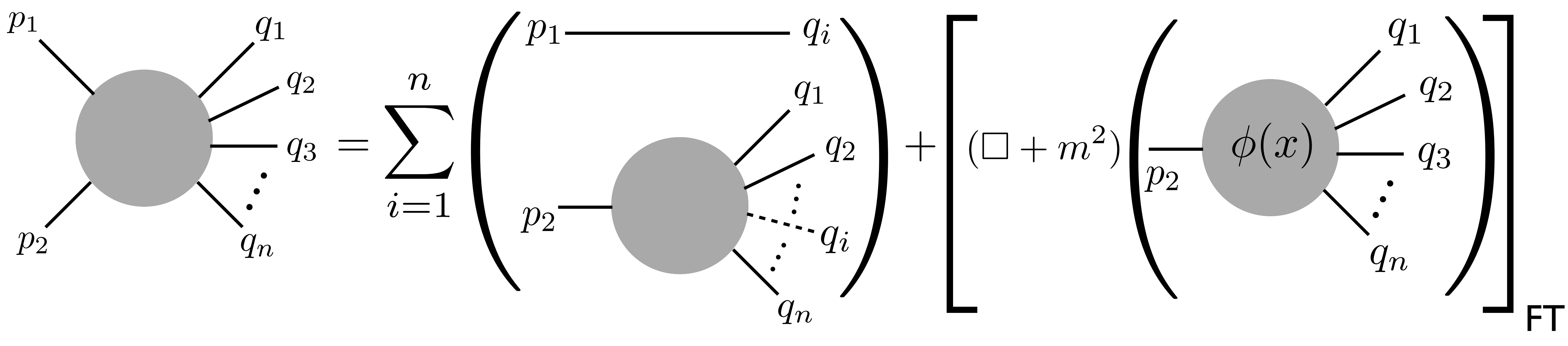
Harry Lehmann
(1924-1998)

Kurt Symanzik
(1923-1983)

Wolfhart Zimmermann
(1928-2016)

$$
\begin{aligned}
{}_{\text{out}}\langle q_1,\ldots,q_n|p_1,p_2\rangle_{\text{in}} &= \sum_{i=1}^{n}(2\pi)^3(2\omega_{\mathbf{q}_i})\delta^{(3)}(\mathbf{q}_i-\mathbf{p}_1)\,{}_{\text{out}}\langle q_1,\ldots,\widehat{q_i},\ldots,q_n|p_2\rangle_{\text{in}} \\
&\quad + iZ^{-1/2}\int d^4x\, e^{-ip_1\cdot x}(\Box+m^2)\,{}_{\text{out}}\langle q_1,\ldots,q_n|\phi(x)|p_2\rangle_{\text{in}}
\end{aligned}
$$

Symbolically:



Iterating the procedure, we **trade** all incoming and outgoing particles by **time-ordered** field insertions:

$$\langle q_1, \ldots, q_n | S | p_1, p_2 \rangle = \text{disconected terms}$$

$$+ i (Z^{-1/2})^{n+2} \int d^4x_1 d^4x_2 e^{-ip_1x_1 - ip_2x_2} \int d^4y_1 \ldots d^4y_n e^{iq_1y_1 + \ldots + iq_ny_n}$$

$$\times (\Box + m^2)_{x_1} (\Box + m^2)_{x_2} (\Box + m^2)_{y_1} \ldots (\Box + m^2)_{y_n} \langle \Omega | T[\phi(x_1)\phi(x_2)\phi(y_1) \ldots \phi(y_n)] | \Omega \rangle$$

Inverse free propagators

S-matrix amplitudes are computed in terms of **time-ordered** (amputated) **correlation functions**

$$G(x_1, \ldots, x_n) = \langle \Omega | T[\phi(x_1) \ldots \phi(x_n)] | \Omega \rangle$$

which can be computed in **perturbation theory**.

→ **Feynman diagrammatics**

Iterating the procedure, we **trade** all incoming ... **ed** field insertions:

$$T[\phi(x)\phi(y)] = \theta(x^0 - y^0)\phi(x)\phi(y) + \theta(y^0 - x^0)\phi(y)\phi(x)$$

$$\langle q_1, \ldots, q_n | S | p_1, p_2 \rangle = \text{disconected terms}$$

$$+ i(Z^{-1/2})^{n+2} \int d^4x_1 d^4x_2 e^{-ip_1x_1 - ip_2x_2} \int d^4y_1 \ldots d^4y_n e^{iq_1y_1 + \ldots + iq_ny_n}$$

$$\times (\Box + m^2)_{x_1} (\Box + m^2)_{x_2} (\Box + m^2)_{y_1} \ldots (\Box + m^2)_{y_n} \langle \Omega | T[\phi(x_1)\phi(x_2)\phi(y_1)\ldots\phi(y_n)] | \Omega \rangle$$

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→ **Feynman diagrammatics**

We can isolate **nontrivial scattering** in the S-matrix by writing

$$S = \mathbf{1} + iT$$

so the matrix elements have the structure

invariant amplitude

$$\begin{aligned}\langle q_1, \ldots, q_n | S | p_1, p_2 \rangle &= \langle q_1, \ldots, q_n | p_1, p_2 \rangle + \langle q_1, \ldots, q_n | iT | p_1, p_2 \rangle \\ &= \langle q_1, \ldots, q_n | p_1, p_2 \rangle + (2\pi)^4 \delta^{(4)} \left(p_1 + p_2 - \sum_{i=1}^{n} q_i \right) i\mathscr{M}_{i\to f}\end{aligned}$$

In terms of the **invariant amplitude**, the **differential cross section** is given by

$$d\sigma = \frac{|i\mathscr{M}_{i\to f}|^2}{\underbrace{4\omega_{\mathbf{p}_1}\omega_{\mathbf{p}_2}|\mathbf{v}_1 - \mathbf{v}_2|}_{\text{observer dependent}}} (2\pi)^4 \delta^{(4)} \left(p_1 + p_2 - \sum_{i=1}^{n} q_i \right) \underbrace{\prod_{k=1}^{n} \frac{d^3 q_k}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{q}_k}}}_{\text{phase space factor}}$$

We can isolate **nontrivial scattering** in the S-matrix by writing

so t

$\langle q_1,$

invariant amplitude

$$\sum_{i=1}^{n} q_i \Bigg) \underbrace{i\mathscr{M}_{i\to f}}$$

In the case of **particle decay**,

$$|p\rangle \longrightarrow |q_1,\ldots,q_n\rangle$$

the **decay width** is given by

$$d\Gamma = \frac{|i\mathscr{M}_{i\to f}|^2}{2\omega_{\mathbf{p}}}(2\pi)^4\delta^{(4)}\left(p_1+p_2-\sum_{i=1}^{n} q_i\right)\prod_{k=1}^{n}\frac{d^3q_k}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{q}_k}}$$

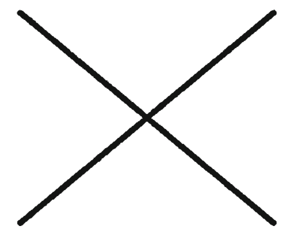
In ter … given by

$$d\sigma = \underbrace{\frac{|i\mathscr{M}_{i\to f}|^2}{4\omega_{\mathbf{p}_1}\omega_{\mathbf{p}_2}|\mathbf{v}_1-\mathbf{v}_2|}}_{\text{observer dependent}}(2\pi)^4\delta^{(4)}\left(p_1+p_2-\sum_{i=1}^{n} q_i\right)\underbrace{\prod_{k=1}^{n}\frac{d^3q_k}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{q}_k}}}_{\text{phase space factor}}$$

The **perturbative computation** of correlation functions in **momentum space** is carried out using **Feynman diagrammatics**

For a ϕ^4 scalar theory, the **Feynman rules** are

$$\Longrightarrow \quad \frac{i}{p^2 - m^2 + i\varepsilon}$$



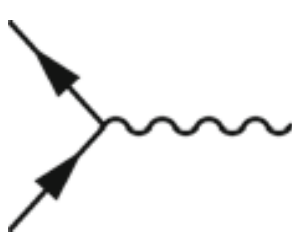
$$\Longrightarrow \quad -i\lambda$$

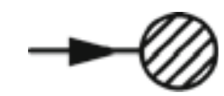
Richard Feynman
(1981-1988)

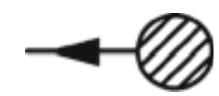
whereas for **QED**

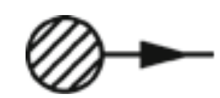
α  β $\Longrightarrow$ $\left(\frac{i}{\not{p} - m + i\varepsilon}\right)_{\beta\alpha}$

μ  ν $\Longrightarrow$ $\frac{-i\eta_{\mu\nu}}{p^2 + i\varepsilon}$

β, α  μ $\Longrightarrow$ $-ie\gamma^{\mu}_{\beta\alpha}.$

Incoming fermion: α  $\Longrightarrow$ $u_\alpha(\mathbf{p}, s)$

Incoming antifermion: α  $\Longrightarrow$ $\overline{v}_\alpha(\mathbf{p}, s)$

Outgoing fermion:  α $\Longrightarrow$ $\overline{u}_\alpha(\mathbf{p}, s)$

Outgoing antifermion:  α $\Longrightarrow$ $v_\alpha(\mathbf{p}, s)$

Incoming photon: μ  $\Longrightarrow$ $\varepsilon_\mu(\mathbf{p})$

Outgoing photon:  μ $\Longrightarrow$ $\varepsilon_\mu(\mathbf{p})^*$

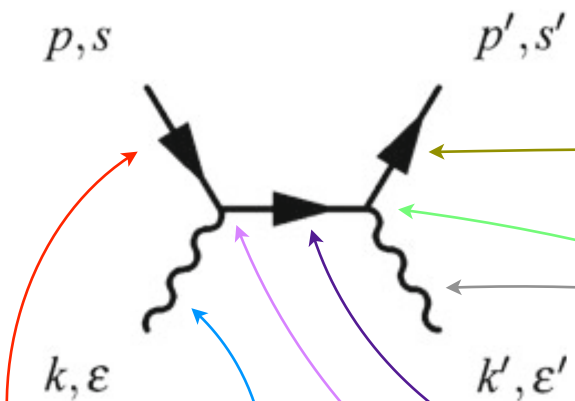
+ integration over internal momenta, **a delta function momentum conservation** at each **vertex**, a **factor of –1** for each fermion loop, and a **combinatorial factor**.

As an example, for **Compton scattering**

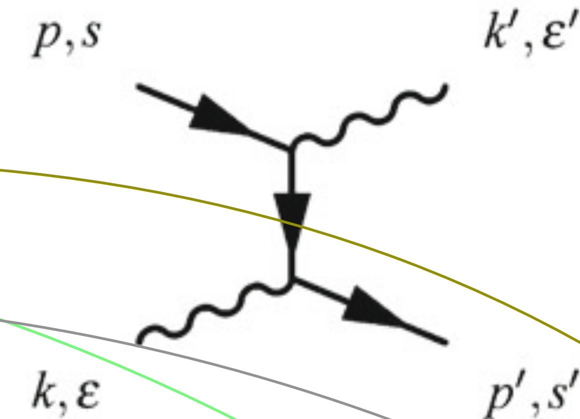
$$\gamma(k,\varepsilon) + e^-(p,s) \longrightarrow \gamma(k',\epsilon') + e^-(p',s')$$

the invariant amplitude at leading $\mathcal{O}(e^2)$ is given by

$$i\mathscr{M}_{i\to f} =$$



$+$



$$= u_\alpha(\mathbf{p},s)\varepsilon_\mu(\mathbf{k})(-ie\gamma^\mu_{\beta\alpha})\left(\frac{i}{\not{p}+\not{k}-m}\right)_{\sigma\beta}(-ie\gamma^\nu_{\lambda\sigma})\varepsilon'_\nu(\mathbf{k}')^*\overline{u}_\lambda(\mathbf{p}',s')$$

$$+ u_\alpha(\mathbf{p},s)\varepsilon'_\mu(\mathbf{k}')^*(-ie\gamma^\mu_{\beta\alpha})\left(\frac{i}{\not{p}-\not{k}-m}\right)_{\sigma\beta}(-ie\gamma^\nu_{\lambda\sigma})\varepsilon_\nu(\mathbf{k})\overline{u}_\lambda(\mathbf{p}',s')$$

$$= -ie^2\overline{u}(\mathbf{p}',s')\not{\varepsilon}'(\mathbf{k}')^*\frac{\not{p}+\not{k}+m}{(p+k)^2-m^2}\not{\varepsilon}(\mathbf{k})u(\mathbf{p},s)$$

$$-ie^2\overline{u}(\mathbf{p}',s')\not{\varepsilon}(\mathbf{k})\frac{\not{p}-\not{k}+m}{(p-k)^2-m^2}\not{\varepsilon}'(\mathbf{k}')^*u(\mathbf{p},s)$$

Remember:

$$\not{A} \equiv A_\mu\gamma^\mu$$

$$p^2 = p'^2 = m^2$$

$$k^2 = k'^2 = 0$$

$$k\cdot\varepsilon(\mathbf{k}) = k'\cdot\varepsilon(\mathbf{k}') = 0$$

In the **low energy limit** $\mathbf{p}^2, \mathbf{p}'^2, \mathbf{k}^2, \mathbf{k}'^2 \ll m^2$ the invariant amplitude is

$$i\mathscr{M}_{i\to f} = \frac{ie^2}{m}\Big[\varepsilon(\mathbf{k})\cdot\varepsilon'(\mathbf{k}')\Big]\overline{u}(\mathbf{p}',s')\frac{\not{k}}{|\mathbf{k}|}u(\mathbf{p},s)$$ (exercise)

If our experiment is **blind** to the **electron spin**, we have to **average** over the **incoming electron spin** and **sum** over the **spin of the outgoing electron**

$$\overline{|i\mathscr{M}_{i\to f}|^2} = \underbrace{\frac{1}{2}}_{\text{average over incoming } e^- \text{ spin}}\left(\frac{e^2}{m|\mathbf{k}|}\right)^2|\varepsilon(\mathbf{k})\cdot\varepsilon'(\mathbf{k}')^*|^2\underbrace{\sum_{s=\pm\frac{1}{2}}}_{\text{average over incoming } e^- \text{ spin}}\ \underbrace{\sum_{s'=\pm\frac{1}{2}}}_{\text{sum over outgoing } e^- \text{ spin}}|\overline{u}(\mathbf{p}',s')\not{k}u(\mathbf{p},s)|^2$$ (exercise)

$$= 4e^4|\varepsilon(\mathbf{k})\cdot\varepsilon'(\mathbf{k}')^*|^2$$

For an electron at **rest**, the differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{3e^4}{48\pi m^2}|\varepsilon(\mathbf{k})\cdot\varepsilon'(\mathbf{k}')^*|^2$$