Lattice QCD

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References:

Textbooks:

- M. Creutz, "*Quarks, gluons and Lattices*," Cambridge, 1983
- I. Montvay and G. Münster, "*Quantum Fields on a Lattice*," Cambridge, 1994
- J. Smit, "*Introduction to Quantum Fields on a Lattice*," Cambridge, 2002
- T. DeGrand and C. DeTar, "*Lattice Methods for Quantum Chromodynamics*," World Scientific, 2006

Les Houches Summer School (2009) ("LH"):

- L. Lellouch, R. Sommer, B. Svetitsky, A. Vladikas and L. Cugliandolo, "*Modern Perspectives in Lattice QCD*," Oxford, 2011

- S. Aoki *et al*. (FLAG), "*Review of lattice results concerning low-energy particle physics,*" arXiv:1607.00299 (EPJC)

Plan:

- Lattice field theory: what does field theory look like on the lattice
- Important: lattice fermions ("species doubling")
- Examples of non-perturbative applications
  (main reason to be interested in lattice field theory!)
- What is lattice gauge theory good for? (examples)

  Numerical simulations: I'll explain (examples of) what is computed,
  not how it's done (see references)

Continuum QCD in euclidean space

Lagrangian: $$\mathcal{L}_{\rm QCD} = \frac{1}{2}{\rm tr}(F_{\mu\nu}F_{\mu\nu}) + \overline{\psi}(\not{D} + m)\psi$$ (one quark)

$$\not{D} = \not{\partial} + ig\not{A}\ , \qquad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$$

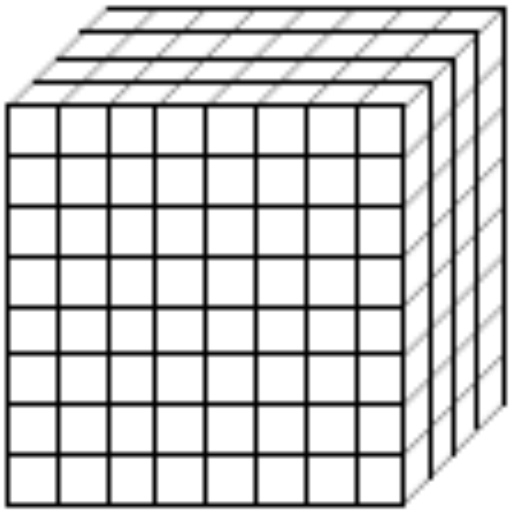
Free fermion propagator:

$$S^{-1}(p) = i\not{p} + m$$

Running coupling: $\mu \frac{dg^2}{d\mu} = -\beta(g^2) < 0$,

hence $g^2(\mu)$ decreases with increasing energy: asymptotic freedom

Lattice: covariant derivative



$\leftrightarrow \; a$

Hypercubic, lattice spacing a
points $x_\mu = n_\mu a\,,\qquad n_\mu \in \mathbb{Z}$
$x+\mu$ is neighbor of x in μ direction

$$\partial_\mu \psi(x) \quad \rightarrow \quad \partial^+_\mu \psi(x) \equiv \frac{1}{a}(\psi(x+\mu) - \psi(x))$$

Make covariant: parallel transport $\psi(x+\mu)$ back to x

$$D^+_\mu \psi(x) = \frac{1}{a}\left(U_\mu(x)\psi(x+\mu) - \psi(x)\right)\ , \quad U_\mu(x) \in SU(N)$$

Gauge transformations:

$$\psi(x+\mu) \rightarrow g(x+\mu)\psi(x+\mu)$$

$g(x) \in SU(N)$

$$\overline{\psi}(x) \rightarrow \overline{\psi}(x) g^\dagger(x)$$

$$U_\mu(x) \rightarrow g(x) U_\mu(x) g^\dagger(x+\mu)$$

then $\overline{\psi}(x)\gamma_\mu D^+_\mu \psi(x)$ is gauge invariant

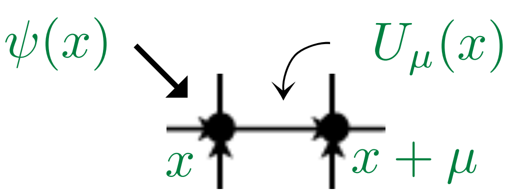
Lattice quark lagrangian

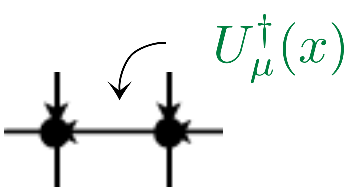
Write $U_\mu(x) = \exp(iagA_\mu(x)) = 1 + iagA_\mu(x) + O(a^2)$

then
$$\frac{1}{a}(U_\mu(x)\psi(x+\mu) - \psi(x)) = \frac{1}{a}\big((1 + iagA_\mu(x) + \dots) \times (\psi(x) + a\partial_\mu\psi(x) + \dots) - \psi(x)\big)$$
$$= \partial_\mu\psi(x) + igA_\mu(x)\psi(x) + O(a)$$

$$\mathcal{L}_{\mathrm{quark}} = \frac{1}{2a}\left(\overline{\psi}(x)\gamma_\mu U_\mu(x)\psi(x+\mu) - \overline{\psi}(x+\mu)\gamma_\mu U_\mu^\dagger(x)\psi(x)\right) + m\overline{\psi}(x)\psi(x)$$

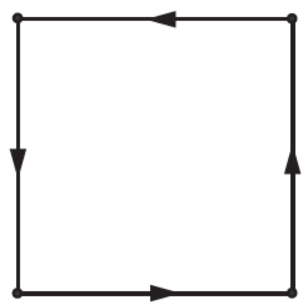
(choice guarantees hermiticity of lattice hamiltonian)





Yang-Mills part

Make smallest gauge-invariant object out of the link variables:



$$
\begin{aligned}
U_{\mu\nu}(x) &= U_\mu(x)U_\nu(x+\mu)U_\mu^\dagger(x+\nu)U_\nu^\dagger(x) \\
&= e^{iagA_\mu(x)}e^{iagA_\nu(x+\mu)}e^{-iagA_\mu(x+\nu)}e^{-iagA_\nu(x)} \\
&= e^{-ia^2g\partial_\nu A_\mu + ia^2 g\partial_\mu A_\nu - a^2g^2[A_\mu,A_\nu]+\dots} \\
&= e^{ia^2gF_{\mu\nu}(x)+\dots} \qquad \to g(x)U_{\mu\nu}(x)g^\dagger(x)
\end{aligned}
$$

hence $$\sum_{x,\mu<\nu}\left(1-\frac{1}{N}\mathrm{Re}\,\mathrm{tr}\,U_{\mu\nu}(x)\right) = \frac{1}{N}\sum_{x,\mu<\nu}\mathrm{tr}\left(\frac{1}{2}a^4g^2F_{\mu\nu}^2+\dots\right)$$

$$\to \frac{g^2}{4N}\int d^4x\sum_{\mu\nu}\mathrm{tr}\,F_{\mu\nu}^2$$

Lattice QCD action

Take $S_{\text{plaquette}} = \beta \sum_{x,\mu<\nu} \left(1 - \frac{1}{N} \operatorname{Re}\operatorname{tr} U_{\mu\nu}(x)\right)$ with $\beta = \frac{2N}{g^2}$

then
$$S_{\text{QCD}} = S_{\text{plaquette}}(U) + a^4 \sum_{x,\mu} \frac{1}{2a} \left(\overline{\psi}(x)\gamma_\mu U_\mu(x)\psi(x+\mu) - \overline{\psi}(x+\mu)\gamma_\mu U_\mu^\dagger(x)\psi(x)\right) + a^4 \sum_x m\overline{\psi}(x)\psi(x)$$

- Simplest lattice action; many other possibilities!
- Symanzik improvement program: can design other actions with different $O(a)$ terms and take linear combinations without them, these are closer to the continuum limit. (Weisz in LH)

Free lattice fermions

Free fermion action:

$$S = \frac{1}{2a}\, a^4 \sum_{x,\mu} \left(\overline{\psi}(x)\gamma_\mu \psi(x+\mu) - \overline{\psi}(x+\mu)\gamma_\mu \psi(x) \right) + m \text{ term}$$

Fourier transform:

$$\psi(x) = \int_p e^{ipx} \psi(p) \ , \qquad \overline{\psi}(x) = \int_p e^{-ipx} \overline{\psi}(p) \ , \qquad \int_p \equiv \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4}$$

then

$$S = \frac{1}{2a}\, a^4 \sum_{x,\mu} \int_p \int_q \overline{\psi}(p)\gamma_\mu \psi(q) \left(e^{-ipx+iq(x+\mu)} - e^{-ip(x+\mu)+iqx} \right) + m \text{ term}$$

With $a^4 \sum_x e^{-ipx+iqx} = (2\pi)^4 \overline{\delta}(p-q) = (2\pi)^4 \sum_n \delta\left(p - q + \frac{2\pi n}{a} \right)$

this yields

$$S = \frac{1}{2a} \int_p \sum_\mu \overline{\psi}(p)\gamma_\mu \psi(p) \left(e^{ip_\mu a} - e^{-ip_\mu a} \right) + m \int_p \overline{\psi}(p)\psi(p)$$

Species doublers

Fermion propagator: $S^{-1}(p) = \sum_{\mu} \frac{i}{a} \gamma_\mu \sin(ap_\mu) + m \xrightarrow{a \to 0} i\not{p} + m$

But, massless propagator has poles not only at $p = 0$ but at all

$$p = \overline{p} \in \{(0,0,0,0), (\pi/a, 0,0,0), \ldots, (\pi/a, \pi/a, \pi/a, \pi/a)\}$$

Take $p_\mu = \overline{p}_\mu + q_\mu$ then $\sin(ap_\mu) = S_\mu \sin(aq_\mu)\ , \quad S_\mu = \pm 1$

$$\Rightarrow \sum_{\mu} \frac{i}{a} \gamma_\mu \sin(ap_\mu) \xrightarrow{a \to 0} \sum_{\mu} i(S_\mu \gamma_\mu) q_\mu + m$$

$$\{S_\mu \gamma_\mu, S_\nu \gamma_\nu\} = 2 S_\mu S_\nu \delta_{\mu\nu} = 2\delta_{\mu\nu}$$

- Continuum limit contains 16 relativistic quarks: fermion doubling problem
- Cannot project onto $\overline{p} = 0$: all species get pair-produced!
- Can we choose a smarter action without this problem?

No: Species doublers and the anomaly

For $m = 0$ invariance under $U(1)_A$: $\psi \to e^{i\alpha\gamma_5}\psi\ ,\quad \overline{\psi} \to \overline{\psi}e^{i\alpha\gamma_5}$

However, this symmetry is anomalous in continuum, $\partial_\mu j^A_\mu = \frac{g^2}{8\pi^2}\,\mathrm{tr}(F\tilde{F})$
in conflict with the lattice!

Assign $Q_A = +1$ to $\overline{p} = 0$ quark, then for quark near other $\overline{p}$

$$\tilde{\gamma}_\mu = S_\mu \gamma_\mu \quad \Rightarrow \quad \tilde{\gamma}_5 = \tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3\tilde{\gamma}_4 = S_1 S_2 S_3 S_4 \gamma_5 = \pm\gamma_5$$

eight quarks with $Q_A = +1$
eight quarks with $Q_A = -1$

and $\partial_\mu j^A_\mu = \sum_A Q_A \frac{g^2}{8\pi^2}\,\mathrm{tr}(F\tilde{F}) = 0$: doublers provide anomaly-free representation
(Karsten&Smit, Nielsen&Ninomiya)

Yes: Wilson fermions

Regulator needs to break chiral symmetry to recover anomaly!
(PV: fermion with large mass; dimreg: $[\gamma_\mu, \gamma_5] = 0$ in extra dimensions)

Here: give the doublers a mass! Momentum-dependent mass term (Wilson)

Replace $a^4 \sum_x m\overline{\psi}\psi \to$

$$a^4 \sum_x \Big(m\overline{\psi}(x)\psi(x) + \frac{1}{2a} \underbrace{\sum_\mu (2\overline{\psi}(x)\psi(x) - \overline{\psi}(x)\psi(x+\mu) - \overline{\psi}(x+\mu)\psi(x))}_{=-a^2\overline{\psi}\Box\psi} \Big)$$

$$= \int_p \overline{\psi}(p)\psi(p) \underbrace{\frac{1}{a}\left(am + 1 - \cos(ap_\mu)\right)}_{=m+ap^2+\ldots}$$

Wilson fermions, cont'd

Take $p = \overline{p} + q$ then mass term is

$$m + \frac{1}{a}\sum_{\mu}(1 - S_{\mu}\cos(aq_{\mu})) \to m + \frac{2n}{a} + O(a)$$

with n equal to # components $\frac{\pi}{a}$ in $\overline{p}$

Removes (decouples) species doublers, breaks chiral symmetry even for $m = 0$

- No multiplicative renormalization of mass, need $\frac{1}{a}$ counter term
- Tune bare mass to set renormalized mass to zero, for $g^2 \neq 0$ ("tune $\kappa_c = (8 + 2am)^{-1}$ ")

Lattice fermions

- LQCD with n_f Wilson fermions gives correct continuum limit
  (proof to all orders in perturbation theory (Reisz))
  But, not the only game in town!
- "clover" fermions: Symanzik-improved Wilson fermions, no $O(a)$
  add (latticized version) of $c_{\rm SW} a\overline{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi$ (Sheikholeslami&Wohlert)
- domain-wall fermions: use extra dimension to exponentially suppress
  $1/a$ counter term, $g^2/a \to g^2 e^{-L_5/a}/a$
  L_5 extent of lattice in 5th direction (Kaplan, Shamir)
- overlap fermions: take limit $L_5 \to \infty$ (very non-trivial limit!) (Neuberger)
- staggered fermions: reduce & make use of doublers
  keeps an exact (flavored) chiral symmetry on the lattice (Kogut&Susskind)

Path integral

$$Z = \int \prod_{x,\mu} dU_\mu(x) \int \prod_x d\psi(x) d\overline{\psi}(x)\, e^{-S(U,\psi,\overline{\psi})}$$

$$= \int \prod_{x,\mu} dU_\mu(x)\, \mathrm{Det}(D(U) + m)\, e^{-S_{\mathrm{plaquette}}}$$

dU is gauge-invariant Haar measure:

$U(1):$ $U = e^{i\phi}$ $\int dU = \frac{1}{2\pi} \int_0^{2\pi} d\phi$ invariant under $\phi \to \phi + \alpha$

$SU(2):$ $U = \sigma + i\vec{\tau} \cdot \vec{\pi}\ , \quad \sigma^2 + \vec{\pi}^2 = 1$

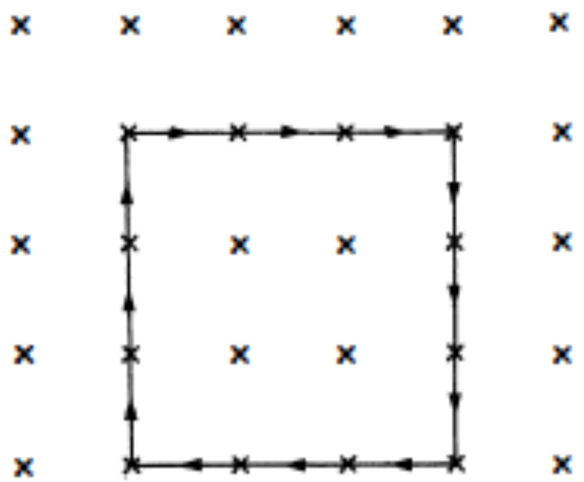
$$dU = d\sigma d^3\pi\, \delta(\sigma^2 + \vec{\pi}^2 - 1) = \frac{1}{\sqrt{1 - \vec{\pi}^2}}\, d^3\pi$$

$SU(N):$ $U = U(\vec{\alpha}) \quad dU = \mathrm{norm}\, \sqrt{\det g} \prod_k d\alpha^k\ , \ g_{k\ell} = \frac{1}{2} \mathrm{tr} \left(\frac{\partial U}{\partial \alpha^k} \frac{\partial U^\dagger}{\partial \alpha^\ell} \right)$

Note: no gauge fixing needed!

Pure Yang-Mills: expansion in $\beta = 2N/g^2$

Consider large $R \times T$ Wilson loop ($R = T = 3a$ in figure)



Interpretation:

static quark-antiquark pair

created at $t = 0$

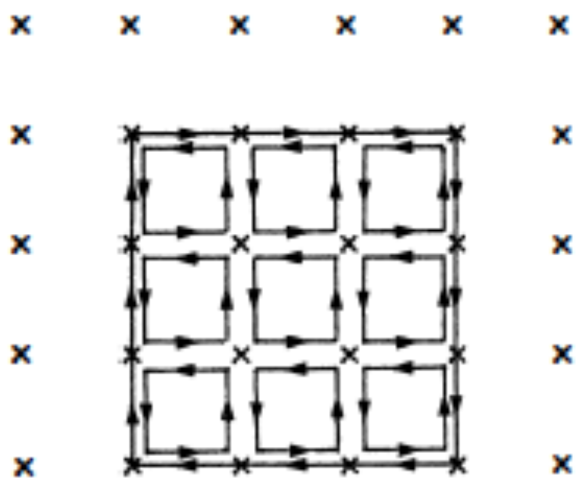
annihilated at $t = T$

distance R apart

Use $\int dU = 1$, $\int dU\, U = 0$, expand $e^{-\beta S_{\text{plaquette}}} = 1 - \beta S_{\text{plaquette}} + \dots$

Pure Yang-Mills: expansion in $\beta = 2N/g^2$

Lowest-order contribution when plaquettes tile Wilson loop



Hence

$\langle W(R \times T \text{ loop})\rangle \propto \beta^{RT}$ area law

interpretation: $e^{-TV(R)}$

hence linear potential $V(R) \propto R$

quark confinement! (Wilson)

However, result at strong coupling, $g^2 \sim \infty$

Continuum limit: take $g^2(1/a) \to 0$ (asymptotic freedom)

No phase transition observed between strong and weak coupling (numerical)

Back to QCD: pion propagator

Take $\pi^+(x) = \overline{d}(x)\gamma_5 u(x)$

$$\langle \pi^+(x)\pi^-(0)\rangle = \frac{1}{Z}\int DU \prod_{\psi=u,d} D\psi D\overline{\psi}\ \underbrace{\overline{d}(x)\gamma_5 u(x)}_{\pi^+(x)}\ \underbrace{\overline{u}(0)\gamma_5 d(0)}_{\pi^-(0)}\ e^{-S(U,\psi,\overline{\psi})}$$

$$= -\frac{1}{Z}\int dU\,\mathrm{Det}(D(U)+m_u)\mathrm{Det}(D(U)+m_d)\,e^{-S_{\mathrm{plaquette}}}$$

$$\times\ \mathrm{tr}\left((D(U)+m_u)^{-1}(x,0)\gamma_5(D(U)+m_d)^{-1}(0,x)\right)$$

Do the integral over gauge fields numerically and probabilistically: generate gauge-field configurations with $p(U) = \frac{1}{Z}\,e^{-S_{\mathrm{plaquette}}(U)}\,\mathrm{Det}^2(U)$

This is the reason for euclidean space!

State of the art lattice: 144^3 x 288 x 4 x 8 ≈ 3 x 10^{10} links!

$$\langle \pi^+(x)\pi^-(0)\rangle = \frac{1}{Z}\int DU \prod_{\psi=u,d} D\psi D\overline{\psi}\; \underbrace{\overline{d}(x)\gamma_5 u(x)}_{\pi^+(x)}\; \underbrace{\overline{u}(0)\gamma_5 d(0)}_{\pi^-(0)}\; e^{-S(U,\psi,\overline{\psi})}$$

$$= -\frac{1}{Z}\int dU\, \mathrm{Det}(D(U)+m_u)\mathrm{Det}(D(U)+m_d)\, e^{-S_{\mathrm{plaquette}}}$$

$$\times\, \mathrm{tr}\left((D(U)+m_u)^{-1}(x,0)\gamma_5(D(U)+m_d)^{-1}(0,x)\right)$$

Need to invert $D(U)+m$ for the propagators

Need to compute $\mathrm{Det}(D(U)+m)$, or rather, variation with gauge field:

$$\delta \log \mathrm{Det}(D(U)+m) = \delta \mathrm{Tr}\, \log(D(U)+m)$$

$$= \mathrm{Tr}\left((D(U)+m)^{-1}\delta D(U)\right)$$

Again, we need the inverse: need reliable and fast inverter for very large (sparse) matrices! (Lüscher in LH)

- Note: can vary "valence" mass (inside propagator) and "sea" mass (inside determinant) independently!

  "Partial quenching" -- can be useful (MG in LH)

Pion propagator -- interpretation

Take $\vec{p} = 0$, *i.e.*, define $\pi^+(t) = \sum_{\vec{x}} \pi^+(\vec{x}, t)$

In QM, for a lattice of extent $T = Na$ in the time direction

$$Z = \mathrm{Tr}\, e^{-\hat{H}T} = \int dq \langle q | e^{-\hat{H}T} | q \rangle = \mathrm{Tr}\, \hat{T}^N = \int_{\mathrm{pbc}} Dq\, e^{-S}$$

with $\hat{T} = e^{-\hat{H}a}$ the “transfer matrix” and $\hat{H}$ the (lattice) hamiltonian

- In Minkowski space $e^{-\hat{H}a} \to e^{i\hat{H}a}$ becomes the evolution operator
- Can get spectrum of theory directly from $\hat{T}$ (Osterwalder&Schrader)

Pion propagator, cont'd

$$C(t) = \langle \pi^+(t)\pi^-(0)\rangle = \frac{1}{Z}\mathrm{Tr}\left(\hat{T}^{N-t/a}\hat{\pi}^+\hat{T}^{t/a}\hat{\pi}^-\right)$$

$$= \frac{1}{Z}\sum_{n,m}\langle n|\hat{\pi}^-\hat{T}^{N-t/a}|m\rangle\langle m|\hat{\pi}^+\hat{T}^{t/a}|n\rangle$$

$$= \frac{\sum_{n,m} e^{-E_m(T-t)}\, e^{-E_n t}\langle m|\hat{\pi}^+|n\rangle\langle n|\hat{\pi}^-|m\rangle}{\sum_m e^{-E_m T}}$$

$$\xrightarrow{T\to\infty} \sum_n e^{-(E_n-E_0)t}|\langle n|\hat{\pi}^-|0\rangle|^2$$

$$|n\rangle = \quad |\pi^-\rangle\ , \quad |(3\pi)^-\rangle\ , \quad |\pi^-(1300)\rangle\ , \ldots$$

Extract pion mass from large-t behavior,

excited states from multi-exponential fits (in principle!)

Propagator at finite T

$$
\begin{aligned}
C(t) = \langle \pi^+(t)\pi^-(0)\rangle &= \frac{1}{Z}\mathrm{Tr}\left(\hat{T}^{N-t/a}\hat{\pi}^+\hat{T}^{t/a}\hat{\pi}^-\right) \\
&= \frac{1}{Z}\sum_{n,m}\langle n|\hat{\pi}^-\hat{T}^{N-t/a}|m\rangle\langle m|\hat{\pi}^+\hat{T}^{t/a}|n\rangle \\
&= \frac{\sum_{n,m} e^{-E_m(T-t)}\, e^{-E_n t}\langle m|\hat{\pi}^+|n\rangle\langle n|\hat{\pi}^-|m\rangle}{\sum_m e^{-E_m T}} \\
&= \frac{1}{Z(T)}\sum_n \left(e^{-E_n t} + e^{-E_n(T-t)}\right)|\langle n|\hat{\pi}^-|0\rangle|^2 + \ldots \\
&= \sum_n A_n(T)\cosh\left[(T/2-t)(E_n-E_0)\right] + \ldots
\end{aligned}
$$

Connection with statistical mechanics

Note that for large (euclidean) time, and vanishing spatial momenta our propagator falls off like

$$e^{-mt} \qquad (\vec{p} = 0 \to E_n = m \text{ , assume } E_0 = 0)$$

In stat.mech., correlation functions decay like $e^{-t/\xi}$, with ξ the correlation length, over a (euclidean) distance t -- note the correspondence $m \leftrightarrow 1/\xi$

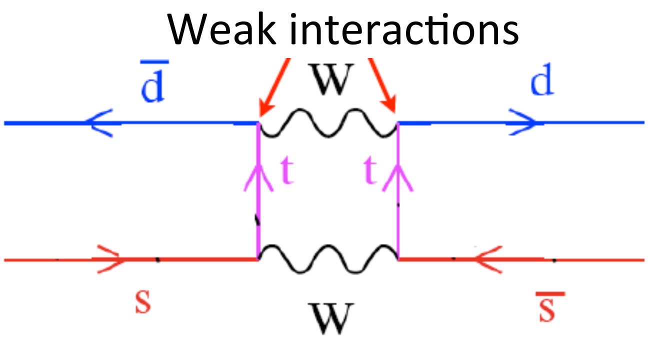
QFT: continuum limit taken by sending $m = am_{\text{phys}} \to 0$

Stat.mech.: $\xi = \xi_{\text{phys}}/a \to \infty$

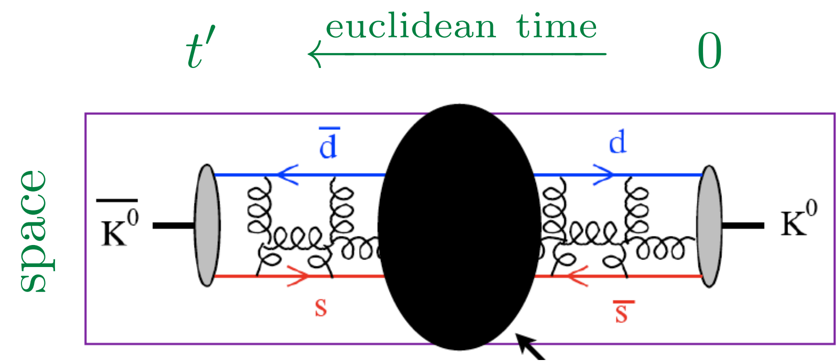
⇒ continuum limit corresponds to a second-order phase transition in d=4 equilibrium statistical mechanics system!

Matrix elements

Example: $\langle \overline{K}^0 | \mathcal{O}_{\Delta S=2} | K^0 \rangle$, $\quad \mathcal{O}_{\Delta S=2} = (\bar{s}\gamma_\mu(1-\gamma_5)d)(\bar{s}\gamma_\mu(1-\gamma_5)d)$



$\Rightarrow$



$\mathcal{O}_{\Delta S=2}(t)$

$$\langle \overline{K}^0(t')\mathcal{O}_{\Delta S=2}(t)\overline{K}^0(0)\rangle$$

$$\rightarrow e^{-m_K(t'-t)}e^{-m_K t}\langle 0|\overline{K}^0|\overline{K}^0\rangle\langle \overline{K}^0|\mathcal{O}_{\Delta S=2}|K^0\rangle\langle K^0|\overline{K}^0|0\rangle$$

Measure m_K , $\langle 0|\overline{K}^0|\overline{K}^0\rangle$, $\langle K^0|\overline{K}^0|0\rangle$ from two-point functions
and extract $\langle \overline{K}^0 | \mathcal{O}_{\Delta S=2} | K^0 \rangle$ (Lellouch in LH)

(fig. credit S. Sharpe)

Matrix elements, cont'd

Then $B_K = \dfrac{\langle \overline{K}^0 | \mathcal{O}_{\Delta S=2} | K^0 \rangle}{\frac{8}{3} m_K^2 f_K^2}$

factor in SM expression for $\varepsilon_K = \dfrac{\Gamma(K_L \to (\pi\pi)_{I=0})}{\Gamma(K_S \to (\pi\pi)_{I=0})}$, measure of indirect $\not{C}\!P$

- Need to match $\mathcal{O}^{\mathrm{lattice}}_{\Delta S=2}$ with $\mathcal{O}^{\overline{\mathrm{MS}}}_{\Delta S=2}$

- In continuum this operator renormalizes multiplicatively:
  $\overline{s}_L \Gamma d_L$ has $\Gamma = \gamma_\mu$ because of $SU(3)_L \times SU(3)_R$ symmetry

- Wilson fermions: only $SU(3)_V$, mixing with $(\overline{s}\Gamma d)(\overline{s}\Gamma d)$, $\Gamma = 1,\ \gamma_\mu,\ \gamma_5, \dots$

- No mixing with lower-dimension, because $(\overline{s}d)(\overline{s}d)$ in $\mathbf{27}$ of $SU(3)_V$

(Vladikas in LH)

Errors -- statistical

Suppose we have K gauge-field configurations distributed according to

$$\frac{1}{Z}\, e^{S_{\rm gauge}(U)}\, {\rm Det}(D(U)+M) > 0$$

(Schaefer in LH)

then $\langle \mathcal{O} \rangle = \lim_{K\to\infty} \sum_{i=1}^{K} \mathcal{O}(U_i)$

statistical error $\sqrt{\frac{1}{K-1}\sum_{i=1}^{K}\left(\mathcal{O}(U_i) - \langle\mathcal{O}\rangle\right)^2}$

(if all U_i and U_j for $i \neq j$ are uncorrelated)

Error in the average $\sim 1/\sqrt{K}$; increasing #configs. by 100 reduces error by 10

Errors -- systematic

- $a \neq 0$: need multiple lattice spacings to extrapolate

- $T < \infty$: excited-state contamination (nucleons!)

- $L < \infty$: finite-volume effects, for simple quantities $\sim e^{-m_\pi L}$

  pions going "around the world" (periodic boundary conditions)

- $m_u = m_d$ too large, (often) still needed (for a number of reasons)

  inversion of $D + m$ more expensive as $m \to 0$

  ($\langle \overline{\psi}\psi \rangle = -\pi\rho(0)$, $\rho(\lambda)$ density of eigenvalues (Banks&Casher))

  extrapolate to physical values with chiral perturbation theory (MG in LH)

- operator mixing: use lattice fermions with good chiral symmetry!

Banks-Casher formula

$$
\begin{aligned}
\langle\overline{\psi}(x)\psi(x)\rangle &= \lim_{m\to 0}\lim_{V\to\infty} -\frac{1}{V}\sum_x \langle \mathrm{tr}(\psi(x)\overline{\psi}(x))\rangle \\
&= -\lim_{m\to 0}\lim_{V\to\infty}\frac{1}{V}\sum_x \left\langle \sum_\lambda \frac{\mathrm{tr}(f_\lambda(x) f_\lambda^\dagger(x))}{i\lambda + m}\right\rangle \\
&= -\lim_{m\to 0}\int_{-\infty}^{\infty} d\lambda\,\frac{\rho(\lambda)}{i\lambda + m} \\
&= -\int_{-\infty}^{\infty} d\lambda\,\rho(\lambda)\left(P\frac{1}{i\lambda} + \pi\delta(\lambda)\right) \\
&= -\pi\rho(0)
\end{aligned}
$$

if $\rho(\lambda) = \rho(-\lambda)$ which is true if $\{D, \gamma_5\} = 0$

so that $Df_\lambda = i\lambda f_\lambda \;\Rightarrow\; D\gamma_5 f_\lambda = -\gamma_5 D f_\lambda = -i\gamma_5\lambda f_\lambda = -i\lambda f_{-\lambda}$

Domain-wall fermions

(Kaplan, Shamir; Kaplan in LH)

Consider five-dimensional fermions $\psi(x,s),\ \ s\geq 0$ with $M\sim 1/a$

$$(\partial\!\!\!/ + \gamma_5\partial_s + M)\psi(x,s) = 0$$

Note: no chiral symmetry, γ_5 is one of the gamma matrices!

⇒ construction can be discretized using Wilson fermions in five dimensions with no change in conclusions

Typical solutions have mass $\sim M$ but ∃ zero modes bound to boundary:

$\psi(x,s) = \chi_\pm(x)u_\pm(s)$ with $\partial\!\!\!/\chi_\pm = 0\ ,\quad P_\pm\chi_\pm = \chi_\pm \Rightarrow u_\pm(s) = u_\pm(0)e^{\mp Ms}$

u_- is not normalizable, but u_+ is!

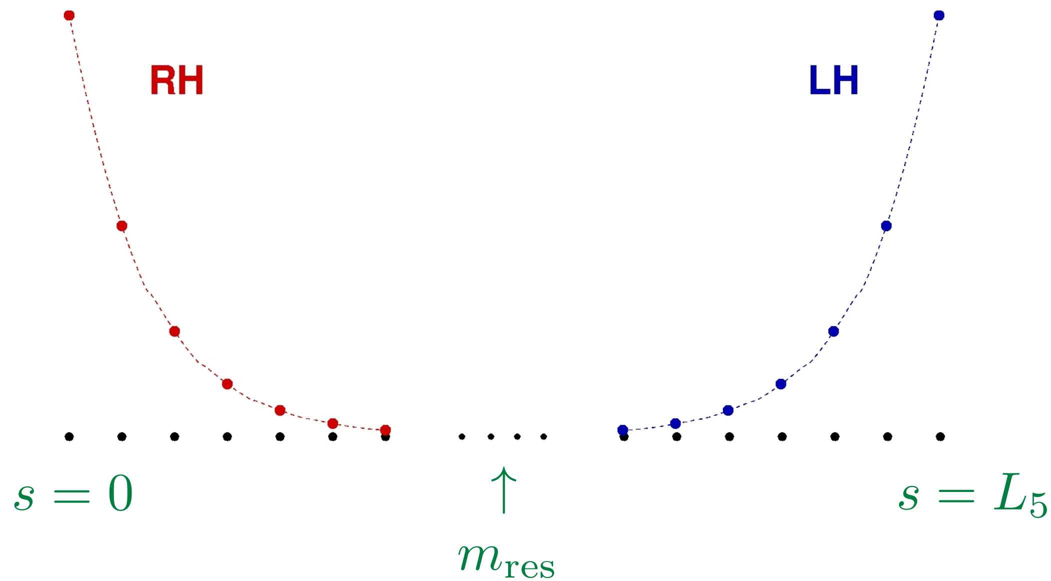
Chiral massless fermion in four dimensions ($s=0$) ("surface mode")

$U(1)_A$ anomaly produced by 5d massive fermions in the bulk (Callan&Harvey)

Now take $0 \le s \le L_5$

$\gamma_5 = +1$ (RH) @ $s = 0$

$\gamma_5 = -1$ (LH) @ $s = L_5$



zero modes now approximate:

$u_+(s)u_-(s) \sim M e^{-ML_5} \to 0$ for $L_5 \to \infty$

$\Rightarrow$ "residual mass" $m_{\rm res} \sim \frac{1}{a} e^{-ML_5}$ ("chirally improved Wilson fermion")

- quark mass: couple $m\left(\overline{\psi}(x, L_5)P_+\psi(x,0) + \overline{\psi}(x,0)P_-\psi(x,L_5)\right)$
- gauge fields: keep in four dimensions, $D_5 \equiv D_4(U) + \gamma_5\partial_s + M$
- wrong-chirality mixing for B_K : $\sim m_{\rm res}^2 \sim e^{-2ML_5}$

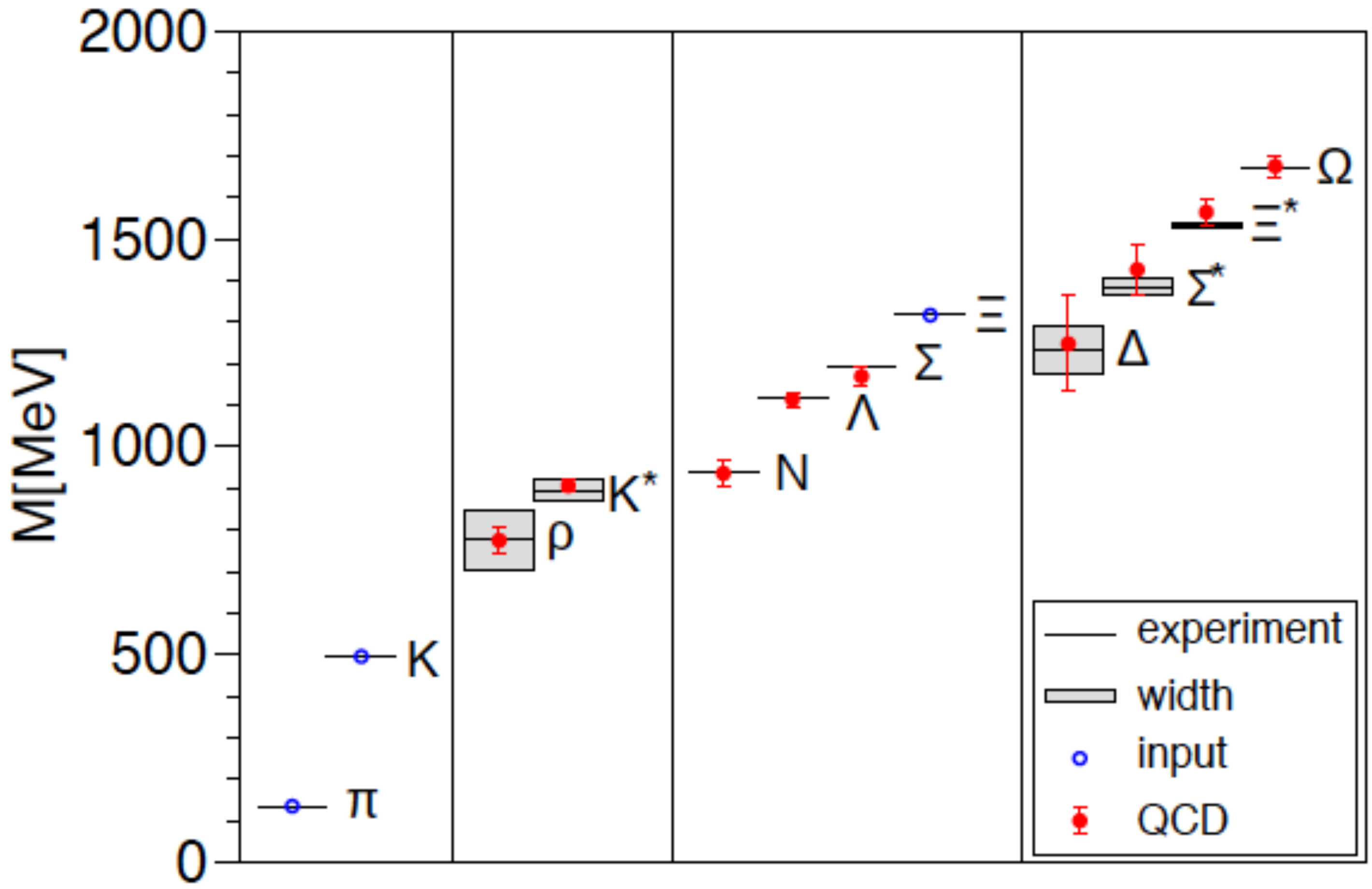
  need two chirality flips, *i.e.*, two "crossings"

What is Lattice QCD (lattice gauge theory) good for?

- Verification of QCD, *e.g.*, spectrum, incl. resonances(!) – glueballs, hybrids??
- Parameters of Standard Model and flavor physics (FLAG report)
- Nuclear Physics from QCD(!) (*e.g.*, Savage in Lattice 2016)
- QCD at finite temperature, equation of state (Philipsen in LH)
  (famous problem: QCD at finite density)
- Beyond the Standard Model:
  - high precision flavor physics
  - muon anomalous magnetic moment: hadronic contributions
  - non-SM *B* parameters
  - vary gauge group and fermion representations:
    strongly-coupled theories (composite Higgs, DeGrand, Rev. Mod. Phys.'16)
- Lattice: an "honest" non-perturbative regulator

Spectrum

BMW coll.'08

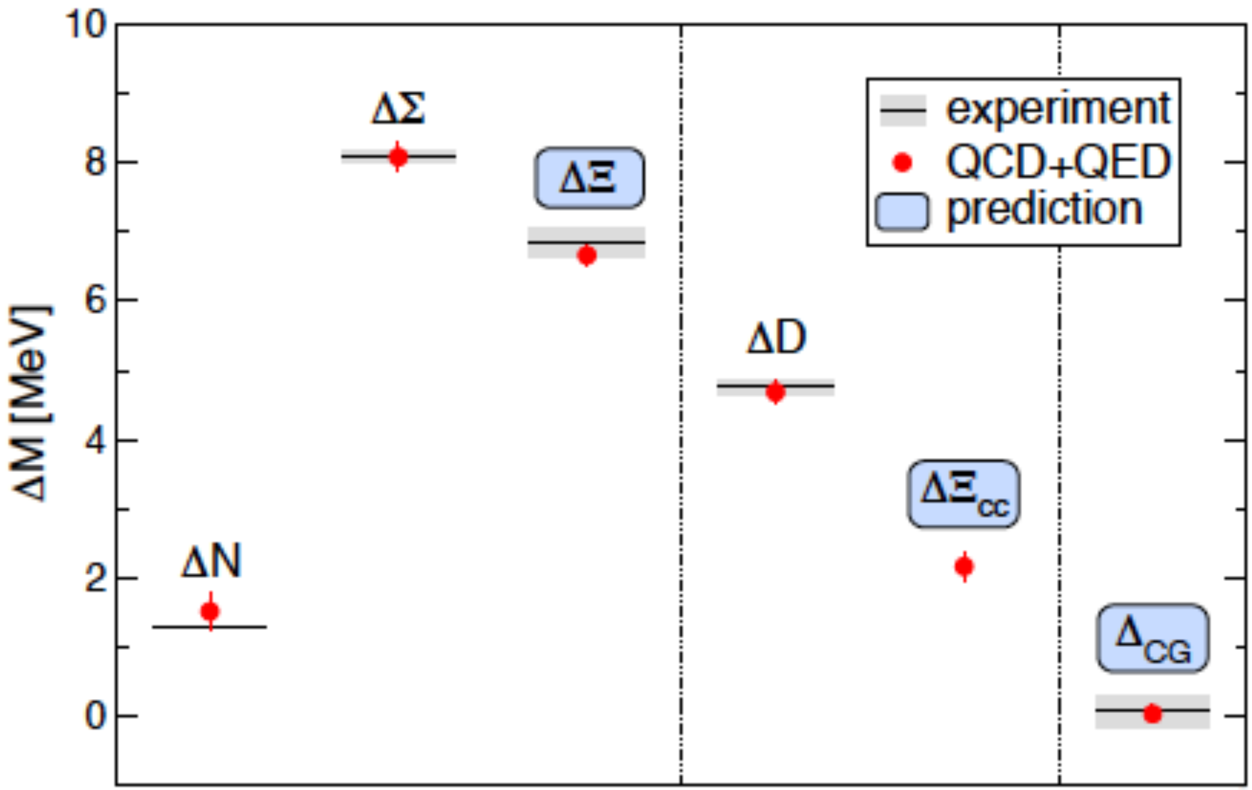


Set scale and strange/light quark masses ($m_u = m_d$) with $m_\Xi,\ m_\pi,\ m_K$

Three lattice spacings, physical strange quark mass, pion mass down to $190\ \mathrm{MeV}$

Mass splittings

BMW coll. ’14

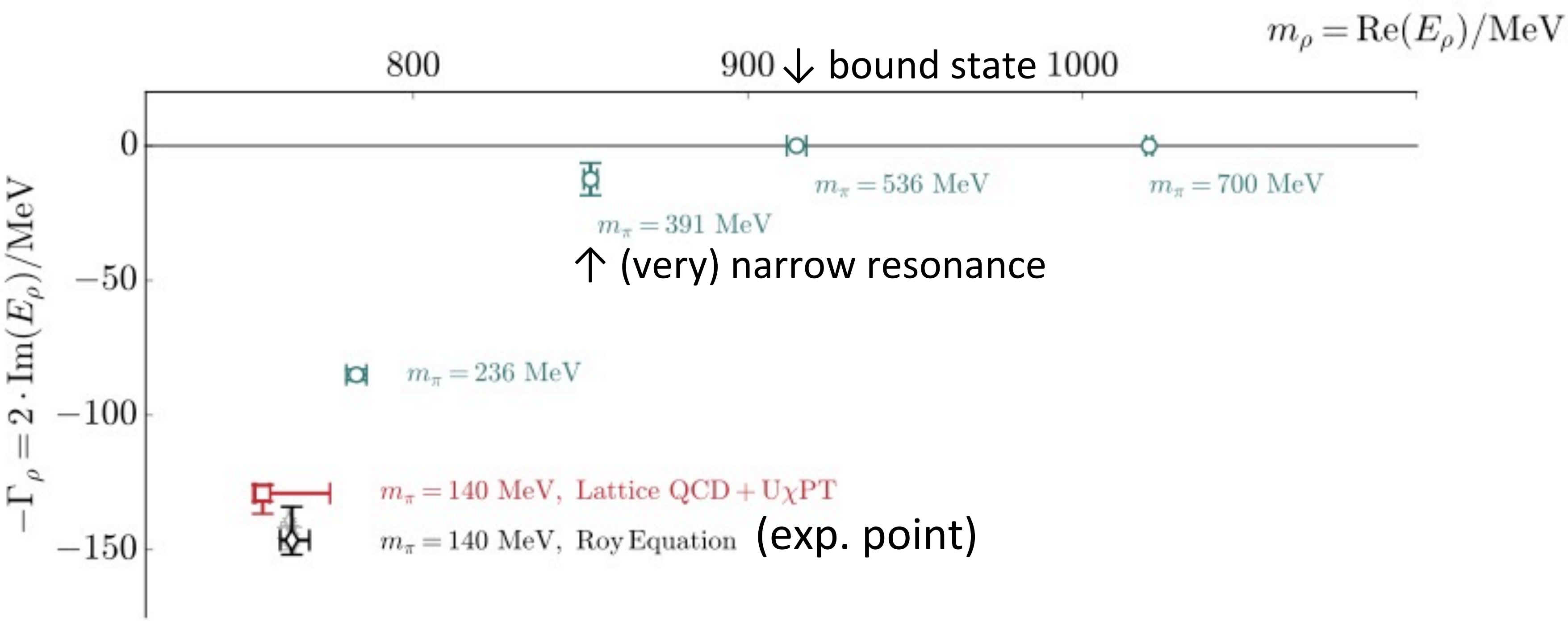


Includes charm (“1+1+1+1”), QED, isospin breaking, 4 lattice spacings

smallest charm mass $am_c = 0.35$, smallest pion mass $195\ \mathrm{MeV}$

Note lattice errors smaller than experimental errors for two splittings!

The ρ resonance



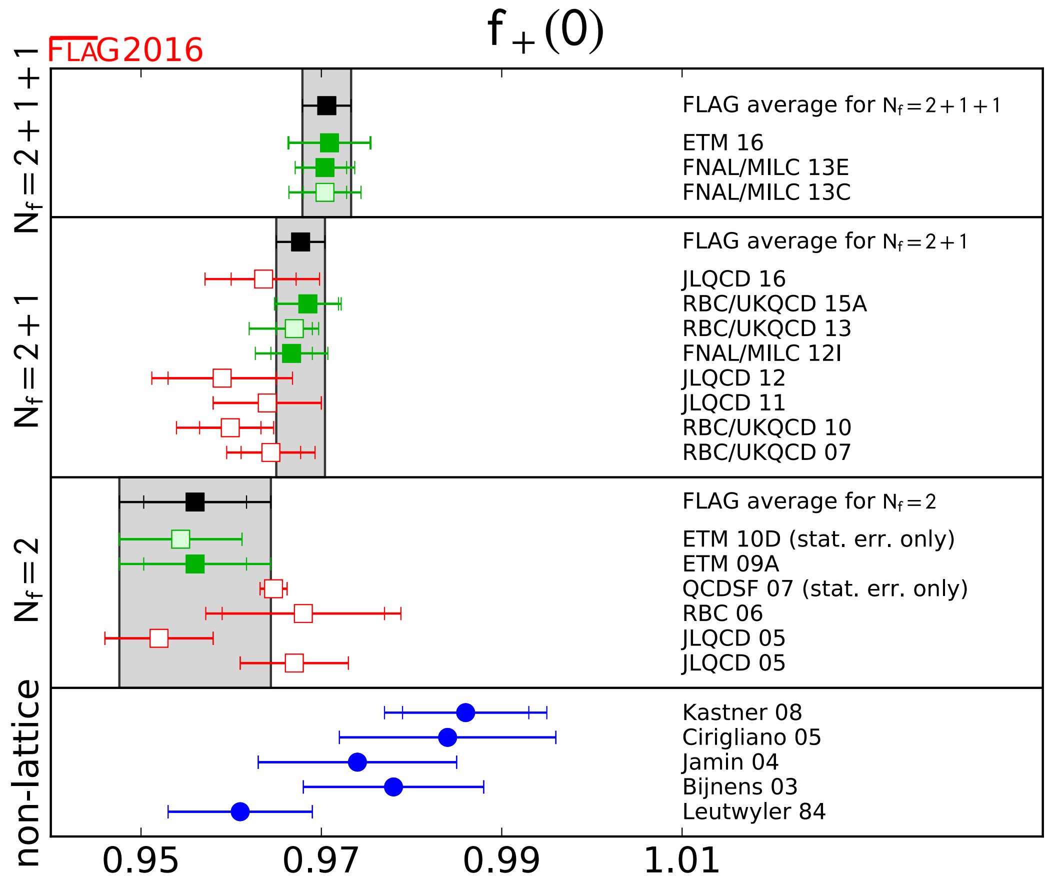
From Briceno, Dudek&Young, '17 review

Idea: spectrum in finite box is related to resonances in infinite volume (same hamiltonian! (Lüscher))

Flavor physics, I

FLAG '16

$K \to \pi$ form factor
($K^- \to \pi^0 e \overline{\nu}_e$ decay)



$$f_+(0) = 0.9704(33) \ (n_f = 2+1+1)$$

Note precision of lattice computations
green: incorporated in FLAG average; red: not all syst. errors controlled

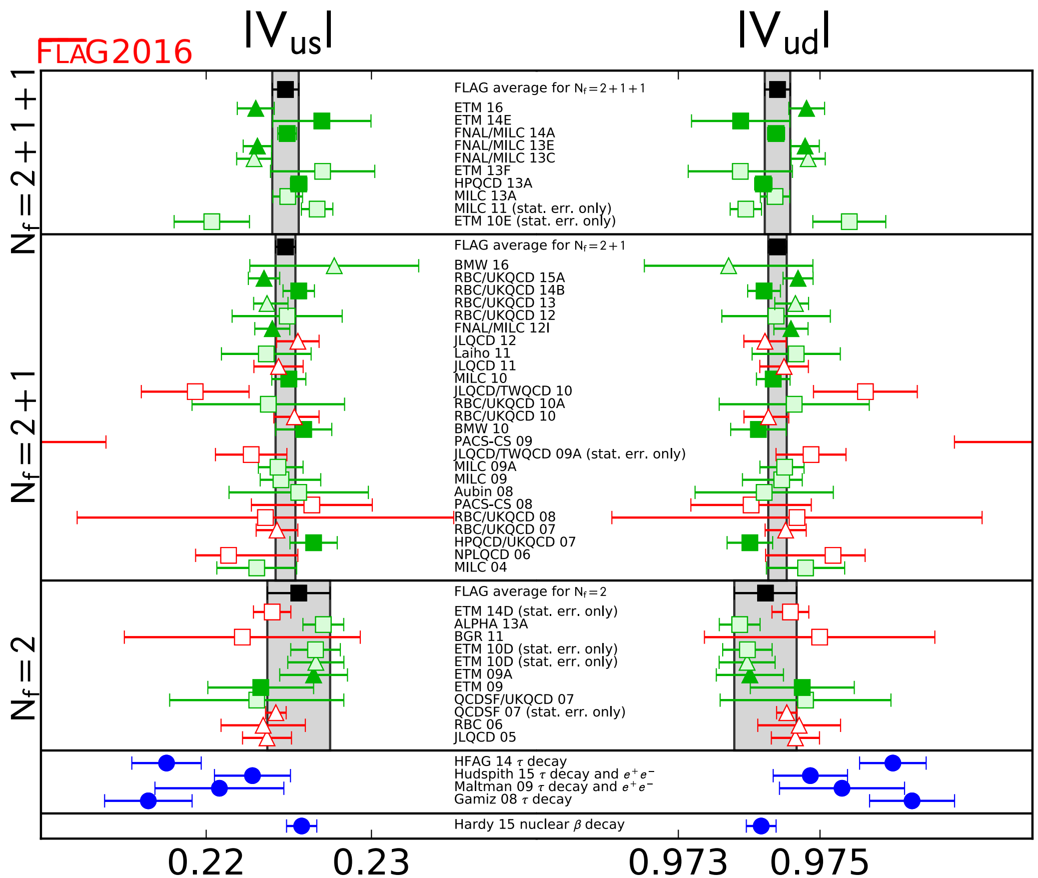
Flavor physics, II

FLAG ’16

from

$f_+(0)$ squares

f_{K^+}/f_{π^+} triangles



assumes Standard Model, *i.e.*, unitarity of CKM matrix

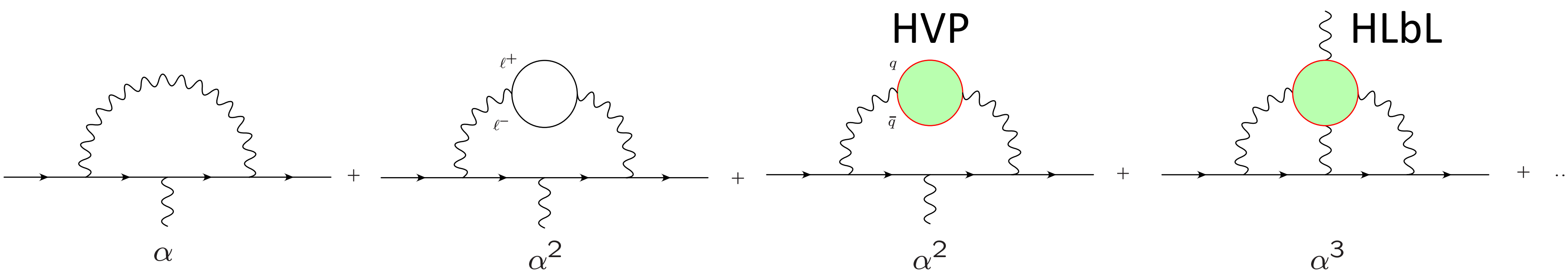
Note precision of lattice computations

green: incorporated in FLAG average; red: not all syst. errors controlled

Muon anomalous magnetic moment I

$$\vec{\mu} = g\left(\frac{e}{2m}\right)\vec{S} \qquad \rightarrow \qquad V(\vec{x}) = -\vec{\mu}\cdot\vec{B}_{\text{ext}}$$

$$a_\mu = (g-2)/2 \propto \left(\alpha = \frac{e^2}{4\pi\hbar c}\right) \neq 0$$



↑ Schwinger term $a_\mu = \dfrac{\alpha}{2\pi} = 0.0011614$

Hadronic corrections:

1st green blob: H(adronic) V(acuum) P(olarization) (HVP)

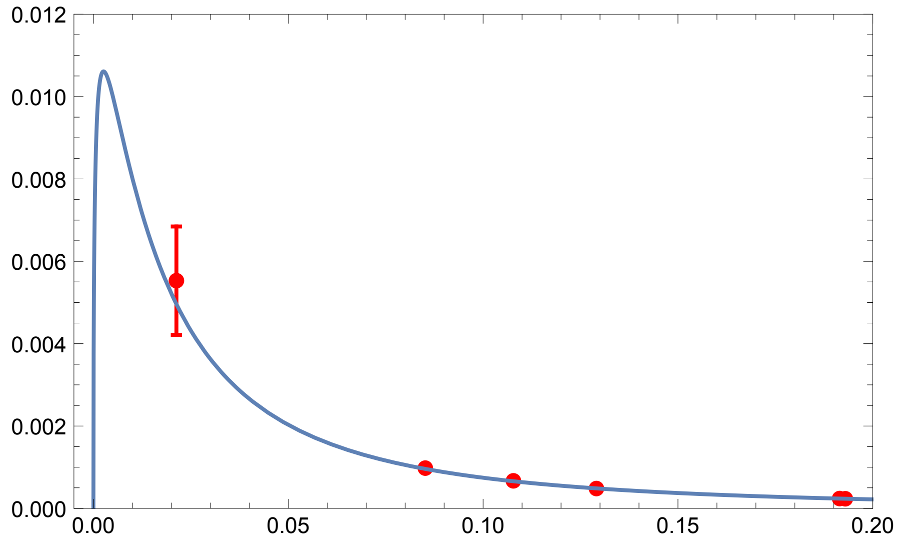
2nd green blob: H(adronic) L(ight) b(y) L(ight) (HLbL)

Low energy: LQCD is the only game in town for a theory computation!

HVP precision issue:

$$a_\mu^{\rm HVP} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{dQ^2}{Q^2}\, w(Q^2)[\Pi(Q^2) - \Pi(0)]$$

Integrand looks like:



Smallest momentum is $2\pi/T$

Small error in vacuum polarization can lead to large error!
The weight function $w(Q^2)/Q^2 \sim 1/\sqrt{Q^2}$ is a "magnifying glass"
Note that data are very sparse in the region with most curvature

Muon anomalous magnetic moment II

- Experimental value $a_\mu^{\text{exp}} = 116592089(63) \times 10^{-11}$ (dominated by BNL E821)
- Standard-Model value (from Blum *et al.* review '13)

	Value ($\times 10^{-11}$) units
QED ($\gamma + \ell$)	$116\,584\,718.951 \pm 0.009 \pm 0.019 \pm 0.007 \pm 0.077_\alpha$
HVP(lo) [20]	$6\,923 \pm 42$
HVP(lo) [21]	$6\,949 \pm 43$
HVP(ho) [21]	-98.4 ± 0.7
HLbL	105 ± 26
EW	154 ± 1
Total SM [20]	$116\,591\,802 \pm 42_{\text{H-LO}} \pm 26_{\text{H-HO}} \pm 2_{\text{other}}\ (\pm 49_{\text{tot}})$
Total SM [21]	$116\,591\,828 \pm 43_{\text{H-LO}} \pm 26_{\text{H-HO}} \pm 2_{\text{other}}\ (\pm 50_{\text{tot}})$

HVP(lo): from $e^+e^- \to$ hadrons

HLbL: model estimate!

Total SM [20]: Davier *et al.*

Total SM [21]: Hagiwara *et al.*

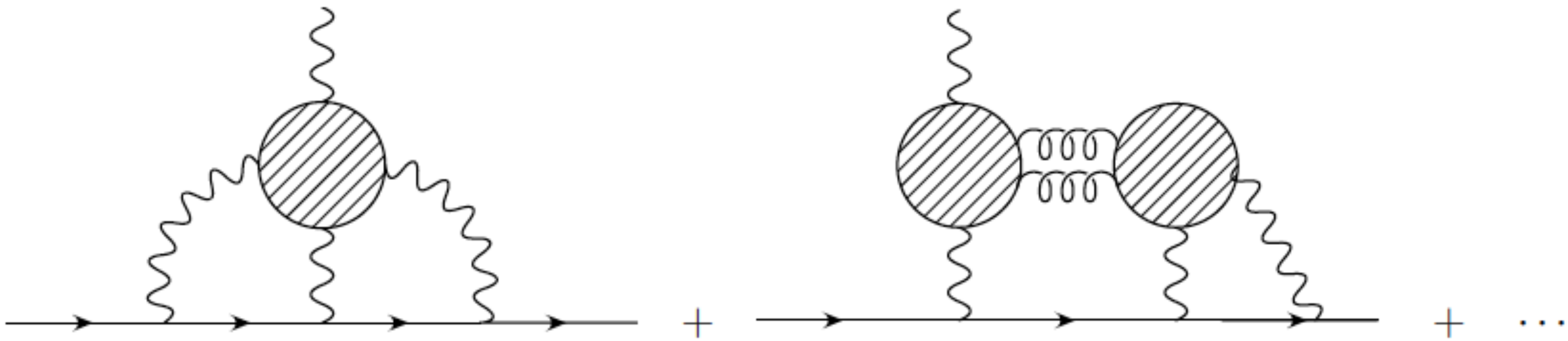
- Difference: $\Delta a_\mu(\text{E821} - \text{SM}) = 287(80) \times 10^{-11}\ [20] \quad (3.6\ \sigma)$

$$= 261(80) \times 10^{-11}\ [21] \quad (3.3\ \sigma)$$

- New Fermilab experiment: error reduction with factor 4 expected!

Muon anomalous magnetic moment III

HLbL (Blum *et al.*, at Lattice 2017, see also Mainz group)



$$a_\mu^{\rm cHLbL} + a_\mu^{\rm dHLbL} = (116.0 \pm 9.6 - 62.5 \pm 8.0) \times 10^{-10}$$

- One lattice spacing ($0.114\ \mathrm{fm}$), one volume ($5.5\ \mathrm{fm}$), physical pion mass
- More "disconnected" diagrams (but suppressed by powers of $m_s - m_{u,d}$)
- Fermilab exp.: reproduction of BNL result end of 2018 (Roberts@g-2 workshop '17)