- **Renormalizability?**

For many years, **renormalizability** was considered to be a **must** for any decent QFT…

To make **predictions**, we would need to add an **infinite number of local counterterms**, including an **infinite number of couplings** to be physically determined.

dimensionfull couplings

$$\mathscr{L} = \mathscr{L}_{\dim\leq 4} + \sum_{i=1}^{\infty} g_i \mathscr{O}_i \qquad\qquad (\dim \mathscr{O}_i > 4)$$

However, these **higher-dimensional** ($>$ 4) **operators**, are **suppressed** by some characteristic **energy scale** M

dimensionless couplings

$$\mathscr{L} = \mathscr{L}_{\dim\leq 4} + \sum_{i=1}^{\infty} \frac{\lambda_i}{M^{\dim \mathscr{O}_i - 4}} \mathscr{O}_i$$

$$\mathscr{L} = \mathscr{L}_{\dim\leq 4} + \sum_{i=1}^{\infty} \frac{\lambda_i}{M^{\dim\mathscr{O}_i - 4}} \mathscr{O}_i \qquad\qquad (\dim\mathscr{O}_i > 4)$$

Considering processes at **energies** $E \ll M$ and working at a **given accuracy**, only a **finite number** of these operators are **important**

$\Downarrow$

The theory is **predictive** and can be seen as an **effective field theory** valid for **energies well below** the scale M

- **Fermi** four-fermion theory of **weak interactions** $\longrightarrow$ $M \sim G_F^{-\frac{1}{2}}$

$$\mathscr{L}_{\text{int}} = -\frac{G_F}{\sqrt{2}} J^\mu J^\dagger_\mu$$

- **Pion effective Lagrangian** $\longrightarrow$ $M \sim f_\pi$

$$\mathscr{L} = \frac{1}{2}\text{tr}\left(\partial_\mu \boldsymbol{\pi} \partial^\mu \boldsymbol{\pi}\right) - \frac{1}{3f_\pi^2}\text{tr}\Big(\partial_\mu \boldsymbol{\pi}[\boldsymbol{\pi}, [\boldsymbol{\pi}, \partial^\mu \boldsymbol{\pi}]]\Big) + \ldots$$

Part IV

Symmetries and their breaking

In **classical mechanics**, **Noether's theorem** states the existence of a **conserved charge** for each continuous symmetry of the Lagrangian

Emmy Noether
(1882-1935)

$$q_i(t) \to q_i'(t,\varepsilon)$$
$$L(q',\dot{q}') = L(q,\dot{q}) + \frac{d}{dt}f(q,\varepsilon)$$

$$\Longrightarrow \quad \dot{Q} = 0 \quad \text{with} \quad Q = \frac{\partial L}{\partial \dot{q}_i}\delta_\varepsilon q_i - f(q,\delta\varepsilon)$$

Similarly, in **classical field theory**, continuous symmetries are associated with **conserved currents**

$$\phi(x) \to \phi(x) + \delta_\varepsilon\phi(x)$$
$$\delta_\varepsilon \mathscr{L} = \partial_\mu K^\mu$$

$$\Longrightarrow \quad \partial_\mu J^\mu = 0 \quad \text{where} \quad J^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu\phi)}\delta_\varepsilon\phi - K^\mu$$

In this case, the associated **conserved charge** is defined by

$$Q \equiv \int d^3x\, J^0(t,\mathbf{x}) \quad \Longrightarrow \quad \dot{Q} = 0$$

At the level of the **quantum theory**, symmetries lead to **identities** for **correlation functions**.

Consider a field theory with **action** $S[\phi]$ **invariant** under infinitesimal continuous transformations

$$\delta_\varepsilon \phi = \varepsilon F(\phi)$$

To compute the current, we use Noether's trick: assume that ε depends on the **position**. Then

$$\begin{aligned} S[\phi + \delta_\varepsilon \phi] &= S[\phi] + \int d^4x\, \partial_\mu \varepsilon(x) J^\mu(x) \\ &= S[\phi] - \int d^4x\, \varepsilon(x) \partial_\mu J^\mu(x) \end{aligned}$$

integration by parts

Now, if the field is **on-shell**, the **action** should remain invariant, even for a local ε

$$\int d^4x\, \varepsilon(x)\partial_\mu J^\mu(x) = 0 \quad \Longrightarrow \quad \partial_\mu J^\mu(x) = 0$$

Let us look now at the **quantum theory** and in particular at the **correlator**

$$\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle = \frac{1}{Z}\int \mathscr{D}\phi\,\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)e^{iS[\phi]}$$

and make the following **change of variables** in the functional integral

$$\phi(x)\longrightarrow\phi'(x)=\phi(x)+\varepsilon(x)F(\phi)$$

Under this,

$$S[\phi]\longrightarrow S[\phi]-\int d^4x\,\varepsilon(x)\partial_\mu J^\mu(x)$$

$$\mathscr{O}_a(x)\longrightarrow\mathscr{O}_a(x)+\delta_\varepsilon\mathscr{O}_a(x)$$

Let us further **assume** that the change of variables **does not** induce a field-dependent **Jacobian**

$$\mathscr{D}\phi'=\mathscr{D}\phi$$

If this is **not** the case, we have **anomalies**

$$\boxed{S[\phi] \longrightarrow S[\phi] - \int d^4x\, \varepsilon(x)\partial_\mu J^\mu(x)}$$

$$\boxed{\mathscr{O}_a(x) \longrightarrow \mathscr{O}_a(x) + \delta_\varepsilon \mathscr{O}_a(x)}$$

$$\boxed{\mathscr{D}\phi' = \mathscr{D}\phi}$$

Now, since this is a **mere** change of variables, it **does not change the value** of the functional **integral**!

$$\int \mathscr{D}\phi\, \mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)e^{iS[\phi]} = \int \mathscr{D}\phi'\, \mathscr{O}_1'(x_1)\ldots\mathscr{O}_n'(x_n)e^{iS[\phi']}$$

$$\int \mathscr{D}\phi'\, \mathscr{O}_1'(x_1)\ldots\mathscr{O}_n'(x_n)e^{iS[\phi']}\bigg|_\varepsilon = 0$$

This last expression gives the **Ward identity** (restoring $\hbar$)

$$\frac{i}{\hbar}\int d^4x\, \varepsilon(x)\partial_\mu^{(x)}\langle\Omega|T\Big[J^\mu(x)\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x)\Big]|\Omega\rangle = \sum_{a=1}^{n}\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\delta_\varepsilon\mathscr{O}_a(x_a)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle$$

For the case $\mathscr{O}_a(x) = \mathbf{1}$

$$\int d^4x\, \varepsilon(x)\partial_\mu\langle J^\mu(x)\rangle = 0 \quad \longrightarrow \quad \boxed{\partial_\mu\langle J^\mu(x)\rangle = 0}$$

The Noether current is conserved quantum mechanically

$$S[\phi] \longrightarrow S[\phi] - \int d^4x\, \varepsilon(x)\partial_\mu J^\mu(x)$$

$$\mathscr{O}_a(x) \longrightarrow \mathscr{O}_a(x) + \delta_\varepsilon \mathscr{O}_a(x)$$

$$\mathscr{D}\phi' = \mathscr{D}\phi$$

In the case of having a **nontrivial Jacobian**

$$\mathscr{D}\phi' = \left[1 + \int d^4x\, \varepsilon(x)\, \mathscr{J}(x)\right] \mathscr{D}\phi$$

the **Ward identity** gets an **additional term**

$$\partial_\mu \langle J^\mu(x)\rangle = i\hbar\, \mathscr{J}(x)$$

Conservation is **spoiled** quantum mechanically and we have an **anomaly**.

N... **change the value** of the fu...

$$\ldots_1) \ldots \mathscr{O}'_n(x_n) e^{iS[\phi']}$$

$$\ldots\Big|_\varepsilon = 0$$

This last expression gives the **Ward identity** (restoring $\hbar$)

$$\frac{i}{\hbar}\int d^4x\, \varepsilon(x)\partial^{(x)}_\mu \langle\Omega|T\Big[J^\mu(x)\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x)\Big]|\Omega\rangle = \sum_{a=1}^n \langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\delta_\varepsilon\mathscr{O}_a(x_a)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle$$

For the case $\mathscr{O}_a(x) = \mathbf{1}$

$$\int d^4x\, \varepsilon(x)\partial_\mu\langle J^\mu(x)\rangle = 0 \quad \Longrightarrow \quad \boxed{\partial_\mu\langle J^\mu(x)\rangle = 0}$$

The Noether current is conserved quantum mechanically

In a **quantum theory**, the **conserved charges** generate the continuous **symmetry** acting on the **Hilbert space**

$$\mathscr{U}(\alpha) = e^{i\alpha^a Q^a} \longrightarrow \begin{array}{l} \mathscr{U}(\alpha) H \mathscr{U}(\alpha)^\dagger = H \\ [Q^a, H] = 0 \end{array}$$

assuming no anomalies!

On the Hilbert space, the **symmetry** admits **two** possible **implementations**:

- **Weyl-Wigner realization**: the ground state remains invariant under the symmetry

$$\mathscr{U}(\alpha)|\Omega\rangle = |\Omega\rangle \longrightarrow Q^a|\Omega\rangle = 0$$

Then, the spectrum is **classified** in **multiplets** transforming in **irreducible representations** of the symmetry group.

E.g., the **hydrogen atom**:

$$|\alpha, j, m\rangle \xrightarrow{\mathrm{SO}(3)} |\alpha, j, m'\rangle = \sum_{m=-j}^{j} \mathscr{D}^{(j)}_{m'm}(\theta, \phi)|\alpha, j, m\rangle$$

total spin (orbital+spin+nuclear)

rotation matrices

In fact, the system has a **larger** SO(4) **symmetry** generated by **rotations** and the **Laplace-Runge-Lenz vector**.

- **Nambu-Goldstone realization**: the ground state is **not** preserved by the symmetry:

$$\mathscr{U}(\alpha)|\Omega\rangle \neq |\Omega\rangle \quad \longrightarrow \quad Q^a|\Omega\rangle \neq 0 \quad \text{(at least for some } a\text{'s)}$$

Yoichiro Nambu
(1921-2015)

This fact has **important consequences** for the theory. Let us consider a theory with a single conserved charge

$$Q(t) = \int d^3x\, J^0(x) \quad \longrightarrow \quad \dot{Q}(t) = 0$$

Jeffrey Goldstone
(b. 1933)

and given an observable $\mathscr{O}(x)$ we compute

$$\begin{aligned}\langle\Omega|[Q(t),\mathscr{O}(0)]|\Omega\rangle &= \int d^3x\, \langle\Omega|[J^0(t,\mathbf{x}),\mathscr{O}(0)]\Omega\rangle \\ &= \int d^3x \Big[\langle\Omega|J^0(t,\mathbf{x})\mathscr{O}(0)|\Omega\rangle - \langle\Omega|\mathscr{O}(0)J^0(t,\mathbf{x})|\Omega\rangle\Big]\end{aligned}$$

We insert next a **basis** of **four-momentum eigenstates** $|n\rangle$

$$\boxed{J^0(x) = e^{iP\cdot x}J^0(0)e^{-iP\cdot x}}$$

$$\begin{aligned}\langle\Omega|[Q(t),\mathscr{O}(0)]|\Omega\rangle &= \sum_n \int d^3x \Big[\langle\Omega|J^0(x)|n\rangle\,\langle n|\mathscr{O}(0)|\Omega\rangle - \langle\Omega|\mathscr{O}(0)|n\rangle\,\langle\Omega|J^0(x)|\Omega\rangle\Big] \\ &= \sum_n \int d^3x \Big[e^{-iP_n\cdot x}\langle\Omega|J^0(0)|n\rangle\,\langle n|\mathscr{O}(0)|\Omega\rangle - e^{iP_n\cdot x}\langle\Omega|\mathscr{O}(0)|n\rangle\,\langle n|J^0(0)|\Omega\rangle\Big]\end{aligned}$$

$$\langle\Omega|[Q(t),\mathscr{O}(0)]|\Omega\rangle=\sum_n\int d^3x\left[e^{-iP_n\cdot x}\langle\Omega|J^0(0)|n\rangle\,\langle n|\mathscr{O}(0)|\Omega\rangle-e^{iP_n\cdot x}\langle\Omega|\mathscr{O}(0)|n\rangle\,\langle n|J^0(0)|\Omega\rangle\right]$$

The integral can be explicitly **evaluated** to give

$$\begin{aligned}\langle\Omega|[Q(t),\mathscr{O}(0)]|\Omega\rangle=\sum_n(2\pi)^3\delta^{(3)}(\mathbf{P}_n)\Big[&e^{-iE_nt}\langle\Omega|J^0(0)|n\rangle\,\langle n|\mathscr{O}(0)|\Omega\rangle\\&-e^{iE_nt}\langle\Omega|\mathscr{O}(0)|n\rangle\,\langle n|J^0(0)|\Omega\rangle\Big]\end{aligned}$$

However, when the **symmetry** is **not preserved** by the **vacuum**, $Q(t)|\Omega\rangle\neq 0$

$$\sum_n\delta^{(3)}(\mathbf{P}_n)\Big[e^{-iE_nt}\langle\Omega|J^0(0)|n\rangle\,\langle n|\mathscr{O}(0)|\Omega\rangle-e^{iE_nt}\langle\Omega|\mathscr{O}(0)|n\rangle\,\langle n|J^0(0)|\Omega\rangle\Big]\neq 0$$

Now, since $\dot{Q}(t)=0$ we can take the **time derivative** to write

$$\sum_n E_n\delta^{(3)}(\mathbf{P}_n)\Big[e^{-iE_nt}\langle\Omega|J^0(0)|n\rangle\,\langle n|\mathscr{O}(0)|\Omega\rangle+e^{iE_nt}\langle\Omega|\mathscr{O}(0)|n\rangle\,\langle n|J^0(0)|\Omega\rangle\Big]=0$$

$$\boxed{\sum_n \delta^{(3)}(\mathbf{P}_n)\Big[e^{-iE_nt}\langle\Omega|J^0(0)|n\rangle\,\langle n|\mathscr{O}(0)|\Omega\rangle - e^{iE_nt}\langle\Omega|\mathscr{O}(0)|n\rangle\,\langle n|J^0(0)|\Omega\rangle\Big] \neq 0}$$

$$\boxed{\sum_n E_n\delta^{(3)}(\mathbf{P}_n)\Big[e^{-iE_nt}\langle\Omega|J^0(0)|n\rangle\,\langle n|\mathscr{O}(0)|\Omega\rangle + e^{iE_nt}\langle\Omega|\mathscr{O}(0)|n\rangle\,\langle n|J^0(0)|\Omega\rangle\Big] = 0}$$

Since both equations involve both **positive** and **negative frequencies**, they can be satisfied only if there **exists** a state $|m\rangle$ such that

$$\begin{array}{c}\langle\Omega|J^0(0)|m\rangle \neq 0 \\ \langle m|\mathscr{O}(0)|\Omega\rangle \neq 0\end{array} \quad \text{and} \quad E_m\delta^{(3)}(\mathbf{P}_m) = 0 \;\Longrightarrow\; E_m(\mathbf{P}_m = \mathbf{0}) = 0$$

This is the content of the **Goldstone theorem**:

Whenever a symmetry **generator** is **broken by the vacuum**, a **state exists** with the following properties:

- It is **massless** and has **zero spin**.
- It is **created** by the **current** from the vacuum $\langle m|J^\mu(x)|\Omega\rangle \neq 0$
- It has the **same quantum numbers** as the conserved **current**.

This state is called a **Nambu-Goldstone boson**.

Pions are a typical example of **Goldstone bosons**: let us look at **QCD** with two flavors:

$$\mathscr{L} = -\frac{1}{2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \overline{u}\left(i\not{D} - m_u\right)u + \overline{d}\left(i\not{D} - m_d\right)d$$

In the **chiral limit** ($m_u = m_d = 0$), the theory has a **global** $\mathbf{SU(2)}_L\times\mathbf{SU(2)}_R$ **symmetry**

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix} \longrightarrow M_L \begin{pmatrix} u_L \\ d_L \end{pmatrix} \qquad \begin{pmatrix} u_R \\ d_R \end{pmatrix} \longrightarrow M_R \begin{pmatrix} u_R \\ d_R \end{pmatrix} \qquad M_L, M_R \in \mathrm{SU}(2)$$

At **low energies**, the dynamics of **QCD** produces a **condensate**:

$$\langle \overline{q}q \rangle = \langle \overline{q}_L q_R + \overline{q}_R q_L \rangle \sim \Lambda^3_{\mathrm{QCD}}$$

In this vacuum, the $\mathbf{SU(2)}_L\times\mathbf{SU(2)}_R$ symmetry is **broken** according to

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R = \mathrm{SU}(2)_V \times \mathrm{SU}(2)_A \longrightarrow \mathrm{SU}(2)_V$$

$$J^\mu_a = (\overline{u}, \overline{d})\gamma^\mu \frac{\sigma_a}{2} \begin{pmatrix} u \\ d \end{pmatrix}$$ (preserved)

$$J^\mu_{a,5} = (\overline{u}, \overline{d})\gamma^\mu \gamma_5 \frac{\sigma_a}{2} \begin{pmatrix} u \\ d \end{pmatrix}$$ (**broken**)

The **axial current** creates **pions** (i.e., **pseudo** Goldstone bosons) out of the vacuum

$$\langle \Omega | J^\mu_{a5}(x) | \pi^b(\mathbf{p}) \rangle = -if_\pi \delta^{ab} p^\mu e^{-i\omega_{\mathbf{p}}t + i\mathbf{p}\cdot\mathbf{x}} \neq 0$$

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$$\begin{pmatrix} u_L \\ d_L \end{pmatrix} \longrightarrow M_L \begin{pmatrix} u_L \\ d_L \end{pmatrix} \qquad \begin{pmatrix} u_R \\ d_R \end{pmatrix} \longrightarrow M \qquad \mathrm{U}(2)$$

$$\mathrm{SU}(2)_V : \begin{cases} q_L \longrightarrow M q_L \\ q_R \longrightarrow M q_R \end{cases}$$

$$\mathrm{SU}(2)_A : \begin{cases} q_L \longrightarrow M' q_L \\ q_R \longrightarrow M'^{-1} q_R \end{cases}$$

$$M, M' \in \mathrm{SU}(2)$$

At **low energies**, the dynamics of **QCD** produces a

$$\langle \overline{q}q \rangle = \langle \overline{q}_L q_R + \overline{q}_R q_L \rangle \sim \Lambda^3_{\mathrm{QCD}}$$

In this vacuum, the $\mathbf{SU(2)}_L\times\mathbf{SU(2)}_R$ symmetry is **broken** according to

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R = \mathrm{SU}(2)_V \times \mathrm{SU}(2)_A \longrightarrow \mathrm{SU}(2)_V$$

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Pions are a typical example of **Goldstone bosons**: let us look at **QCD** with two flavors:

$$\mathscr{L} = -\frac{1}{2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \overline{u}\Big(i\not{D}$$

$$\mathrm{SU}(2)_V : \begin{cases} q_L \longrightarrow M q_L \\ q_R \longrightarrow M q_R \end{cases}$$

$$\mathrm{SU}(2)_A : \begin{cases} q_L \longrightarrow M' q_L \\ q_R \longrightarrow M'^{-1} q_R \end{cases}$$

$$M, M' \in \mathrm{SU}(2)$$

In the **chiral limit** ($m_u = m_d = 0$), the theory ha **etry**

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix} \longrightarrow M_L \begin{pmatrix} u_L \\ d_L \end{pmatrix} \qquad \begin{pmatrix} u_R \\ d_R \end{pmatrix} \longrightarrow M \qquad \mathrm{U}(2)$$

At **low energies**, the dynamics of **QCD** produces a

$$\langle \overline{q}q \rangle = \langle \overline{q}_L q_R + \overline{q}_R q_L \rangle \sim \Lambda^3_{\mathrm{QCD}}$$

In this vacuum, the **SU(2)*L*×SU(2)*R*** symmetry is **broken** according to

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R = \mathrm{SU}(2)$$

$$m_u, m_d \neq 0 \Longrightarrow m_\pi \neq 0$$

$$J^\mu_a = (\overline{u}, \overline{d})\gamma^\mu \frac{\sigma_a}{2}\begin{pmatrix} u \\ d \end{pmatrix} \quad \text{(preserved)}$$

$$J^\mu_{a,5} = (\overline{u}, \overline{d})\gamma^\mu \gamma_5 \frac{\sigma_a}{2}\begin{pmatrix} u \\ d \end{pmatrix} \quad \textbf{(broken)}$$

The **axial current** creates **pions** (i.e., **pseudo** Goldstone bosons) out of the vacuum

$$\langle \Omega | J^\mu_{a5}(x) | \pi^b(\mathbf{p}) \rangle = -i f_\pi \delta^{ab} p^\mu e^{-i\omega_{\mathbf{p}} t + i\mathbf{p}\cdot\mathbf{x}} \neq 0$$

Part V

Gauge theories

Gauge invariance is the **prize** we pay to describe a **massless spin-one** field in way compatible with **Lorentz invariance** and **locality**

Let us write the **wave function** of a **photon** with momentum *p*. **Locality** and **Lorentz invariance**, leads to the Ansatz

$$A_\mu(x) = \epsilon_\mu(p) e^{-ip\cdot x} \qquad \text{with} \qquad p^2 = 0$$

As it stands, this contains **four** independent **polarizations**, while the **real photon** only has **two**. We can impose transversality:

$$p^\mu \epsilon_\mu(p) = 0$$

but we **still** have **tree polarizations**.

To **get rid** of the unwanted one, we have to impose **gauge invariance**:

$$\boxed{\epsilon_\mu(p) \quad \text{and} \quad \epsilon_\mu(p) + \lambda p_\mu \quad \text{represent the same state}}$$

With this we are left with just **two transverse polarizations**!

Gauge invariance is not a symmetry, but a **redundancy**!

Ordinary symmetries transform a physical state into a different one, e.g.

$$|\alpha,j,m\rangle \xrightarrow{\mathrm{SO}(3)} |\alpha,j,m'\rangle = \sum_{m=-j}^{j} \mathscr{D}^{(j)}_{m'm}(\theta,\phi)|\alpha,j,m\rangle$$

Gauge invariance, however, does not **change** the physical state itself, just the **label**

$$|\mathrm{phys}\rangle \xrightarrow{\mathscr{G}} |\mathrm{phys}'\rangle \sim |\mathrm{phys}\rangle$$

Thus, the **Hilbert space** of **physical states** is **smaller** than the "naive" Hilbert space of the theory

$$\mathscr{H}_{\mathrm{phys}} = \mathscr{H}/\mathscr{G}$$

As a class of states in the Hilbert space, **physical states** are **gauge invariant**

$$|\psi\rangle_{\mathrm{phys}} \in \mathscr{H}_{\mathrm{phys}} \quad \Longrightarrow \quad \delta_{\mathrm{gauge}}|\psi\rangle_{\mathrm{phys}} = 0$$

This **eliminates** from the spectrum the **spurious states** introduced to preserve Lorentz invariance and locality.

The **zero mass** of the **gauge field** is **crucial** for gauge invariance:

Alexandru Proca
(1897-1955)

$$S_{\text{Proca}} = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu - e j_\mu A^\mu \right) \quad \text{with} \quad \partial_\mu j^\mu = 0$$

Under a **gauge transformation** $\delta A_\mu = \partial_\mu \epsilon$

$$\delta S_{\text{Proca}} = m^2 \int d^4x \, A^\mu \partial_\mu \epsilon \neq 0$$

We can write the **equations of motion**

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = e j^\nu$$

Now, if we take the **divergence** of this equation,

$$\partial_\nu \partial_\mu F^{\mu\nu} + m^2 \partial_\nu A^\nu = e \partial_\nu j^\nu \quad \Longrightarrow \quad \partial_\mu A^\mu = 0$$

The Lorenz (**transversality**) condition allows the elimination of the temporal polarization

$$\Downarrow$$

A **massive** gauge field has **three polarizations** (one longitudinal + two transverse)

Gauge invariance, however, can always be **faked**… Let us introduce a new **U(1) scalar field**, $U(x) = [U(x)^*]^{-1}$

Ernst Stückelberg
(1905-1984)

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2e^2}(D_\mu U)^\dagger D^\mu U - ej_\mu A^\mu$$

where $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative.

The theory is **invariant** under **gauge transformations**

$$A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu \xi(x) \qquad\qquad U(x) \longrightarrow e^{-ie\xi(x)}U(x)$$

To see that this theory is **equivalent** to the Proca Lagrangian, we **fix the gauge** to the **unitary gauge**

$$U(x) = 1$$

so the **gauge fixed Lagrangian** is given by

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2e^2}(ieA_\mu)^\dagger(ieA^\mu) - ej_\mu A^\mu = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu - ej_\mu A^\mu \equiv \mathscr{L}_{\text{Proca}}$$

Moral: any theory without gauge invariant can be seen as a gauge-fixed gauge theory!

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2e^2}(D_\mu U)^\dagger D^\mu U - e j_\mu A^\mu$$

In fact, the **breaking of gauge invariance** in the **unitary gauge** $U(x) = 1$ is similar to **spontaneous symmetry breaking**.

The corresponding **Nambu-Goldstone bosons** can be identified by writing

$$U(x) = e^{\frac{ie}{m}\pi(x)}$$

Derivative couplings

In terms of the new field, the Lagrangian reads

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}\left(A_\mu + \frac{1}{m}\partial_\mu\pi\right)\left(A^\mu + \frac{1}{m}\partial^\mu\pi\right) - e j_\mu A^\mu$$

and the **gauge transformations** read

$$A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu\xi(x) \qquad\qquad \pi(x) \longrightarrow \pi(x) - m\xi(x)$$

The **counting** of degrees of freedom **matches** the ones of the original theory:

Three polarizations of a **massive** gauge field

Stückelberg

$A_\mu \longrightarrow$ two **transverse** polarizations

$\pi \longrightarrow$ one **longitudinal** polarization

We can try the **Stückelberg trick** with a SU(2) "**toy** standard model":

$$\mathscr{L} = -\frac{1}{2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + M^2\mathrm{Tr}\left(A_\mu A^\mu\right) + i\overline{\Psi}_L \not{D}\Psi_L + i\overline{\Psi}_R \not{\partial}\,\Psi_R - m\Big(\overline{\Psi}_L\Psi_R + \overline{\Psi}_R\Psi_L\Big)$$

For $M = m = 0$, the theory is **invariant** under **chiral** SU(2) gauge transformations

$$\Psi_L(x) \longrightarrow g(x)\Psi_L(x)$$

$$\Psi_R(x) \longrightarrow \Psi_R(x)$$

$$A_\mu(x) \longrightarrow g(x)A_\mu(x)g(x)^{-1} - \frac{1}{ig_{\mathrm{YM}}}g(x)\partial_\mu g(x)^{-1}$$

where

$$g(x) = e^{i\chi(x)} \in \mathrm{SU}(2)$$

However, in the presence of **masses**, gauge invariance is **broken**

$$\delta\mathscr{L} = \frac{2M^2}{g_{\mathrm{YM}}}\mathrm{Tr}\left(A^\mu D_\mu\chi\right) - im\Big(\overline{\Psi}_L\chi\Psi_R - \overline{\Psi}_R\chi\Psi_L\Big) \neq 0$$

$$\mathscr{L} = -\frac{1}{2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + M^2\mathrm{Tr}\left(A_\mu A^\mu\right) + i\overline{\Psi}_L \not{D}\Psi_L + i\overline{\Psi}_R \not{\partial}\,\Psi_R - m\Big(\overline{\Psi}_L\Psi_R + \overline{\Psi}_R\Psi_L\Big)$$

Introducing now a scalar field $U(x) \in \mathrm{SU}(2)$

$$\boxed{D_\mu U = \partial_\mu U - ig_{\mathrm{YM}}A_\mu U}$$

$$\mathscr{L} = -\frac{1}{2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) - \frac{M^2}{g_{\mathrm{YM}}^2}\mathrm{Tr}\left[(U^\dagger D_\mu U)(U^\dagger D_\mu U)\right]$$
$$+ i\overline{\Psi}_L \not{D}\Psi_L + i\overline{\Psi}_R \not{\partial}\,\Psi_R - m\Big(\overline{\Psi}_L U\Psi_R + \overline{\Psi}_R U^\dagger\Psi_L\Big)$$

which is now **invariant** under the **chiral** SU(2) gauge transformations

$$\Psi_L(x) \longrightarrow g(x)\Psi_L(x)$$
$$\Psi_R(x) \longrightarrow \Psi_R(x)$$
$$A_\mu(x) \longrightarrow g(x)A_\mu(x)g(x)^{-1} - \frac{1}{ig_{\mathrm{YM}}}g(x)\partial_\mu g(x)^{-1}$$
$$U(x) \longrightarrow g(x)U(x)$$

with $g(x) = e^{i\chi(x)} \in \mathrm{SU}(2)$

The original theory is **restored** in the **unitary gauge** $U(x) = \mathbf{1}$

$$U^\dagger D_\mu U \xrightarrow{U=\mathbf{1}} -ig_{\mathrm{YM}}A_\mu$$

$$\mathscr{L} = -\frac{1}{2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) - \frac{M^2}{g_{\mathrm{YM}}^2}\mathrm{Tr}\left[(U^\dagger D_\mu U)(U^\dagger D_\mu U)\right]$$
$$+i\overline{\Psi}_L \not{D}\Psi_L + i\overline{\Psi}_R \not{\partial}\,\Psi_R - m\left(\overline{\Psi}_L U\Psi_R + \overline{\Psi}_R U^\dagger \Psi_L\right)$$

This theory, however, has a **problem**: it **violates unitarity** at a scale

$$\Lambda \sim \frac{M}{g_{\mathrm{YM}}}$$

It should be **completed** in the **UV**…but how?

Technicolor?

The **Stückelberg field** emerges as the **Goldstone boson** resulting from **chiral symmetry breaking** of strongly interacting "technifermions"

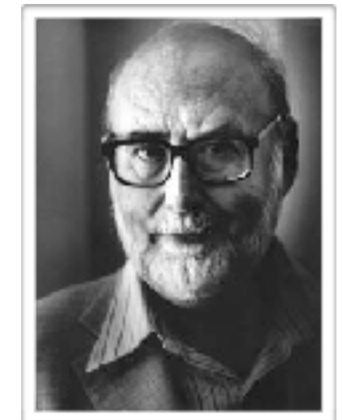
$$\langle\overline{\psi}\psi\rangle \sim \Lambda_{\mathrm{TC}}^3 \quad \longrightarrow \quad U(x)$$

Nature seems to have chosen the **Brout-Englert-Higgs mechanism**.

Let us add the **gauge invariant**, self-interaction **potential**

$$V(U^{\dagger}U) = \frac{\lambda}{4}\left(\frac{M}{g_{\mathrm{YM}}}\right)^4\left[\frac{1}{2}\mathrm{Tr}\left(U^{\dagger}U\right) - 1\right]^2$$

Robert Brout (1928-2011)



François Englert (b. 1932)

Peter Higgs (b. 1929)

and **include** a **new excitation**

Higgs field

$$U(x) = U_0(x)\left[1 + \frac{g_{\mathrm{YM}}}{\sqrt{2}M}h(x)\right]$$

Stückelberg mode

$$U_0(x) \in \mathrm{SU}(2)$$

$$U(x) \notin \mathrm{SU}(2)$$

In the **unitary gauge**, we **get rid** of the **Stückelberg mode** but we still have the **Higgs** field

Stückelberg mode → longitudinal components of massive vectors

Higgs particle → new physical excitation

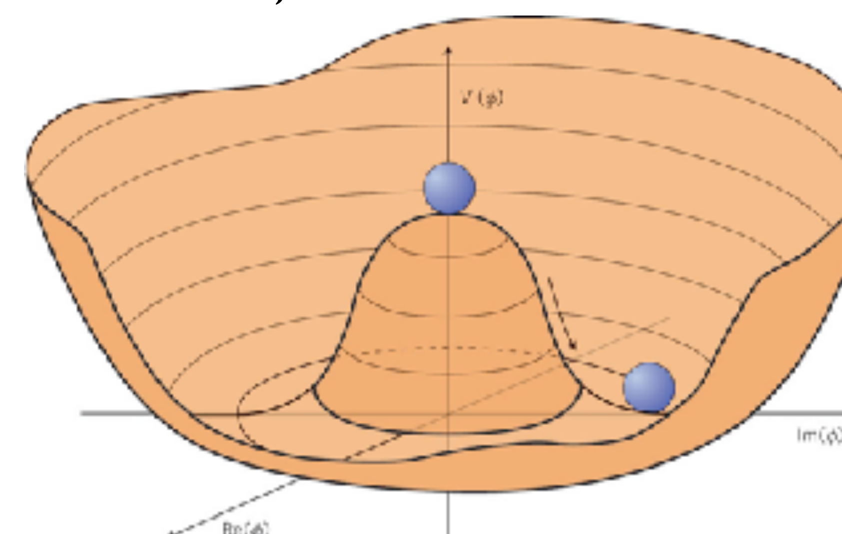
To be explicit, we **parametrize** $U(x)$ as

$$\boxed{U(x) = U_0(x)\left[1 + \frac{g_{\rm YM}}{\sqrt{2}M}h(x)\right]}$$

$$U(x) = \frac{g_{\rm YM}}{M}\left(\begin{array}{cc} \varphi^{0*} & \varphi^{+} \\ -\varphi^{+*} & \varphi^{0}\end{array}\right) \qquad \Phi = \left(\begin{array}{c} \varphi^{+} \\ \varphi^{0}\end{array}\right)$$

SU(2) doublet

In terms of the **Higgs doublet**, the potential has a more familiar form,



$$V(\Phi) = \frac{\lambda}{4}\left(\Phi^\dagger\Phi - \frac{M^2}{g_{\rm YM}^2}\right)^2$$

At the **bottom** of the potential,

$$U(x) = \mathbf{1} \qquad \langle\Phi\rangle = \left(\begin{array}{c} 0 \\ \frac{\nu}{\sqrt{2}}\end{array}\right)$$

with $\nu = \dfrac{\sqrt{2}M}{g_{\rm YM}}$

Excitations around this vacuum are **parametrized** by

$$\Phi(x) = \frac{1}{\sqrt{2}}U_0(x)\left(\begin{array}{c} 0 \\ \nu + h(x)\end{array}\right)$$

Stückelberg mode ⟵ "angular" excitation

"radial" excitation ⟶ Higgs particle

Higgs-gauge field coupling

Higgs self interaction

Expanding

$$\mathscr{L} = -\frac{1}{2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) - \frac{M^2}{g_{\mathrm{YM}}^2}\mathrm{Tr}\left[(U^\dagger D_\mu U)(U^\dagger D_\mu U)\right] - \frac{\lambda}{4}\left(\frac{M}{g_{\mathrm{YM}}}\right)^4\left[\frac{1}{2}\mathrm{Tr}\left(U^\dagger U\right) - 1\right]^2$$

$$+i\overline{\Psi}_L \not{D}\Psi_L + i\overline{\Psi}_R \not{\partial}\,\Psi_R - m\left(\overline{\Psi}_L U \Psi_R + \overline{\Psi}_R U^\dagger \Psi_L\right)$$

Yukawa couplings

to **second order** in the Higgs field $h(x)$, we find the **mass** of the **Higgs particle**

$$m_H = \nu\sqrt{\frac{\lambda}{2}} = \frac{M\sqrt{\lambda}}{g_{\mathrm{YM}}}$$

For the **real standard model**, until 2012, high energy experiments had only detected the Stückelberg mode (a.k.a., the longitudinal components of the W, Z gauge bosons)

With the discovery of the Higgs, the non-Stückelberg, **“radial” mode** was finally observed.

Thank you