Cosmology: Inflation

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Outline

Cosmology

- Cosmology has many open questions
- A way to find new fundamental physics, new particles, ...using the sky as a detector
- An advantage: it can go up to extremely high energy (early times)

Open questions

- Cosmology has many open questions related to high energy physics:
  - What is Dark Matter?
  - What is Dark Energy?
  - Why there is an initial asymmetry between matter and antimatter?
  - Are there extra light particles (extra neutrinos, axions...)?
  - What is the mass of the neutrino?
  - What happened at the beginning of the radiation dominated era, at extremely high energy?
  - How do we explain the Cosmic Microwave Background (CMB) and its fluctuations?

Inflation: some bibliography

- E. W. Kolb & M. Turner, *The Early Universe*, Series “Frontiers in Physics", Westview Press , 1994. (book)
- V. Mukhanov, *Physical Foundations of Cosmology*, Cambridge University Press, 2005. (book)
- S. Weinberg, *Cosmology* : Oxford University Press; 1 edition (2008). (book)
- Mukhanov & Brandenberger, “Theory of Cosmological Perturbations", Physics Reports 215, n.5-6, 1992 (review)
- A. Riotto, hep-ph/0210162 (shorter review, pedagogical)
- Some recent papers

Assumptions & Conventions

- I will use +,-,-,-
- Natural units $\hbar = c = \kappa_B = 1$
- Familiarity with basic GR
- Knowledge of FLRW cosmology: Matter, Radiation
- For example: Recombination and Last Scattering:

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- For example: Recombination and Last Scattering:
  - at $\boxed{T_{LS} \approx 0.2eV}$ photons decouple from primordial plasma
  - We observe them at $\boxed{T_0 \approx 2.7K \approx 2.3 \cdot 10^{-4} eV}$ (Cosmic Microwave Background, CMB)
  - $\boxed{1+z_{LS} \equiv T_{LS}/T_0 \approx 1100}$

FLRW metric

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- More in general (curvature):

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right] \qquad k = \pm 1$$

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$$T^{\mu}{}_{\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}$$

($T^{\mu}{}_{\nu}$ stress-energy tensor, ρ energy density, p pressure)

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- $\boxed{H \equiv \frac{\dot{a}}{a}}$ Hubble rate

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- Such an angle is seen in the CMB (1st peak)

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- Root of the problem: ρ diluted too fast!

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$\Longrightarrow$ N=ln $\left(\frac{T_0}{H_0}\right) + \ln\left(\frac{H_I}{T_E}\right)$

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- If at least $N \gtrsim 60 - 70$, solves Horizon problem

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- If we consider a generic observed length today λ_*:

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- With CMB we see scales down to $\ln\left(\frac{\lambda_*}{H_0^{-1}}\right) \approx 5 - 10$
- "Small" visible window of scales (*e.g.*, $55 \lesssim N \lesssim 65$)

Curvature problem

- If we reintroduce k in the metric

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right] \qquad k = \pm 1$$

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- If Inflation a bit longer $\Omega_{k,0} \to 0$ (generic prediction of Inflation)

Scalar field models

- We need a “fluid" with no preferred direction and $w \simeq -1$
- Simple candidate: a scalar field:

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- Gradients decay $\frac{\nabla^2\phi}{a^2} \to 0$

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- Useful relations: $\boxed{\epsilon = \frac{\dot\phi^2}{2H^2M_P^2}}$, $\boxed{\dot H = -\epsilon H^2}$

Number of efolds

- Check for a given potential $V(\phi)$ if slow-roll conditions are satisfied for at least $N \gtrsim 60 - 70$ efolds.
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- Integrating backwards we find $N(\phi)$ and $\phi(N)$

Number of efolds

- A given observed scale length today λ_* corresponds to a given value of ϕ_*.
- In fact $\lambda*$ corresponds to N_* via:

$$N_* \simeq 67 + \ln\left(\frac{H_I}{T_E}\right) - \ln\left(\frac{\lambda_*}{H_0^{-1}}\right)$$

- $\boxed{\lambda_* \leftrightarrow N_* \leftrightarrow \phi_*}$

Observational constraints

- As we will see: Inflation produces a spectrum of metric perturbations around FLRW:

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   Not yet observed: $r \lesssim 0.1$

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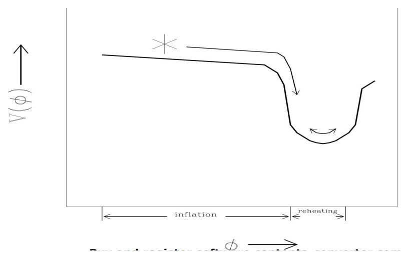

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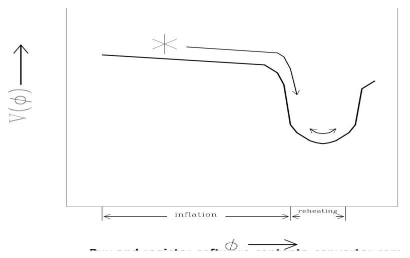

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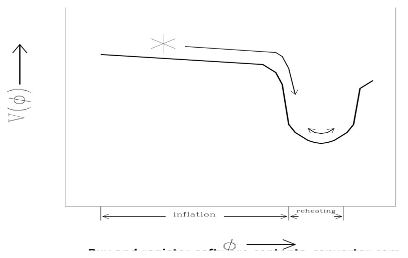

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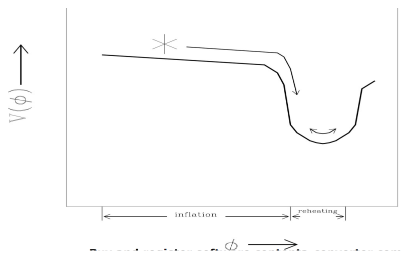

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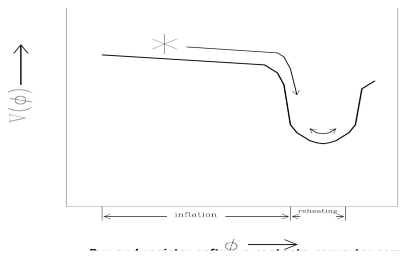

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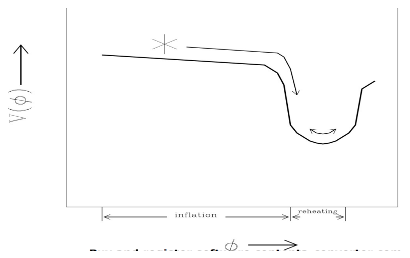

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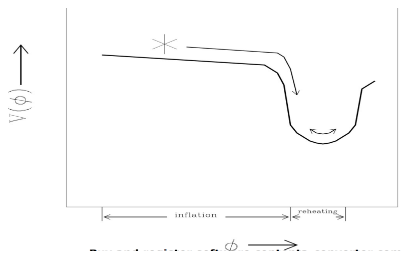

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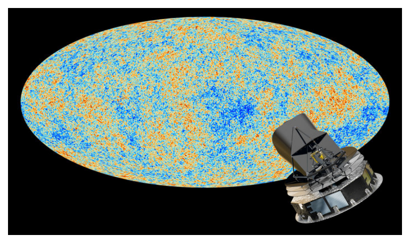
- Quantum fluctuations around $\phi(t) \implies$ Density fluctuations, CMB fluctuations, GW...

CMB from Planck satellite - Power Spectrum



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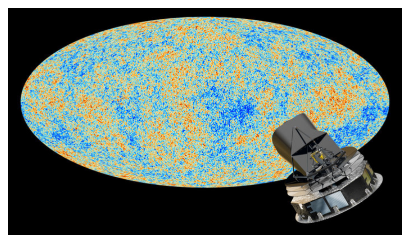
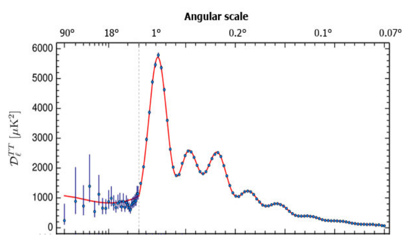
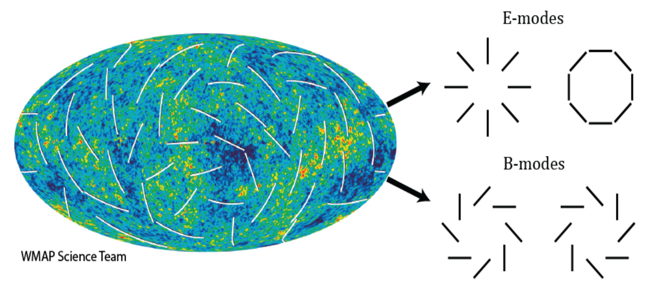




Primordial spectrum: $P_\zeta(k) \propto A^2 k^{n_s - 1}$

CMB Polarization

- CMB light is linearly polarized



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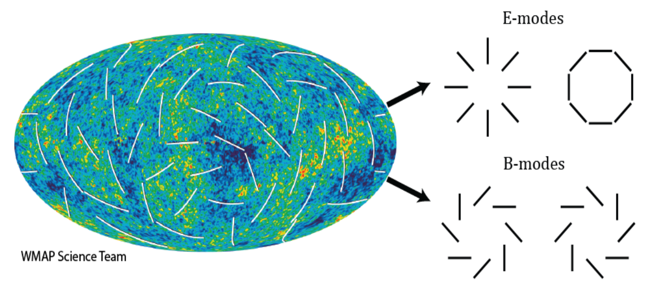
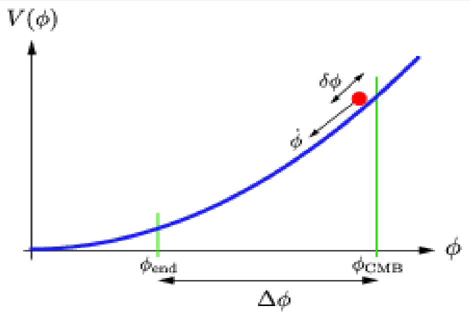


Decomposition of polarization patterns

- Gravitational waves produce a peculiar pattern ("B-modes")
- Unobserved $\Longrightarrow$ $\boxed{r = 16\epsilon \lesssim 0.07}$

Outline

Polynomial models



Examples:

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- $\implies$ Super-Planckian field excursion $\Delta\phi \gtrsim M_{Pl}$
- $\boxed{\epsilon \approx \eta}$

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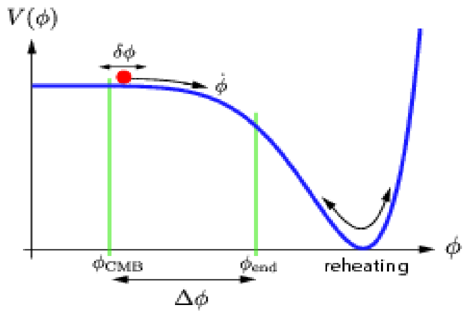
- Crucial problem: $\boxed{\epsilon \approx \eta}$

Outline

Hilltop-Small Field models



Example:

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- Fits better observations

CMB observations: n_S vs. r

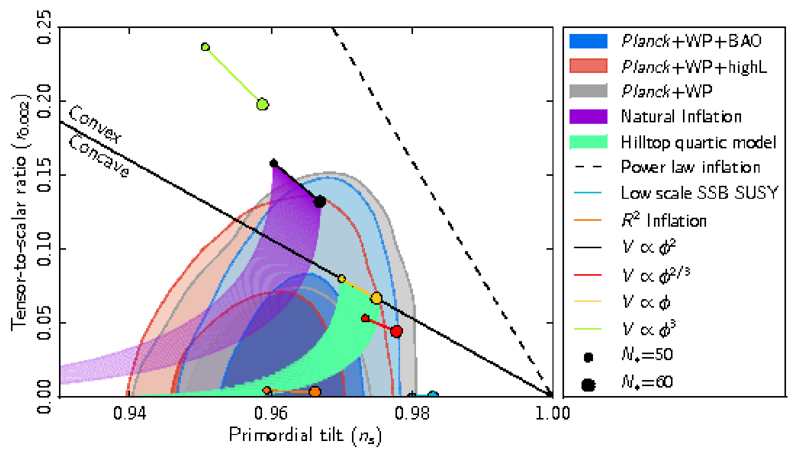

Figure: Planck satellite 2015 data.

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- However a new coupling is possible

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- Unimportant in the late universe:
  - Tiny correction to the Planck mass: $\xi v^2 \ll R$
  - $R \approx \mathcal{O}(H^2) \implies$ tiny correction to $V(\phi)$

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“Einstein frame"

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- And now define a new scalar field σ:

$$K(\phi)\partial_\mu\phi\partial^\mu\phi = \partial_\mu\sigma\partial^\mu\sigma$$

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- At small $\phi \ll M_P/\sqrt{\xi}$, we have $\sigma \approx \phi$ and the usual Higgs potential $U = V$

Higgs inflation

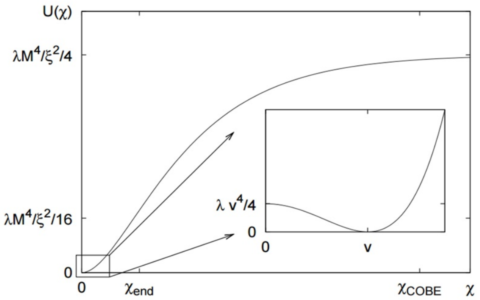

Fig. 1. Effective potential in the Einstein frame.

Fitting data

- Now we have to satisfy phenomenological constraints:

$$\boxed{\left.\frac{H^2}{M_P^2\epsilon}\right|_{\phi=\phi_*} \approx 10^{-9}}$$

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- And this implies: $\begin{cases} n_S|_{\phi=\phi_*} \approx 0.97 \\ r|_{\phi=\phi_*} \approx 0.0033 \end{cases}$

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- Big debate about quantum corrections being large and model not predictive...

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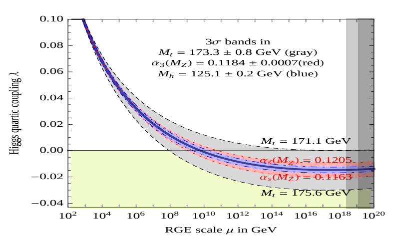
- This depends on precise measurements of m_t, m_H and α_s (top, gluons and self-interactions are the most important in the running equations for $\lambda(\phi)$)



Outline

Inhomogenous scalar field in de Sitter

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  - At late times $\frac{k}{a}$ negligible: $\ddot{\varphi} + 3H\dot{\varphi} \approx 0 \implies$ $\boxed{\varphi \approx const}$

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Inhomogenous scalar field in de Sitter

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- Standard Minkowski case: $\boxed{u_k = \frac{e^{-i\omega_k\tau}}{\sqrt{2k}}}$ $(\omega_k \equiv |\vec{k}|)$
- So that the field is quantized as usual:

$$\boxed{u = \int \frac{d^3k}{(2\pi^3)} \frac{1}{\sqrt{2\omega_k}} \left(e^{i(kx-\omega_k\tau)}a_{\vec{k}} + e^{-i(kx-\omega_k\tau)}a^\dagger_{\vec{k}}\right)}$$

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$$u_k \approx -\frac{1}{\sqrt{2k}}\frac{i}{k\tau} \implies \boxed{\varphi_k \equiv \frac{u_k}{a} \approx \frac{H}{\sqrt{2}}\frac{i}{k^{3/2}}}$$

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- Such initial spectrum evolves through gravity (overdensities grow and form non-linear structures, galaxies, etc...)

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- If we compute the typical fluctuation of the field in one point in Minkowski we get:

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- We can compute $\langle \varphi_V^2 \rangle$ and get:

$$\langle \varphi_V^2 \rangle^{1/2} = \left(\int^{k_L} d^3k \frac{1}{k} \right)^{1/2} \approx k_L \qquad k_L = 1/L$$

Physical importance

- If we compute the typical fluctuation of the field in one point in Minkowski we get:

$$\langle \varphi^2 \rangle^{1/2} = \left(\int^{\Lambda} \frac{d^3k}{(2\pi)^3} |\varphi_k|^2 \right)^{1/2} \propto \left(\int d^3k \frac{1}{k} \right)^{1/2} \propto \Lambda$$

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- In de Sitter , averaging over any region of size $V = L^3 \gtrsim (H^{-1})^3$

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- These are minimal quantum fluctuations
- $\implies$ we cannot impose $\Phi_V = 0$ in the region that corresponds to our universe as initial condition

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$$L = \frac{\partial_\mu \Phi \partial^\mu \Phi}{2} - \frac{\lambda}{3}\left(\frac{\phi^4}{4} + \varphi\phi^3 + \frac{3\varphi^2\phi^2}{2} + \varphi^3\phi + \frac{\varphi^4}{4}\right)$$

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- We necessarily have to include such terms!

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1. Adding a mass $m^2\varphi^2$
2. Inflaton perturbations: Coupling φ to metric perturbations
3. Expansion is not exactly de Sitter
4. Adding a non-minimal coupling: $R\varphi^2$ (R, Ricci scalar)
5. Higher spin cases: 1, 2, 1/2

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- If $m^2 \gg H^2$ it prevents the superhorizon growth (large growing positive mass: damped oscillator)

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$$\boxed{u_k = c_1\sqrt{-\tau}H_\nu^{(1)}(k\tau) + c_2\sqrt{-\tau}H_\nu^{(2)}(k\tau)}$$

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- Subhorizon $k\tau \gg 1$, it is:

$$u_k \approx c_1\left(\sqrt{\frac{2}{\pi k}}e^{-\frac{i\pi\nu}{2}-\frac{i\pi}{4}}\right)e^{-ik\tau} + c_2\left(\sqrt{\frac{2}{\pi k}}e^{-\frac{i\pi\nu}{2}+\frac{i\pi}{4}}\right)e^{ik\tau}$$

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Choosing: $\boxed{c_1 = \frac{\sqrt{\pi}}{2}e^{\frac{i\pi\nu}{2}+\frac{i\pi}{4}}\ , c_2 = 0}$

$$\Longrightarrow \boxed{u_k = \frac{e^{-ik\tau}}{\sqrt{2k}}}$$

Massive case

- With the previous choice we have the mode function at all times:

$$\boxed{u_k = \frac{\sqrt{\pi}}{2} e^{\frac{i\pi\nu}{2}+\frac{i\pi}{4}} \sqrt{-\tau} H_\nu^{(1)}(k\tau)}$$

pause

- Superhorizon $\boxed{k\tau \to 0^-}$, and for $\boxed{\eta \equiv \frac{m^2}{3H^2} \ll 1}$ it goes as :

$$\boxed{P_\varphi} = \frac{k^3|u|^2}{2\pi^2 a^2} \approx \left(\frac{H}{2\pi}\right)^2 (-k\tau)^{2\eta} \approx \boxed{\left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{2\eta}}$$

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- Afterwards (superhorizon) it slowly decays due the mass
- Long scales (small k) have more time to decay

Outline

Including metric

- If the scalar inhomogeneous $\phi = \phi(t) + \varepsilon\, \varphi(t, x^i)$
- Sources an inhomogeneous metric:
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$$\begin{cases} N = 1 + \varepsilon\,\delta N \\ N_i = 0 + \varepsilon\, N_i \\ \phi = \phi(t) + \varepsilon\,\varphi(\vec{x}, t) \\ h_{ij} = a^2[(1 + 2\varepsilon\,\psi(\vec{x}, t))\delta_{ij} + \varepsilon\,\gamma_{ij}(\vec{x}, t)] \end{cases}$$

- $\delta N, N_i, \varphi, \psi, \gamma_{ij}$ are $\mathcal{O}(\varepsilon)$, *first* order perturbations

Including metric

- Quantities can be decomposed in scalar, vector and tensors:

$$N_i = N_i^S + N_i^V$$

$$\gamma_{ij} = \gamma_{ij}^S + \gamma_{ij}^V + \gamma_{ij}^T$$

[2] V.Mukhanov and R.Brandenberger, Part I, sect.3. S.Weinberg, “Cosmology", Oxford University Press, sect.5. A.R.Liddle, D.H.Lyth, “Cosmological Inflation and Large Scale Structure", Cambridge University Press, sect.14.5.

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- *At linear order* in perturbations: scalars only couple to scalar, vectors with vectors and tensors with tensors [2]

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- One can change coordinates by $\mathcal{O}(\epsilon)$:

$$\boxed{x^\mu \to x'^\mu = x^\mu + \epsilon\, \delta x^\mu}$$

- This leaves the metric in the form background + perturbations: $\boxed{g_{\mu\nu} = g^{(0)}_{\mu\nu} + \epsilon\, h_{\mu\nu}}$
- We get “new" perturbations $h_{\mu\nu} \to h'_{\mu\nu}$
- A scalar field $\Phi = \phi(t) + \epsilon\varphi(t, x^i)$ also changes:

$$\boxed{\phi \to \phi_0(t(t')) + \epsilon\varphi(t, x^i) \approx \phi_0(t') + \epsilon(-\phi_0'(t')\delta t + \varphi(t', x^{i\prime}))}$$

- We can use this change of coordinate to fix some conditions (“gauge choice")
- $\boxed{\delta x^0 = \delta t, \qquad \delta x^i = \partial^i Q + Q^i}, \qquad (\partial_i Q^i = 0)$
  (2 scalars and 1 vector)
- We can set 2 scalars and 1 vector to zero for simplicity.
- Tensors are “gauge-invariant".

Including metric

- For example we can set to zero $\varphi = 0 = E = E_i = 0$, leaving only the tensor part of γ_{ij}:

$$\boxed{\varphi = 0, \qquad \partial_i \gamma^T_{ij} = 0, \qquad \gamma^T_{ii} = 0}$$

(“comoving or uniform field gauge")

Including metric

Cosmology: Inflation

Introduction
Inflation

Scalar field models
Large-field polynomial
Hilltop potentials
Higgs & Conformal transformation

Perturbations
Free scalar
Massive scalar
With Metric
Gravitational waves
Gauge fields

End of Inflation

Naturalness issues

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- Using then Einstein equations in this gauge one finds that all quantities can be expressed as functions of ψ (scalar) and γ_{ij}^T (tensor)

$$N = \frac{\dot{\psi}}{H}, \qquad N_i = \partial_i f(\psi) + N_i^V, \qquad N_i^V = 0$$

Including metric

- Start from Einstein-Hilbert action plus scalar field:

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left[-M_P^2 R + (\partial_\mu\phi\partial^\mu\phi)^2 - 2V(\phi)\right]$$

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- It becomes a quadratic action for ψ (ignore γ_{ij}^T for the moment):

$$\boxed{S = \int d^3x\, dt\,\frac{\dot{\phi}^2}{H^2}\left[a(t)^3\dot{\psi}^2 - a(t)(\partial_i\psi\partial^i\psi)^2\right]}$$

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- This looks complicated, but...

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$$P_\psi = \frac{|u|^2}{z^2} = \left(\frac{H}{\dot{\phi}}\right)^2 \left(\frac{H}{2\pi}\right)^2 (-k\tau)^{2\eta - 6\epsilon} \tag{4}$$

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$$\Longrightarrow \boxed{A^2 = \left(\frac{H^2}{2\pi\dot{\phi}}\right)^2} \qquad \boxed{n_s - 1 = 2\eta - 6\epsilon}$$

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Evolution of ζ

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- It can be shown that: if the universe is dominated by one fluid with δP proportional to $\delta\rho$: $\delta P = c_s^2\delta\rho$ (“adiabatic")
- $\implies \zeta$ constant superhorizon

ζ and CMB

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$$\boxed{\zeta \to \frac{\delta T}{T}}$$

- Measuring $\frac{\delta T}{T}$ we measure A^2 and $n_s - 1$

Outline

Spin 2

- The quadratic action for γ_{ij} can be also obtained:

$$S_\gamma = \frac{M_P^2}{8}\int d^3x dt \left(a^3\dot{\gamma}_{ij}\dot{\gamma}_{ij} - a\partial_k\gamma_{ij}\partial_k\gamma_{ij}\right)$$

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- Four constraints: $\delta^{ij}\gamma_{ij} = 0$ (traceless), $\partial^i\gamma_{ij} = 0$ (transverse, 3 equations)
- Write:

$$\gamma_{ij} = \int \frac{d^3k}{(2\pi)^3}\sum_s \varepsilon_{ij}^{(s)}\gamma_{\vec{k}}^{(s)} e^{ik.x}$$

where the polarization tensors obey: $\varepsilon_{ii}^{(s)} = 0$, $k^i\varepsilon_{ij} = 0$ and normalized $\varepsilon_{ij}^{(s)}\varepsilon_{ij}^{(s')} = 2\delta^{ss'}$.

- Two degrees of freedom: each $\gamma_{\vec{k}}^{s}$ evolves as a massless scalar (same equation for both s)

Spin 2

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$$A_T^2 = \left(\frac{8}{M_P^2}\right)\left(\frac{H}{2\pi}\right)^2$$

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- Note that potential energy V decreases:

$$\Longrightarrow \boxed{\epsilon \propto \dot{H}/H^2 < 0} \Longrightarrow \boxed{n_T < 0}$$

- Typical definition $\boxed{r \equiv \frac{A_T^2}{A_S^2} \equiv 16\epsilon}$

Spin 2: measurability

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- For large k it oscillates

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- At Last Scattering of the CMB we can measure scales close to the H_{LSS}^{-1} (large scales)
- Today we can in principle measutre the scale that are re-entering now (close to H_0^{-1})

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- This implies the scale of inflation is

$$\boxed{V^{1/4} \lesssim 1.8 \cdot 10^{16}\mathrm{GeV} \left(\frac{r}{0.07}\right)^{1/4}}$$

Spin 2: measurability

- If a GW reentters today with an amplitude $\frac{H}{M_P}$
- It stretches space by this amount, producing a large-scale quadrupole pattern in the CMB
- By imposing this is less than the observed quadrupole $$\boxed{\frac{\delta T}{T}\Big|_{\ell=2} \lesssim 10^{-5}}$$
- $\Longrightarrow \frac{H}{M_P} \lesssim 10^{-5}$ (similar to previous bound)

Outline

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- An interesting way is to couple ϕ to gauge fields
- Advantages:
  - The coupling can be derivative (it does not induce large corrections to $V(\phi)$)
  - It leads to interesting phenomenology during inflation

Spin 1

- Consider a free gauge field:

$$S = -\frac{1}{4}\int d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu} \qquad (F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu)$$

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- Now Fourier decompose:

$$A_i^T = \int \frac{d^3k}{(2\pi)^3}\sum_s \varepsilon_i^{(s)} A_k^{(s)} e^{ik.x} \qquad s = 1, 2$$

with $\varepsilon^{(s)}$ two polarization vectors

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- Not enhanced superhorizon (small k)

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- However it can be coupled to the inflaton (even if ϕ neutral real field), *e.g.*:
- $\mathcal{L} = J^2(\phi)F_{\mu\nu}F^{\mu\nu} + g(\phi)F_{\mu\nu}\tilde{F}^{\mu\nu}$
- $\tilde{F}^{\mu\nu} \equiv \epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$

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- This leads to a richer equation:

$$\boxed{v''_\pm + \left(k^2 + \frac{J''}{J} \pm g'(\phi) k\right) v_\pm = 0}$$

for the two circular polarizations: the $F\tilde{F}$ term induces a $(\pm)$ polarization dependent term

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- $F_{\mu\nu} \tilde{F}^{\mu\nu} = \partial_\mu K^\mu$ is a total derivative
- Integrating by parts $\Longrightarrow$ it is a derivative coupling: $\frac{\partial_\mu \phi K^\mu}{f}$

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- *Caveat:* for non-abelian gauge fields there is a non-pertubative potential induced by instantons at a strong coupling scale Λ (like for QCD axion)

$$V(\phi) \approx \Lambda^4 \cos(\frac{\phi}{f})$$

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Possibility of resonances. Describes the (non-perturbative) decay of the inflaton into photons

Constant $\dot{\phi}$ and de Sitter

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- Solution at $\tau \to 0^-$ (future):

$$A_+ \approx \frac{1}{\sqrt{2k}} \left(\frac{k|\tau|}{2\xi}\right)^{1/4} e^{\pi\xi - 2\sqrt{2\xi k|\tau|}}$$

Consequences

- (Sorbo & Anber '09) estimated:

$$\frac{\langle F\tilde{F}\rangle}{4} = \frac{1}{2a^4}\int \frac{d^3k}{(2\pi)^3}\, k\, \frac{d\left[|A_+|^2 - |A_-|^2\right]}{d\tau} \approx \frac{H^4}{\xi^4} e^{2\pi\xi}$$

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- New features: [3]
  - Fields are **not** in the vacuum:
    - Possibly large corrections to P_ζ
    - Possibly large corrections to P_γ
    - Possibly large corrections to $\langle\zeta\zeta\zeta\rangle$
    - It can also backreact $\Longrightarrow$ non-trivial dynamics on ϕ:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) + \frac{1}{f}\langle F\tilde{F}\rangle = 0$$

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Spin 1: Resonances

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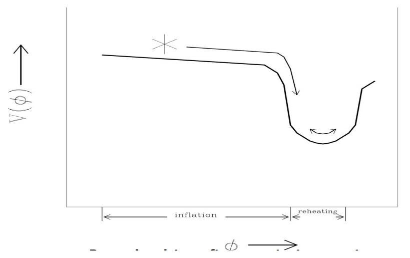
- Eventually the decay products (the gauge fields in the amplified bands) will scatter and produce a plasma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- E.o.m. can be rewritten as:

$$\boxed{\rho_\phi \equiv V + \frac{\dot{\phi}^2}{2}} \qquad \boxed{\dot{\rho_\phi} + 3H\dot{\phi}^2 = 0}$$

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- By conservation of energy, energy in decay products grows, then thermalizes and produces a plasma

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- $\implies$ Resonant production (as we will see..)
- At large amplitude of oscillations can be more efficient than perturbative reheating

How “natural" is the inflaton potential?

- In many inflationary models we have $\Delta\phi \gtrsim M_{Pl}$
- One may expect corrections (*e.g.* due to quantum gravity?)

$$V(\phi) = m^2\phi^2 + \lambda\phi^4 + \sum_n \alpha_n \frac{\phi^{4+n}}{M_{Pl}^n}$$

- Such corrections could spoil flatness!

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- Caveat: such dangerous corrections seem instead present in string theory

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- However one must still
  - Tune the divergence $m^2 \propto \Lambda^2$(as for the Higgs)
  - Fix the non-minimal coupling ($m^2 \propto R = 12H^2$) to be small
- Solved for example by “Natural Inflation"
- $L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\phi}{f}F_{\mu\nu}F^{\mu\nu}$
  - It generates a potential (like the QCD “Axion"):

$$V = \Lambda^4 \cos(\phi/f),$$

  (Λ is a strong coupling scale, non-abelian model)
  - protected against quadratic corrections