International Summer School on High Energy Physics
TAE 2017, Benasque

QCD Tutorial 1

1. (a) The N^2-1 generators $\boldsymbol{t}^A$ of $SU(N)$ and the unit matrix form a basis in the space of $N \times N$ matrices (where of course $N=3$ for QCD). Verify that

$$\mathcal{M}_{ij} = \frac{\mathbf{Tr}[\mathcal{M}]}{N}\delta_{ij} + 2\mathbf{Tr}[\mathcal{M}\boldsymbol{t}^A]t^A_{ij}\,. \tag{1}$$

and use this result to establish the relation

$$t^A_{ij}t^A_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right)\,. \tag{2}$$

(b) Use the relation (2) to show that

$$(\boldsymbol{t}^A\boldsymbol{t}^A)_{ij} = \left(\frac{N^2-1}{2N}\right)\delta_{ij} \equiv C_F\,\delta_{ij} \tag{3}$$

(c) Use again the relation (2) to calculate the color factors for the scattering amplitude for $q_i\bar{q}_\ell \to q_j\bar{q}_k$ with a gluon exchanged in the $t-$channel (the subscripts i, j, ℓ, k denote color). Show that the interaction can be either attractive (positive color factor) or repulsive (negative color factor) according to the $q\bar{q}$ pair being in a color-singlet or a color-octet state. [Hint: color-singlet and color-octet states can be obtained by projecting the $q_j\bar{q}_k$ pair via multiplication of the scattering amplitude with a δ_{jk} or a t^B_{jk} respectively.]

2. (a) Use the QCD Feynman rules and the relations of the colour matrices calculate the colour factor associated to the probability of emission of a gluon by a quark.

(b) Calculate the colour factor associated to the probability of the splitting $g \to q\bar{q}$. Is it the same as the previous one?

(c) Making use of the relation

$$\sum_{bc} f^{abc}f^{ebc} = N\delta^{ae} \tag{4}$$

and of the Feynman rule for the three-gluon vertex (the vertex of the splitting $g^a \to g^b\,g^c$ is proportional to f^{abc}), calculate the colour factor associated to the probability of the splitting $g \to gg$.

(d) Try to derive the relation (4) above (see for instance Chapter 15 of Peskin-Schroeder).

3. Consider a $q(p_1)\bar{q}(p_2)$ pair, produced by a photon of momentum Q. Denote the amplitude of this process by

$$\mathcal{M}_{q\bar{q}} = -\bar{u}(p_1) i e_q \gamma_\mu v(p_2) \tag{5}$$

Consider now a gluon of momentum k emitted by the pair.

(a) Use the QCD Feynman rules to write down the amplitude for this process. Show that you get

$$\mathcal{M}_{q\bar{q}g} = \bar{u}(p_1) i g \not{\epsilon} t^A \frac{i}{\not{p}_1 + \not{k}} i e_q \gamma_\mu v(p_2) - \bar{u}(p_1) i e_q \gamma_\mu \frac{i}{\not{p}_2 + \not{k}} i g \not{\epsilon} t^A v(p_2) \tag{6}$$

(b) Show that, in the soft limit $k \ll p_{1,2}$, you can approximate $\mathcal{M}_{q\bar{q}g}$ as

$$\mathcal{M}_{q\bar{q}g} \simeq \mathcal{M}_{q\bar{q}} t^A g \left(\frac{p_2 \cdot \epsilon}{p_2 \cdot k} - \frac{p_1 \cdot \epsilon}{p_1 \cdot k} \right) \tag{7}$$

(c) Further show that, after summing over the gluon polarisations and colours,

$$\sum |\mathcal{M}_{q\bar{q}g}|^2 \simeq |\mathcal{M}_{q\bar{q}}|^2 C_F g^2 \frac{2 p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)} \tag{8}$$

and that, after including the gluon phase space and denoting E as the gluon energy and θ as the gluon emission angle with respect to the quark momentum, and defining $\alpha_s \equiv g^2/4\pi$, you have

$$\sum |\mathcal{M}_{q\bar{q}g}|^2 d\Phi_{q\bar{q}g} \simeq |\mathcal{M}_{q\bar{q}}|^2 d\Phi_{q\bar{q}}\, d\mathcal{S} \tag{9}$$

where

$$d\mathcal{S} = \frac{2\alpha_s C_F}{\pi} \frac{dE}{E} \frac{d\theta}{d\sin\theta} \frac{d\phi}{2\pi} \tag{10}$$