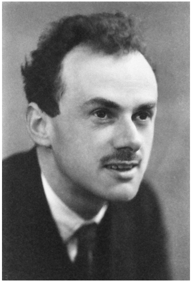
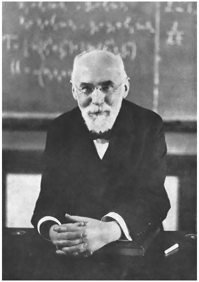
The nature of the neutrino

A general discussion about Chirality and Helicity





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Motivations

- Neutrinos are massless in the Current SM but we know through neutrino oscillations that they are massive. Secure search for BSM.
- It is the most abundant form of matter in the Univers. Key role at astrophysical sclaes.
- Majorana neutrinos unlike those of Dirac violate lepton number conservation and consequently they could be related to the matter-antimatter observed imbalance.
- The peculiar character of neutrinos makes them good candidates to resolve the unaccountable dark matter presence.
- Is one of the most ancient questions in particle physics for which we do not have a successful answer.
- There are lots of experiments trying to distinguish Dirac from Majorana by looking to the famous neutrinoless double beta decays but currently there are not mathematically proven interesting alternatives.

The Lorentz Group

Wondering about being flying over a lightray Einstein realized that the time would be percived as different depending on the observer's speed. Assuming that two independent observers would measure the same speed for the light on its own reference frames, the special relativity was born. Mathematically, we can write:

$$\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2} = c(s-t) \qquad \text{reference frame 1}$$

$$\sqrt{(x_1-x_2')^2+(y_1'-y_2')^2+(z_1'-z_2')^2} = c(s'-t') \qquad \text{reference frame 2}$$

Or rearranging:

$$s^2 = c^2t'^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2$$

That in compact form is usually written as:

$$s^2 = (x'^{\mu})^2 = (x^{\mu})^2$$

There should exist some metric g such that the norm $x^{\mu} \cdot x^{\mu}$ of a general spacetime vector $x^{\mu} = (ct, x, y, z)$ gives us an invariant spacetime interval s^2. Of course the proper election is the Minkowski metric g:

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

But another interesting approach is to think in spacetime vectors as objects with an imaginary time component $x^{\mu} = (ict, x, y, z)$. That helps to intuitively understand why complex numbers continously appear in high energy physics. In any case, we are interested in studying the transformation between objects that fulfills the SR postulates, or equivalently

$$x' = \Lambda x$$

The Lorentz Group

The condition that the transformations $x' = \Lambda x$ have to fulfill is

$$x' \cdot x' = x \cdot x \quad \rightarrow \quad x'^T g x' = (\Lambda x)^T g (\Lambda x) = x^T \Lambda^T g \Lambda x = x^T g x \quad \rightarrow \quad \boxed{\Lambda^T g \Lambda = g}$$

The objects that satisfy this condition form the **Lorentz Group**.

It is very interesting to find which are the generators of the group, or in other terms, the objects L such that

$$\Lambda = e^L \quad \text{where} \quad \det|A| = \det|e^L| = e^{\mathrm{Tr}\, L} = \pm 1$$

From the boxed condition it is trivial to find

$$\Lambda^T g \Lambda = g \quad \rightarrow \quad g A^T g = A^{-1}$$

Since $A^T = e^{L^T}$, $A^{-1} = e^{L^{-1}}$ and $g^2 = I$, we arrive to fundamental condition of the Lorentz generators:

$$(gL)^T = -gL$$

That forces

$$L = \begin{pmatrix} 0 & L_{01} & L_{02} & L_{03} \\ L_{01} & 0 & L_{12} & L_{13} \\ L_{02} & -L_{12} & 0 & L_{23} \\ L_{03} & -L_{13} & -L_{23} & 0 \end{pmatrix}$$

From here we find that there are only 6 generators and accordingly, we can find 6 independent generators.

The Lorentz Group

We can distinguish three *Rotation* generators:

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

As well as three *Boost* generators:

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Where $S_i^3 = -S_i$ and $K_i^3 = K_1$. Now is straightforward how to generate a Lorentz transformation.

$$L = -\omega^{\alpha} S_{\alpha} - \xi^{\beta} K_{\beta}$$

Example:
If we have have a boost in the x_1 direction caracterized by a rapidity $\xi_1 = v_1/c$, and no rotation:

$$\begin{aligned} A = e^L = e^{-\xi_1 K_1} &= I - \xi_1 K_1 + \frac{1}{2!}(\xi_1 K_1)^2 - \frac{1}{3!}(\xi_1 K_1)^3 + \ldots \\ &= (I - K_1)^2 - K_1(I + \frac{1}{3!}x_1^3 + \ldots) + K_2(\frac{1}{2!}x_1^2 + \ldots) \\ &= (I - K_1)^2 - K_1 \sinh\xi + K_1^2 \cosh\xi \end{aligned}$$

Or in matricial form

$$\Lambda = \begin{pmatrix} \cosh\xi & -\sinh\xi & 0 & 0 \\ -\sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Lie Algebra

We have many ways in which we can express the generators of a group, known as representations. The generators of the Lorentz group SO(3,1) satisfy the Lie Algebra:

$$[S_i, S_j] = i\varepsilon_{ijk} S_k, \qquad [K_i, K_j] = -i\varepsilon_{ijk} S_k, \qquad [S_i, K_j] = i\varepsilon_{ijk} K_k$$

However, this is a *reducible* representation. The Lorentz group may be reduced under the following redefinition of its generators:

$$A_i = \frac{S_i + iK_i}{2}, \qquad B_i = \frac{S_i - iK_i}{2}$$

That satisfy the Lie Algebra of SU(2)

$$[A_i, A_j] = i\varepsilon_{ijk} A_k, \qquad [B_i, B_j] = -i\varepsilon_{ijk} B_k, \qquad [A_i, B_j] = 0$$

Consequently, the Lorentz algebra SO(3,1) factorizes in two independent SU(2) algebras, that are the irreducible ones for the Lorentz group. Accordingly, we can clasify the objects, depending on how they transform under lorentz. If we have some n-dimensional object $\Psi^n(x)$ it will transform under Lorentz as

$$\Psi'^n = e^{\phi_1 A + \phi_2 B} \Psi^n$$

And of course it may have a *scalar* structure, meaning that it does not transform (is a Lorentz eigenstate with 0 eigenvalue). It may be a *left spinor*: it transforms with A but not with B (It has 1/2 eigenvalue for A and 0 eigenvalue for B, and accordingly it is a spin 1/2 particle). Or the other way around it may be a *right spinor*: It transforms with B but not with A (It has 1/2 eigenvalue for B and 0 eigenvalue for A, and accordingly it is a spin 1/2 particle). Usually this is sumarized as the (0,0), ($\frac{1}{2}$,0) or (0,$\frac{1}{2}$) representations of the Lorentz group.

Chirality

It is a fundamental property directly related to the Lorentz group representations (abstract mathematics).

The spinor and bispinor representations

Since we are interested in neutrinos, we will focus on the spin $\frac{1}{2}$ representation of the Lorentz group. We have seen that we have two independent SU(2) algebras defining the mathematical space for the physical objects we are interested in. Recalling that the Pauli matrices are a well known basis for SU(2), we can descrive particles as two component objects χ and η with the following properties: For the $(\frac{1}{2},0)$ representation

$$A_i\chi_a = \left(\frac{\sigma}{2}\right)_{ab}\chi_b \qquad \Rightarrow \qquad \frac{1}{2}(J_i + iK_i) = \left(\frac{\sigma}{2}\right)_{ab}\chi_b$$

$$B_i\chi_a = 0 \qquad \Rightarrow \qquad \frac{1}{2}(J_i - iK_i) = 0$$

For the $(0,\frac{1}{2})$ representation

$$A_i\eta_a = 0 \qquad \Rightarrow \qquad \frac{1}{2}(J_i + iK_i) = 0$$

$$B_i\eta_a = \left(\frac{\sigma}{2}\right)_{ab}\eta_b \qquad \Rightarrow \qquad \frac{1}{2}(J_i - iK_i) = \left(\frac{\sigma}{2}\right)_{ab}\eta_b$$

Solving the equations we obtain

$$\vec{J}\chi = \frac{\vec{\sigma}}{2}\chi \qquad \vec{J}\chi = -i\frac{\vec{\sigma}}{2}\chi \qquad \text{and} \qquad \vec{J}\eta = \frac{\vec{\sigma}}{2}\eta \qquad \vec{J}\eta = i\frac{\vec{\sigma}}{2}\eta$$

From this fundamental, well defined left and right handed bilding blocks we can construct a 4-dimensional object Ψ, know as a bispinor, that corresponds to the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation.

$$\Psi = \begin{pmatrix}\chi\\ \eta\end{pmatrix} = \begin{pmatrix}\Psi_L\\ \Psi_R\end{pmatrix} \qquad \text{and} \qquad \vec{J}\Psi = \begin{pmatrix}\frac{\vec{\sigma}}{2} & 0\\ 0 & \frac{\vec{\sigma}}{2}\end{pmatrix}\Psi, \qquad \vec{K}\Psi = \begin{pmatrix}-i\frac{\vec{\sigma}}{2} & 0\\ 0 & i\frac{\vec{\sigma}}{2}\end{pmatrix}\Psi$$

The Mass Role

Up to now we have seen that objects that transform under Lorentz (object that live in the space-time in agreement with the SR) can be clasified depending on its inner structure, namely the way that they transform under the Lorentz group. If all the objects would be massless, namely if all the objects would travel at the speed of the light, that would be the end of the story. However we know for our daily expirience that we are not traveling to the speed of light and consequently the problem is quite a bit more complex.

Let's consider a real problem: Finding the solution for a realtivistic spin $\frac{1}{2}$ free particle. Since it is relativistic, the solutions must have the shape of spin $\frac{1}{2}$ Lorentz representations. The correct equation was found by Dirac and it is:

$$(i\gamma^\mu \partial_\mu - m)\,\Psi = \begin{pmatrix} -m & i\sigma^\mu \partial_\mu \\ i\bar{\sigma}^\mu \partial_\mu & -m \end{pmatrix} \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = 0$$

If I choose this particular γ^μ basis:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \qquad \begin{aligned} \bar{\sigma}^\mu &= (\sigma^0, \vec{\sigma}) \\ \sigma^\mu &= (\sigma^0, -\vec{\sigma}) \end{aligned}$$

In the case of having a **massless** particle the solutions are docoupled, and are trivially given by:

$$\begin{aligned} (\bar{\sigma}^\mu \partial_\mu)\Psi_R &= 0 \\ (\sigma^\mu \partial_\mu)\Psi_L &= 0 \end{aligned}$$

This are the famous Weyl equations that *almost* describe correctly the neutrinos. However, since we know from oscillations that neutrinos are massive, we need to solve the system considering a non-zero mass.

The massive Dirac equation

Of course, the solution for the Dirac equation is well known. Assuming plane-wave soultions of the form:

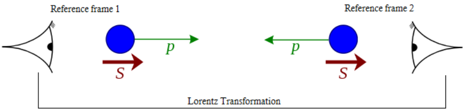
$$\Psi = u(p^\mu)e^{-ip^\mu x_\mu}$$

Gathering this ansatz inside the Dirac equation we can find the necessary conditions to find the shape of $u(p^\mu)$:

$$u_1 = N\begin{pmatrix}1\\0\\\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\end{pmatrix} \qquad u_2 = N\begin{pmatrix}0\\1\\\frac{p_x-ip_y}{E+m}\\\frac{-p_z}{E+m}\end{pmatrix} \qquad u_3 = N\begin{pmatrix}\frac{p_z}{E-m}\\\frac{p_x+ip_y}{E-m}\\1\\0\end{pmatrix} \qquad u_4 = N\begin{pmatrix}\frac{p_x-ip_y}{E-m}\\\frac{-p_z}{E-m}\\0\\1\end{pmatrix}$$

By inspectioning this objects we can realize that, as we could expect, this objects are not Chirality eigenstates, since the left and the right components are now mixed. However, this objects are eigenstates from a slightly different observable, the helicity. It is defined as:

$$\hat{h} = \frac{\vec{S}\vec{p}}{|\vec{S}||\vec{p}|} = \frac{1}{2}\begin{pmatrix}\vec{\sigma}\vec{p} & 0\\ 0 & \vec{\sigma}\vec{p}\end{pmatrix}$$



The funny thing about helicity is that it is RF-dependent quantity! Imagine we want to compute $\hat{h}\Psi_1 = u_1 e^{-ip^\mu x_\mu}$.

$$\text{if} \qquad p_z > 0, p_x = p_y = 0 \qquad \hat{h}\Psi_1 = \Psi_1$$
$$\text{if} \qquad p_z < 0, p_x = p_y = 0 \qquad \hat{h}\Psi_1 = -\Psi_1$$

Majorana

Up to now I just explained how to compute things using Dirac particles, but in any case I presented previously what a Majorana particle is. To do so, lets consider the Dirac Lagrangian.

$$\mathcal{L} = i\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi + M\bar{\Psi}\Psi \tag{1}$$

From which the Dirac equation is derived. Notice that in the lagrangian we have been forced to define a new object $\bar{\Psi}$ in order to ensure that the quantities $\bar{\Psi}\Psi$ and $\bar{\Psi}\gamma^{\mu}\Psi$ are Lorentz invariant, where $\bar{\Psi} = \Psi^{\dagger}\gamma^{0}$.

And here comes the brilliant idea by Majorana. Instead of constructing a field by joining a chiral left and a chiral right fields as Dirac did, he realized that, by redefinition of the field it is possible to define an object using two left, or two right fields!

$$\Psi_D = \begin{pmatrix} \Psi_R \\ \Psi_L \end{pmatrix} \qquad \Psi_M = \begin{pmatrix} -\varepsilon\Psi_L^{*} \\ \Psi_L \end{pmatrix} \quad \text{or} \quad \Psi_M = \begin{pmatrix} -\varepsilon\Psi_R^{*} \\ \Psi_R \end{pmatrix} \tag{2}$$

Either Ψ_D and Ψ_M may construct succesfully a Lorentz inveriant Lagrangian with a mess term. For Dirac we have

$$(\hat{\sigma}^{\mu}\partial_{\mu})\Psi_R - m\varepsilon\Psi_L = 0$$
$$(\hat{\sigma}^{\mu}\partial_{\mu})\Psi_L - m\varepsilon\Psi_R = 0$$

And for Majorana

$$(\hat{\sigma}^{\mu}\partial_{\mu})\Psi_R - m\varepsilon\Psi_R^{*} = 0 \qquad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$(\hat{\sigma}^{\mu}\partial_{\mu})\Psi_L - m'\varepsilon\Psi_L^{*} = 0$$

Helicity Representation

	$m \neq 0$	m=0
Dirac fields	$\Psi_R(x) = \int \left[A(+)e^{-ikx} - B^\dagger(-)e^{ikx}\right]\chi(+)\sqrt{E+k} + \int \left[A(-)e^{-ikx} + B^\dagger(+)e^{ikx}\right]\chi(-)\sqrt{E-k}$ $\Psi_L(x) = \int \left[A(-)e^{-ikx} - B^\dagger(+)e^{ikx}\right]\chi(-)\sqrt{E+k} + \int \left[A(+)e^{-ikx} + B^\dagger(-)e^{ikx}\right]\chi(+)\sqrt{E-k}$	$\Psi_R(x) = \int \left[A(+)e^{-ikx} - B^\dagger(-)e^{ikx}\right]\chi(+)\sqrt{2E}$ $\Psi_L(x) = \int \left[A(-)e^{-ikx} - B^\dagger(+)e^{ikx}\right]\chi(-)\sqrt{2E}$
Majorana field	$\Psi_L(x) = \int \left[a(-)e^{-ikx} - a^\dagger(+)e^{ikx}\right]\chi(-)\sqrt{E+k} + \int \left[a(+)e^{-ikx} + a^\dagger(-)e^{ikx}\right]\chi(+)\sqrt{E-k}$	$\Psi_L(x) = \int \left[a(-)e^{-ikx} - a^\dagger(+)e^{ikx}\right]\chi(-)\sqrt{2E}$

Table 1: Dirac and Majorana Fields in Helicity representation.

If $E \gg m$, $m \neq 0 \Rightarrow \sqrt{E-k} \approx \frac{m}{\sqrt{2E}} + \mathcal{O}(m^2)$. The “wrong” helicity states are supressed. $\chi(\pm)$ are Pauli spinors for helicity (+) or (-).

We use $A, A^\dagger$ for particles and $B, B^\dagger$ for antiparticles. Majorana does not distinguish between particles and operators, in fact

$$a(\pm) = \frac{1}{\sqrt{2}}\left(A(\pm) + B(\pm)\right) \tag{3}$$

Helicity

Helicity is a mathematical deformation of the Chirality for massive particles. When we do $m \to 0$ helicity eigenstates smoothly approach the chirality eigenstates making the difference between Dirac and Majorana almost indistinguishible.

Conceptual Diference

Before study interactions let's see what we should look for.

For Dirac: $A(-)$ and $B(+)$.
For Majorana: $a(-)$ and $a(+)$

Is A=B?

In order to give an answer we need to compare

$$A(-) \quad \text{with} \quad B(-) \qquad \text{or} \qquad A(+) \quad \text{with} \quad B(+) \tag{4}$$

The problem is that A(+) and B(-) appear in the right handed Chiral state Ψ_R which is almost sterile.

Are there other possibilities to distinguish them?

- For Majorana we have an imporant extra condition: $i\gamma^2\Psi_M^* = \Psi_M$.
- Majorana can break lepton number conservation. Hence for a same process there is a diferent number of diagrams for Dirac and Majorana.
- Process involving two neutrinos in the final state should be symmetric for Majorana but not for Dirac.

Does it represent any appreciable difference?