# Problems – Lattice QCD, TAE, Benasque, Sept. 2017

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## 1. Difference operators

Show that the "symmetric" difference operator

$$\frac{1}{2a}\left(\psi(x+\mu)-\psi(x-\mu)\right) \tag{1.1}$$

also goes to $\partial_\mu\psi(x)$ in the continuum limit, $a\to 0$. Show that a possible discretization of the free, massless fermion action using this discetization of the partial derivative $\partial_\mu$ can be written as

$$S_{\rm free}=a^4\sum_x\frac{1}{2a}\left(\overline{\psi}(x)\gamma_\mu\psi(x+\mu)-\overline{\psi}(x+\mu)\gamma_\mu\psi(x)\right)\ . \tag{1.2}$$

## 2. Scalar field on the lattice

Consider a complex scalar field $\phi$ with lattice action

$$S=a^4\sum_{x,\mu}\frac{1}{a^2}\left(-\phi^\dagger(x)\phi(x+\mu)-\phi^\dagger(x+\mu)\phi(x)+2\phi^\dagger(x)\phi(x)\right)+a^4\sum_x m^2\phi^\dagger(x)\phi(x)\ .$$

Show that the action can be written as

$$S=a^4\sum_x\left(-\phi^\dagger(x)\Box\phi(x)+m^2\phi^\dagger(x)\phi(x)\right)\ ,$$

with a suitable definition of the laplacian on the lattice. Show that the propagator takes the form (setting $a=1$ for the rest of this problem)

$$G(x)=\int_{-\pi}^{\pi}\frac{d^4p}{(2\pi)^4}\,\frac{e^{ipx}}{2\sum_\mu(1-\cos p_\mu)+m^2}\ .$$

Carry out the integral over $p_4$ (take $t>0$), and show that the result can be written as

$$G(\vec{x},t)=\int_{-\pi}^{\pi}\frac{d^3p}{(2\pi)^3}\,\frac{e^{i\vec{p}\cdot\vec{x}-\omega t}}{2\sinh\omega}\ ;$$

find the expression for $\omega$. Restoring the lattice spacing $a$, show that in the continuum limit we recover $\omega = \sqrt{m^2 + \vec{p}^2}$.

## 3. Gluon propagator

Find the free gluon propagator on the lattice, from the plaquette action, for instance in Feynman gauge.

## 4. Species doublers

Show that the $\gamma$ matrices $\tilde{\gamma}_\mu = \gamma_\mu \cos(a\overline{p}_\mu)$, with

$$\overline{p} \in \{(0,0,0,0), (\pi/a, 0, 0, 0), \dots\} ,$$

as we defined in the lecture, are unitarily equivalent to the original $\gamma_\mu$.

## 5. Wilson fermions

Start with the free Wilson-fermion action,

$$\begin{aligned} S &= a^4 \sum_{x,\mu} \frac{1}{2a} \Big( \overline{\psi}(x)\gamma_\mu \psi(x+\mu) - \overline{\psi}(x+\mu)\gamma_\mu \psi(x) \Big) \\ &\quad + a^4 \sum_{x,\mu} \frac{r}{2a} \Big( 2\overline{\psi}(x)\psi(x) - \overline{\psi}(x)\psi(x+\mu) - \overline{\psi}(x+\mu)\psi(x) \Big) + a^4 \sum_x m\overline{\psi}(x)\psi(x) \ . \end{aligned}$$

Couple this free theory to lattice gauge fields (in a gauge invariant way), and derive the Feynman rules involving fermions, *i.e.*, the free fermion propagator and the $\psi\overline{\psi}A^n_\mu$ vertices, in momentum space. We have inserted a parameter $r \sim O(1)$ that helps keeping track of the effects of the Wilson term. Then derive an expression for the fermion self-energy at one loop at zero external momentum and for $m = 0$. With a regulator respecting chiral symmetry, this would vanish. Show that it indeed it vanishes for $r = 0$, but that it does not with Wilson fermions, *i.e.*, for $r \neq 0$, and thus that there is a linear divergence in this case.

## 6. Currents

Find the conserved vector current for the symmetry $\psi \to e^{i\alpha}\psi$, $\overline{\psi} \to \overline{\psi}e^{-i\alpha}$ on the lattice. Attempt to do the same thing for the axial current by considering $\psi \to e^{i\alpha\gamma_5}\psi$, $\overline{\psi} \to \overline{\psi}e^{i\alpha\gamma_5}$, and show that there is a conserved axial current for naive fermions (for $m = 0$), but not for Wilson fermions [1].

Note that the conserved currents on the lattice are point-split. Often, one uses a strictly local current instead in numerical computations, which is not conserved, and thus suffers renormalization. For instance, if the vector current is denoted by $V_\mu$, one has in the continuum limit that

$$V_\mu^{\rm cont} = Z_V V_\mu^{\rm local}\ .$$

How might you find $Z_V$, for instance, in perturbation theory?

## 7. $SU(N)$ Haar measure

Parametrizing an element of $SU(N)$ as $U = U(\vec{\alpha})$, we defined the Haar measure for $SU(N)$ (up to a normalization constant) from the metric on the group manifold

$$g_{k\ell} = \frac{1}{2}\,{\rm tr}\left(\frac{\partial U}{\partial\alpha_k}\frac{\partial U^\dagger}{\partial\alpha_\ell}\right)$$

as

$$dU = \sqrt{\det g}\prod_{k=1}^{N^2-1} d\alpha_k\ .$$

Show that this measure is invariant under a general reparametrization of the group manifold (and thus under gauge transformations). For $N = 2$, parametrizing $U = \sigma + i\vec{\tau}\cdot\vec{\pi}$ with $\sigma^2 + \vec{\pi}^2 = 1$, show that we recover

$$dU = \frac{1}{\sqrt{1-\vec{\pi}^2}}\, d^3\pi\ .$$

## 8. Lattice domain-wall fermions

Consider the five-dimensional free Wilson–Dirac equation [2, 3]

$$\frac{1}{2}\sum_{A=1}^{5}\left(\gamma_A(\psi(X+A)-\psi(X-A)) - 2\psi(X) + \psi(X+A) + \psi(X-A)\right) + M\psi(X) = 0\ ,$$

where $X = (x, s)$, $s$ is the coordinate in the fifth direction, and the index $A$ stands for the pair $(\mu, 5)$, with $\mu$ the usual four-dimensional index. We work in units in which $a = 1$, and the 5th dimension is restricted to $s \ge 0$.

Consider solutions of the form $\psi(p, s) = \psi_\pm(p)u_\pm(s)$, where we transitioned to four-dimensional momentum space, and requiring that $\sum_\mu i\gamma_\mu \sin(p_\mu)\psi_\pm(p) = 0$ and $\gamma_5\psi_\pm(p) = \pm\psi_\pm(p)$. Find the normalizable solutions, depending on $M$ and $p$.

## 9. Ginsparg–Wilson fermions

Consider a chiral transformation of the following form:

$$\delta\psi = T_a\hat{\gamma}_5\psi \equiv T_a\gamma_5(1 - a\mathcal{O})\psi\ , \qquad \delta\overline{\psi} = \overline{\psi}\gamma_5 T_a\ , \tag{9.1}$$

where $\mathcal{O}$ is an operator of mass-dimension one, and $T_a$ is a hermitian flavor-symmetry generator. We would like to choose $a\mathcal{O}\psi = 0$ for $p = 0$, but $a\mathcal{O}\psi = 2$ for $p$ equal to $(\pi/a, 0, 0, 0)$, *etc.* This would provide a momentum-dependent definition of chiral symmetry that allows us to give a mass to the doublers. For small physical momenta $p$, for which $a\mathcal{O} \sim ap$, this amounts to an order-$a$ modification of chiral symmetry, which will vanish in the continuum limit.

We would like to maintain the structure of the full chiral flavor symmetry group on the lattice, which means that we should require

$$\hat{\gamma}_5^2 = (\gamma_5(1 - a\mathcal{O}))^2 = 1 \ ,$$

or

$$\{\mathcal{O}, \gamma_5\} = a\mathcal{O}\gamma_5\mathcal{O} \ . \tag{9.2}$$

We say that $\mathcal{O}$ obeys the Ginsparg–Wilson relation [4]. The massless Wilson–Dirac operator $D_{\rm W}$ does not satisfy this relation, with, for $ap \sim 0$, $aD_{\rm W} \sim ia\not{p}$, and $aD_{\rm W} = 2n$, $n = 2,\ 4,\ 6,\ 8$ for $p = (\pi/a, 0, 0, 0)$, *etc.* We can modify this definition such that it precisely satisfies the Ginsparg–Wilson relation by defining the overlap Dirac operator [5]

$$\begin{aligned} aD_{\rm ov} &= 1 - \frac{A}{\sqrt{A^\dagger A}} \ , \qquad A = 1 - aD_{\rm W} \ , \\ D_{\rm W} &= \frac{1}{2}\left(\gamma_\mu(\partial^+_\mu + \partial^-_\mu) - a\partial^+_\mu\partial^-_\mu\right) \ , \end{aligned}$$

where $\partial^\pm_\mu$ is the forward/backward difference operator on the lattice, and a sum over $\mu$ is implied. Give an expression in momentum space and show that the (free) overlap Dirac operator satisfies Eq. (9.2). Inspect the values of $aD_{\rm ov}$ for momenta $\overline{p}$ in the set (4.1). Argue that, while this operator is not strictly local, it is local in the sense that, seen as a matrix $D_{\rm overlap}(x, y)$, it falls exponentially with distance $|x - y|$, with a decay constant of order one in lattice units.

Show that the action

$$S_{\rm ov} = \sum_{x,y} \overline{\psi}(x) D_{\rm overlap}(x, y)\psi(y)$$

is invariant under the symmetry (9.1) [6].[1] This theory of free, undoubled fermions with a lattice chiral symmetry can be gauged by simply gauge the Wilson–Dirac operator from which it is constructed. Show that its eigenvalues lie on a circle in the complex plane centered at 1 and with radius 1, and that for small momenta $D_{\rm overlap} = i\not{p}$. [Hint: use that $D^\dagger_{\rm overlap} = \gamma_5 D_{\rm overlap}\gamma_5$.] The issue of locality of the gauged overlap operator (necessary for the theory to be acceptable as a discretization of QCD) is not trivial, see Refs. [7, 8].

[1] This paper also shows how the $U(1)_A$ anomaly is recovered in this construction. Note that the fermion measure, $\prod_x d\psi(x)d\overline{\psi}(x)$ is not invariant under (9.1) when the $T_a$ are omitted.

Finally, note that the relation (9.2) for the overlap operator can be recast as

$$\{D^{-1}_{\rm overlap}(x,y),\gamma_5\} = a\gamma_5\delta(x-y)\ ,$$

with here $\delta(x-y)$ the Kronecker delta. This relation shows that the fermion propagator anti-commutes everywhere with $\gamma_5$ except at the point $x=y$.

# References


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