Quantum Field Theory and Standard Model

Miguel Á. Vázquez-Mozo
Universidad de Salamanca

Taller de Altas Energías 2017, Benasque
An “ideal” list of topics

* The very basics of QFT
* Loops, divergences and renormalization
* What to ask from a “healthy” QFT
  - Lorentz invariance
  - Locality
  - Unitarity
  - Renormalizability?
* Symmetries and their breaking
* Gauge invariance
* Massive gauge fields
* Building the standard model
Schedule

* **Lectures**: Monday to Thursday (first week), from 9:00 to 10:00.

* **Tutorials**: 
  - Monday 4th, 15:30 to 16:30
  - Tuesday 5th, 16:30 to 17:30
  - Thursday 7th, 18:00 to 19:00

  **Tutor**: Ramon Miravitllas.
A sample of textbooks


A note about conventions:

* We use the “mostly minus” metric (a.k.a. West Coast metric):

\[
\eta_{\mu\nu} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

* Unless otherwise said, natural units are used throughout:

\[\hbar = c = 1\]

* We use Heaviside-Lorentz electromagnetic units:

\[
\nabla \cdot \mathbf{E} = \rho
\]

\[
\nabla \cdot \mathbf{B} = 0
\]

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}
\]

\[
\nabla \times \mathbf{B} = \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t}
\]

\[
\alpha = \frac{e^2}{4\pi} \quad \text{(fine structure constant)}
\]
Part I

From elementary particles to quantum fields
Elementary particles are studied through **scattering experiments**, typically

![Diagram of scattering experiment]

**Quantum mechanics**, even relativistic, is **not enough** to describe these **high energy** experiments...
Let us consider the relativistic quantum evolution of a localized, single-particle wave packet:

\[ H = \sqrt{p^2 + m^2} \]

\[
\psi(t, x) = e^{-it\sqrt{-\nabla^2 + m^2}} \delta(3)(x) = \int \frac{d^3 k}{(2\pi)^3} e^{i k \cdot x - it\sqrt{k^2 + m^2}}
\]

The integral can be regularized by \( t \to t - i\epsilon \), to give

\[
\psi(t, x) = -\frac{i}{4\pi^2 |x|} \int_{-\infty}^{\infty} dk \ e^{i k |x| - it\sqrt{k^2 + m^2}} = \frac{1}{2\pi^2 |x|} \int_{0}^{\infty} dk \ \sin(k |x|) e^{-it\sqrt{k^2 + m^2}}
\]

The probability \( |\psi(t, x)|^2 \) spills outside the light-cone

Causality is violated!
Let us consider the **relativistic** quantum evolution of a **localized, single-particle wave packet**:

\[
H = \sqrt{p^2 + m^2}
\]

\[
\psi(t, x) = e^{-it\sqrt{-\nabla^2 + m^2}} \delta(3)(x) = \int \frac{d^3k}{(2\pi)^3} e^{ik \cdot x - it\sqrt{k^2 + m^2}}
\]

\[
= -\frac{i}{4\pi^2 |x|} \int_{-\infty}^{\infty} dk \sin(k |x|) e^{-it\sqrt{k^2 + m^2}}
\]

What we are computing is the **propagator** of a relativistic particle:

\[
\psi(t, x) = \langle x | e^{-itH} | 0 \rangle \equiv G(t, x; 0, 0)
\]

Relativistic quantum mechanics propagates states **outside** the light cone

\[
G(t', x'; t, x) \neq 0 \text{ when } (t' - t)^2 - (x' - x)^2 < 0
\]

**Causality is violated!**
But when $t^2 - x^2 < 0$ there are frames in which $(t, x)$ happens before $(0, 0)$. Thus,

$$\psi(t, x) = \begin{cases} 
\langle x | e^{-itH} | 0 \rangle & \text{when } t^2 - x^2 > 0 \\
\langle x | e^{-itH} | 0 \rangle + \langle 0 | e^{itH} | x \rangle = 2\text{Re} \langle x | e^{-itH} | 0 \rangle & \text{when } t^2 - x^2 < 0 
\end{cases}$$

But since we have computed

$$\langle x | e^{-itH} | 0 \rangle = -\frac{i}{2\pi^2} \frac{m^2 t}{t^2 - x^2} K_2\left(\frac{im\sqrt{t^2 - x^2}}{t^2 - x^2}\right)$$

the result is

$$\psi(t, x) = -\frac{i}{2\pi^2} \frac{m^2 t}{t^2 - x^2} K_2\left(\frac{im\sqrt{t^2 - x^2}}{t^2 - x^2}\right) \theta(t^2 - x^2)$$

Causality is restored!
But when \( t^2 - x^2 < 0 \) there are \textbf{frames} in which \((t, x)\) \textbf{happens before} \((0, 0)\). Thus,

\[
\psi(t, x) = -\frac{i}{2\pi^2} \frac{m^2 t}{t^2 - x^2} K_2\left(im\sqrt{t^2 - x^2}\right) \theta(t^2 - x^2)
\]

\textbf{Causality is restored!}
But when \( t^2 - x^2 < 0 \) there are **frames** in which \((t, x)\) **happens before** \((0, 0)\). Thus,

\[
\psi(t, x) = \begin{cases} 
\langle x | e^{-i t H} | 0 \rangle & \text{when } t^2 - x^2 > 0 \\
\langle x | e^{-i t H} | 0 \rangle + \langle 0 | e^{i t H} | x \rangle = 2 \text{Re} \langle x | e^{-i t H} | 0 \rangle & \text{when } t^2 - x^2 < 0
\end{cases}
\]

But since we have computed

\[
\langle x | e^{-i t H} | 0 \rangle = -\frac{i}{2\pi^2} \frac{m^2 t}{t^2 - x^2} K_2 \left( i m \sqrt{t^2 - x^2} \right)
\]

the result is

\[
\psi(t, x) = -\frac{i}{2\pi^2} \frac{m^2 t}{t^2 - x^2} K_2 \left( i m \sqrt{t^2 - x^2} \right) \theta(t^2 - x^2)
\]

**Causality is restored!**
We have to allow particles travelling backward in time!!

Their wave functions are

\[ \psi(t, x)_\downarrow = \langle 0 | e^{itH} | x \rangle = \langle x | e^{-itH} | 0 \rangle^* = \psi(t, x)^*_\uparrow \]

Thus, under any global U(1) symmetry

\[ \psi(t, x)_\uparrow \rightarrow e^{iq\theta} \psi(t, x)_\uparrow \quad \text{and} \quad \psi(t, x)_\downarrow \rightarrow e^{-iq\theta} \psi(t, x)_\downarrow \]

these particles have opposite charges, \( q_\downarrow = -q_\uparrow \) (but the same mass! \( H_{\uparrow, \downarrow} = \sqrt{-\nabla^2 + m^2} \))

To restore causality we are forced to introduce antiparticles!!
We have to allow **particles travelling backward in time!!**

Their wave functions are

\[ \psi(t, x)_\downarrow = \langle 0 | e^{itH} | x \rangle = \langle x | e^{-itH} | 0 \rangle^* = \psi(t, x)_\uparrow \]

States moving backward in time can be reinterpreted as **negative frequency states** with reversed momentum, propagating **forward** in time:

\[
\psi(t, x)_\downarrow = \int \frac{d^3k}{(2\pi)^3} e^{-ik \cdot x + it\sqrt{k^2 + m^2}}
\]

\[ = \int \frac{d^3k}{(2\pi)^3} e^{i(-k) \cdot x - it(-\sqrt{k^2 + m^2})} \]

To **restore causality** we are forced to introduce **antiparticles!!**
Switching on **interactions**, charge conservation allows the creation of **particle-antiparticle pairs**, provided **enough energy** is available.

For example, localizing particle below their **Compton wavelength**

\[
\Delta x \sim \frac{1}{m} \quad \Delta x \Delta p \sim 1 \quad \Delta p \sim m \quad \Delta E \sim m
\]

and due to energy **quantum fluctuations** the creation of particle-antiparticle pairs **cannot be prevented**.

---

We have to **give up the single-particle description!**
Relativistic quantum mechanics is a **dead end** for high energy particle physics…

To handle many particles, **second quantization** seems the best approach, introducing **creation-annihilation operators** for particles with **on-shell momentum** $p$

\[
a(p), \ a(p)\dagger \quad p^2 = m^2 \quad \omega_p = \sqrt{p^2 + m^2} \quad [a(p), a(p')\dagger] = (2\pi)^3 (2\omega_p) \delta^{(3)}(p - p')
\]

\[
[a(p), a(p')] = [a(p)^\dagger, a(p')^\dagger] = 0
\]

Lorentz invariant (exercise)

(Multi-)particle states are obtained from the **Poincaré-invariant vacuum** $|0\rangle$

\[
|p\rangle = a(p)\dagger |0\rangle \quad \langle p|p'\rangle = (2\pi)^3 (2\omega_p) \delta^{(3)}(p - p')
\]

Lorentz invariant (exercise)

\[
|f\rangle = \int \left[ \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2\omega_{p_i}} \right] f(p_1, \ldots, p_n) a(p_1)^\dagger \cdots a(p_n)^\dagger |0\rangle
\]
Relativistic quantum mechanics is a **dead end** for high energy particle physics…

To handle many particles, **second quantization** seems the best approach, introducing **creation-annihilation operators** for particles with **on-shell momentum** $p$

\[
\begin{align*}
 a(p), & \quad a(p)\dagger \\
p^2 & = m^2 \\
\omega_p & = \sqrt{p^2 + m^2} \quad [a(p), a(p')\dagger] = (2\pi)^3 (2\omega_p)\delta^3(p - p') \\
[a(p), a(p')] & = [a(p)\dagger, a(p')\dagger] = 0
\end{align*}
\]

Lorentz invariant (exercise)

(Multi-)$\,$particle states are obtained from the **Poincaré-invariant vacuum** $|0\rangle$

\[
|p\rangle = a(p)\dagger |0\rangle \quad \langle p | p'\rangle = (2\pi)^3 (2\omega_p)\delta^3(p - p')
\]

Lorentz invariant (exercise)

\[
\mathcal{U}(\Lambda)|0\rangle = e^{-ia\cdot P} |0\rangle = |0\rangle \\
\mathcal{U}(\Lambda)a(p)\mathcal{U}(\Lambda)\dagger = a(\Lambda p) \\
\mathcal{U}(\Lambda)|p\rangle = |\Lambda p\rangle
\]

where $\mathcal{U}(\Lambda) \in \text{SO}(1,3)$

\[
1, \ldots, p_n a(p_1)\dagger \ldots a(p_n)\dagger |0\rangle
\]
Relativistic quantum mechanics is a **dead end** for high energy particle physics...

To handle many particles, **second quantization** seems the best approach, introducing **creation-annihilation operators** for particles with **on-shell momentum** $p$

\[
\begin{align*}
[a(p), a(p')^\dagger] &= (2\pi)^3 (2\omega_p) \delta^{(3)}(p - p') \\
[a(p), a(p')] &= [a(p)^\dagger, a(p')^\dagger] = 0
\end{align*}
\]

Lorentz invariant (exercise)

(Multi-)particle states are obtained from the **Poincaré-invariant vacuum** $|0\rangle$

\[
\begin{align*}
|p\rangle &= a(p)^\dagger |0\rangle \\
\langle p'|p\rangle &= (2\pi)^3 (2\omega_p) \delta^{(3)}(p - p') \\
\end{align*}
\]

Lorentz invariant (exercise)

\[
|f\rangle = \int \left[ \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2\omega_{p_i}} \right] f(p_1, \ldots, p_n) a(p_1)^\dagger \cdots a(p_n)^\dagger |0\rangle
\]
Free fields are linear combinations of creation-annihilation operators. E.g., for a free Hermitian scalar field

$$\phi(x) = \phi(x)\dagger$$

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left[ f(x, p)a(p) + f(x, p)^*a(p)\dagger \right]$$

Imposing the equations of motion,

$$(\Box + m^2)\phi(x) = 0$$

$$f(x, p) = e^{-i\omega_p t + ip\cdot x}$$

The free quantum field satisfies:

* **Equal-time** canonical commutation relations

$$[\phi(t, x), \dot{\phi}(t, x')] = i\delta^{(3)}(x - x'), \quad [\phi(t, x), \phi(t, x')] = [\dot{\phi}(t, x), \dot{\phi}(t, x')] = 0$$

* **Microcausality**

$$[\phi(x), \phi(x')] = 0 \quad \text{when} \quad (x - x')^2 < 0$$
Free fields are linear combinations of creation-annihilation operators. E.g., for a free Hermitian scalar field

$$\phi(x) = \phi(x)^\dagger$$  \hspace{1cm}  $$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left[ e^{-i\omega_p t + ip \cdot x} a(p) + e^{i\omega_p t - ip \cdot x} a(p)^\dagger \right]$$

Imposing the equations of motion,

$$\Box + m^2 \phi(x) = 0$$  \hspace{1cm}  $$f(x, p) = e^{-i\omega_p t + ip \cdot x}$$

The free quantum field satisfies:

* Equal-time canonical commutation relations

$$[\phi(t, x), \dot{\phi}(t, x')] = i\delta^{(3)}(x - x'), \quad [\phi(t, x), \phi(t, x')] = [\dot{\phi}(t, x), \dot{\phi}(t, x')] = 0$$

* Microcausality

$$[\phi(x), \phi(x')] = 0 \quad \text{when} \quad (x - x')^2 < 0$$
The many-particle Fock states diagonalize the free field Hamiltonian:

\[
H = \frac{1}{2} \int d^3x \left[ \dot{\phi}^2 + (\nabla \phi)^2 + m^2 \phi^2 \right]
\]

\[
\overset{(exercise)}{=} H = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \left[ a(p)\dagger a(p) + (2\pi)^3 \omega_p \delta^{(3)}(0) \right]
\]

\[
= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left[ \omega_p a(p)\dagger a(p) \right] + E_0
\]

**Subtracting** the (divergent) zero-point energy \( E_0 \)

\[
[a(p), a(p')\dagger] = (2\pi)^3 (2\omega_p) \delta^{(3)}(p - p')
\]

\[
H|p\rangle = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} a(k)\dagger a(k)a(p)\dagger|0\rangle = \omega_p |p\rangle
\]

\[
H|p_1, \ldots, p_n\rangle \equiv Ha(p_1)\dagger \ldots a(p_n)\dagger|0\rangle = \left( \sum_{i=1}^{n} \omega_{p_i} \right) |p_1, \ldots, p_n\rangle
\]

**Particles** are the low-lying excitations of quantum fields.
The **many-particle** Fock states **diagonalize** the free field **Hamiltonian**

\[ H = \frac{1}{2} \int d^3x \left[ \dot{\phi}^2 + (\nabla \phi)^2 + m^2 \phi^2 \right] \]

\[ \rightarrow \quad H = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \left[ a(p)^\dagger a(p) + (2\pi)^3 \omega_p \delta^{(3)}(0) \right] \]

\[ E_0 = \langle 0 | H | 0 \rangle = \sum_p \frac{1}{2} \omega_p \]

\[ = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left[ \omega_p a(p)^\dagger a(p) \right] + E_0 \]

**Subtracting** the (divergent) zero-point energy \( E_0 \)

\[ [a(p), a(p')^\dagger] = (2\pi)^3 (2\omega_p) \delta^{(3)}(p - p') \]

\[ H | p \rangle = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} a(k)^\dagger a(k) a(p)^\dagger | 0 \rangle = \omega_p | p \rangle \]

\[ H | p_1, \ldots, p_n \rangle \equiv Ha(p_1)^\dagger \ldots a(p_n)^\dagger | 0 \rangle = \left( \sum_{i=1}^n \omega_{p_i} \right) | p_1, \ldots, p_n \rangle \]

**Particles** are the **low-lying excitations** of **quantum fields**
A particle is characterized by a number of “Casimirs”:

\[
\begin{align*}
\text{Poincaré group:} & \\
& \begin{cases} 
\ P_\mu P^\mu = m^2 & \text{Mass} \\
\ W_\mu W^\mu = -m^2 s(s+1) & \text{Spin}
\end{cases}
\]

\[
W^\mu = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} J_{\nu\alpha} P_\beta
\]

\text{vector de Pauli-Lubański}

\text{_internal symmetry groups:} \quad \text{electric charge} \quad : \quad : \\

To do particle physics, we have to choose the appropriate interpolating field:

\* It transforms correctly (i.e., the right value for the “Casimirs”)

\* It creates the corresponding particle out of the vacuum:

\[
\langle 0 | \phi(x) | p \rangle \neq 0
\]
The $x$-dependence is **fixed** by the Poincaré invariance of the vacuum

$$\langle 0 | \phi(x) | p \rangle = \langle 0 | e^{iP \cdot x} \phi(0) e^{-iP \cdot x} | p \rangle = \langle 0 | \phi(0) | p \rangle e^{-ip \cdot x}$$

The fields can be **canonically normalized**, such that:

**Scalar field:** $\langle 0 | \phi(0) | p \rangle = 1$

**Dirac field:**

\[
\begin{cases}
\langle 0 | \psi_\alpha(0) | p, \sigma; 0 \rangle = u_{\alpha}^{(\sigma)} (p) \\
\langle 0 | \bar{\psi}_\alpha(0) | 0; p, \sigma \rangle = \bar{v}_{\alpha}^{(\sigma)} (p)
\end{cases}
\]

**Photon field:** $\langle 0 | A_\mu(0) | p, \lambda \rangle = \varepsilon_{\mu}^{(\lambda)} (p)$

**Any** properly normalized interpolating field does the job, provided it satisfies **microcausality**

$$[\phi(x), \phi(x')] = 0 \quad \text{when} \quad (x - x')^2 < 0$$

Borchers classes
The $x$-dependence is fixed by the Poincaré invariance of the vacuum

$$\langle 0| \phi(x) | p \rangle = \langle 0| e^{iP \cdot x} \phi(0) e^{-iP \cdot x} | p \rangle = \langle 0| \phi(0) | p \rangle e^{-ip \cdot x}$$

The fields can be canonically normalized, such that:

*Scalar field:* $\langle 0| \phi(0) | p \rangle = 1$

To describe scalar particles, instead of $\phi(x)$ we can also use

$$\Phi(x) = -\frac{1}{m^2} \square \phi(x)$$

$$\langle 0| \Phi(x) | p \rangle = -\frac{1}{m^2} \square \langle 0| \phi(x) | p \rangle = -\frac{1}{m^2} \square e^{-ip \cdot x} = e^{-ip \cdot x}$$

and

$$[\Phi(x), \Phi(x')] = \frac{1}{m^4} \square x \square x' [\phi(x), \phi(x')] = 0$$

for $(x - x')^2 < 0$
Still, to study particle physics we need to **introduce interactions**…

In interacting field theories, **particles** still emerge as **weakly coupled excitations:**

\[
\mathcal{L}(\phi, \partial \phi) = \mathcal{L}(\phi, \partial \phi)_{\text{free}} + \sum_i g_i \mathcal{O}_i(\phi, \partial \phi)
\]

where \(g_i\) is “small” and

\[
\langle \Omega | \phi(x) | p \rangle = \phi(p) e^{-i p \cdot x}
\]

Thus:

* Particles are identified by **quantizing** the **free theory**.
* Interactions are treated in **perturbation theory**.
Still, to study particle physics we need to introduce interactions…

In interacting field theories, particles still emerge as weakly coupled excitations:

\[ \mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (\gamma^\mu \partial_\mu - m) \psi - q A_\mu \bar{\psi} \gamma^\mu \psi \]

where \( g \) is "small" and dimensionless effective couplings.

**Examples**

**QED**: electrons+photons

\[ \mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (\gamma^\mu \partial_\mu - m) \psi - q A_\mu \bar{\psi} \gamma^\mu \psi \]

**QCD (high energies)**: quarks+gluons

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \sum_{f=1}^6 \bar{Q}_f (\gamma^\mu \partial_\mu - m_f) Q_f - g A_\mu^a \sum_{f=1}^6 \bar{Q}_f T^a \gamma^\mu Q_f \]

**QCD (low energies)**: pions (+nucleons)

\[ \mathcal{L} = \frac{1}{2} \text{tr} \left( \partial_\mu \pi \partial^\mu \pi \right) - \frac{1}{3 f_\pi^2} \text{tr} \left( \partial_\mu \pi \left[ \pi, [\pi, \partial^\mu \pi] \right] \right) + \ldots \]

* Interactions are treated in perturbation theory.
A **scattering** experiment is characterized by its initial (in) and final (out) multiparticle state:

\[ |p_1, p_2 \rangle_{\text{in}} \quad |q_1, q_2, \ldots, q_{n-1}, q_n \rangle_{\text{out}} \]

Both are **Heisenberg-picture** (i.e., time-independent) **states** in a very complicated **interacting theory**.

Our aim is to **compute** the **probability amplitude**:

\[ S(i \rightarrow f) = \text{out} \langle q_1, \ldots, q_n | p_1, p_2 \rangle_{\text{in}} \]
These states can also be seen as belonging to the **free, multiparticle Fock space**

\[
|p_1, p_2\rangle_{\text{in}} \quad |q_1, q_2, \ldots, q_{n-1}, q_n\rangle_{\text{out}}
\]

\[
|p_1, p_2\rangle, |q_1, q_2, \ldots, q_n\rangle \in \mathcal{F} \equiv \bigoplus_{n=0}^{\infty} \mathcal{H}_1 \otimes (\mathcal{H}_1)^{\otimes n} \otimes \mathcal{H}_1
\]

The scattering experiment is then described by the **S-matrix operator**

\[
S : \mathcal{F} \longrightarrow \mathcal{F}
\]

\[
S(i \rightarrow f) = \text{out} \langle q_1, \ldots, q_n | p_1, p_2 \rangle_{\text{in}} \equiv \langle q_1, \ldots, q_n | S | p_1, p_2 \rangle
\]

The **S-matrix** operator satisfies a number of **properties**:

* **Unitarity**: \( S^\dagger = S^{-1} \)

* **Lorentz invariance**: \( \mathcal{U}(\Lambda)S\mathcal{U}(\Lambda)^\dagger = S \) with \( \Lambda \in \text{SO}(1,3) \)

* \( \langle q_1, \ldots, q_n | S | p_1, p_2 \rangle \) is **analytic** in the external momenta.
The **S-matrix** is a kind of **holographic** quantity in **Minkowski** space-time: **in-** and **out-** states live on its **boundary**.

The **massive states** and **massless states** are depicted on the diagram.
The computation of the S-matrix in terms of the interacting field theory is done using the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula.

\[
\text{out} \langle q_1, \ldots, q_n | p_1, p_2 \rangle_{\text{in}} = \sum_{i=1}^{n} (2\pi)^3 (2\omega_{q_i}) \delta^{(3)}(q_i - p_1) \text{out} \langle q_1, \ldots, \hat{q}_i, \ldots, q_n | p_2 \rangle_{\text{in}}
\]

\[
+i Z^{-1/2} \int d^4 x e^{-i p_1 \cdot x} (\Box + m^2) \text{out} \langle q_1, \ldots, q_n | \phi(x) | p_2 \rangle_{\text{in}}
\]

Symbolically:

\[
\begin{pmatrix}
q_1 \\
q_2 \\
q_3 \\
\vdots \\
q_n
\end{pmatrix}
= \sum_{i=1}^{n}
\begin{pmatrix}
p_1 \\
p_2 \\
q_1 \\
q_2 \\
q_3 \\
\vdots \\
q_n
\end{pmatrix} + (\Box + m^2)
\begin{pmatrix}
\phi(x) \\
p_2 \\
q_1 \\
q_2 \\
q_3 \\
\vdots \\
q_n
\end{pmatrix}
\]

FT
Iterating the procedure, we **trade** all incoming and outgoing particles by **time-ordered** field insertions:

\[
\langle q_1, \ldots, q_n | S | p_1, p_2 \rangle = \text{disconnected terms}
\]

\[
+i(Z^{-1/2})^{n+2} \int d^4x_1 d^4x_2 e^{-ip_1 x_1 - ip_2 x_2} \int d^4y_1 \ldots d^4y_n e^{iq_1 y_1 + \ldots + iq_n y_n}
\]

\[
\times (\Box + m^2)_{x_1} (\Box + m^2)_{x_2} (\Box + m^2)_{y_1} \ldots (\Box + m^2)_{y_n} \langle \Omega | T[\phi(x_1)\phi(x_2)\phi(y_1) \ldots \phi(y_n)] | \Omega \rangle
\]

S-matrix amplitudes are computed in terms of **time-ordered** (amputated) **correlation functions**

\[
G(x_1, \ldots, x_n) = \langle \Omega | T[\phi(x_1) \ldots \phi(x_n)] | \Omega \rangle
\]

which can be computed in **perturbation theory**.

**Feynman diagrammatics**
Iterating the procedure, we trade all incoming and outgoing particles by time-ordered field insertions:

\[
\langle q_1, \ldots, q_n | S | p_1, p_2 \rangle = \text{disconected terms}
\]

\[
+ i \left( Z^{-1/2} \right)^{n+2} \int d^4 x_1 d^4 x_2 e^{-i p_1 x_1 - i p_2 x_2} \int d^4 y_1 \ldots d^4 y_n e^{i q_1 y_1 + \ldots + i q_n y_n}
\]

\[
\times (\Box + m^2)_{x_1} (\Box + m^2)_{x_2} (\Box + m^2)_{y_1} \ldots (\Box + m^2)_{y_n} \langle \Omega | T[\phi(x_1)\phi(x_2)\phi(y_1)\ldots\phi(y_n)] | \Omega \rangle
\]

Inverse free propagators

S-matrix amplitudes are computed in terms of **time-ordered** (amputated) **correlation functions**

\[
G(x_1, \ldots, x_n) = \langle \Omega | T[\phi(x_1)\ldots\phi(x_n)] | \Omega \rangle
\]

which can be computed in **perturbation theory**.

**Feynman diagrammatics**
We can isolate **nontrivial scattering** in the $S$-matrix by writing

$$S = 1 + iT$$

so the matrix elements have the structure

$$\langle q_1, \ldots, q_n | S | p_1, p_2 \rangle = \langle q_1, \ldots, q_n | p_1, p_2 \rangle + \langle q_1, \ldots, q_n | iT | p_1, p_2 \rangle$$

$$= \langle q_1, \ldots, q_n | p_1, p_2 \rangle + (2\pi)^4 \delta^{(4)} \left( p_1 + p_2 - \sum_{i=1}^{n} q_i \right) iM_{i\rightarrow f}$$

In terms of the **invariant amplitude**, the **differential cross section** is given by

$$d\sigma = \frac{|iM_{i\rightarrow f}|^2}{4\omega_{p_1} \omega_{p_2} |\mathbf{v}_1 - \mathbf{v}_2|} (2\pi)^4 \delta^{(4)} \left( p_1 + p_2 - \sum_{i=1}^{n} q_i \right) \prod_{k=1}^{n} \frac{d^3q_k}{(2\pi)^3} \frac{1}{2\omega_{q_k}}$$

(observer dependent) (phase space factor)
We can isolate **nontrivial scattering** in the S-matrix by writing

\[ S = 1 + iT \]

so the matrix elements have the structure

\[ |p_1, p_2 i = |q_1, \ldots, q_n i + \frac{(2\pi)^4}{4} (p_1 + p_2 - \sum_{i=1}^{n} q_i) \prod_{k=1}^{n} \frac{d^3q_k}{(2\pi)^3} \frac{1}{2\omega_{q_k}} \]

In the case of **particle decay**, the **decay width** is given by

\[ d\Gamma = \frac{|i\mathcal{M}_{i\rightarrow f}|^2}{2\omega_p} (2\pi)^4 \delta(4) \left( p_1 + p_2 - \sum_{i=1}^{n} q_i \right) \prod_{k=1}^{n} \frac{d^3q_k}{(2\pi)^3} \frac{1}{2\omega_{q_k}} \]

In terms of the **invariant amplitude**, the **differential cross section** is given by

\[ d\sigma = \frac{|i\mathcal{M}_{i\rightarrow f}|^2}{4\omega_{p_1}\omega_{p_2}|v_1 - v_2|} (2\pi)^4 \delta(4) \left( p_1 + p_2 - \sum_{i=1}^{n} q_i \right) \prod_{k=1}^{n} \frac{d^3q_k}{(2\pi)^3} \frac{1}{2\omega_{q_k}} \]

\[ \text{observer dependent} \]

\[ \text{phase space factor} \]
The perturbative computation of correlation functions in momentum space is carried out using Feynman diagrammatics.

For a $\phi^4$ scalar theory, the **Feynman rules** are:

\[
\begin{align*}
\text{Incoming fermion:} & \quad \alpha \rightarrow & \quad \Rightarrow & \quad u_\alpha(p,s) \\
\text{Incoming antifermion:} & \quad \alpha \rightarrow & \quad \Rightarrow & \quad \bar{v}_\alpha(p,s) \\
\text{Outgoing fermion:} & \quad \alpha \rightarrow & \quad \Rightarrow & \quad \bar{u}_\alpha(p,s) \\
\text{Outgoing antifermion:} & \quad \alpha \rightarrow & \quad \Rightarrow & \quad v_\alpha(p,s) \\
\text{Incoming photon:} & \quad \mu \rightarrow & \quad \Rightarrow & \quad \varepsilon_\mu(p) \\
\text{Outgoing photon:} & \quad \mu \rightarrow & \quad \Rightarrow & \quad \varepsilon_\mu(p)^* \\
\end{align*}
\]

+ integration over internal momenta, a delta function momentum conservation at each vertex, a factor of $-1$ for each fermion loop, and a combinatorial factor.
As an example, for **Compton scattering**

\[ \gamma(k, \varepsilon) + e^-(p, s) \rightarrow \gamma(k', \varepsilon') + e^-(p', s') \]

the invariant amplitude at leading \( \mathcal{O}(e^2) \) is given by

\[
i \mathcal{M}_{i \rightarrow f} = u_\alpha(p, s) \varepsilon_\mu(k)(-ie\gamma^\mu_{\beta\alpha})(\frac{i}{\not{p} + k - m})_{\sigma\beta}(-ie\gamma^\nu_{\lambda\sigma})\varepsilon'_{\nu}(k')^* \bar{u}_\lambda(p', s')
\]

\[\]

\[
= u_\alpha(p, s)\varepsilon_\mu(k')( -ie\gamma^\mu_{\beta\alpha})(\frac{i}{\not{p'} - k' - m})_{\sigma\beta}(-ie\gamma^\nu_{\lambda\sigma})\varepsilon_{\nu}(k)\bar{u}_\lambda(p', s')
\]

\[\]

\[
= -ie^2\bar{u}(p', s')^\nu(k')^* \frac{\not{p'} + \not{k'} + m}{(p + k')^2 - m^2} \hat{\gamma}(k)u(p, s)
\]

\[\]

\[-ie^2\bar{u}(p', s')^\nu(k) \frac{\not{p} - \not{k} + m}{(p - k)^2 - m^2} \hat{\gamma}'(k')^* u(p, s)\]

---

**Remember:**

\[ \mathcal{A} \equiv A_\mu \gamma^\mu \]

\[ p^2 = p'^2 = m^2 \]

\[ k^2 = k'^2 = 0 \]

\[ k \cdot \varepsilon(k) = k' \cdot \varepsilon(k') = 0 \]
In the **low energy limit** \( p^2, p'^2, k^2, k'^2 \ll m^2 \) the invariant amplitude is

\[
i.\mathcal{M}_{i\rightarrow f} = \frac{ie^2}{m} \left[ \varepsilon(k) \cdot \varepsilon'(k') \right] \bar{u}(p', s') \frac{k}{|k|} u(p, s)
\]

If our experiment is **blind** to the **electron spin**, we have to **average** over the **incoming electron spin** and **sum** over the **spin of the outgoing electron**

\[
|i.\mathcal{M}_{i\rightarrow f}|^2 = \frac{1}{2} \left( \frac{e^2}{m|k|} \right)^2 |\varepsilon(k) \cdot \varepsilon'(k')|^2 \sum_{s=\pm \frac{1}{2}} \sum_{s'=\pm \frac{1}{2}} |\bar{u}(p', s') \cdot u(p, s)|^2
\]

\[
= 4e^4 |\varepsilon(k) \cdot \varepsilon'(k')|^2
\]

For an electron at **rest**, the differential cross section is

\[
\frac{d\sigma}{d\Omega} = \frac{3e^4}{48\pi m^2} |\varepsilon(k) \cdot \varepsilon'(k')|^2
\]