Matter under extreme conditions

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EXCELENCIA MARÍA DE MAEZTU

Today's menu

- Matter at finite baryon density
- Matter at finite isospin density
- Matter with a chiral imbalance

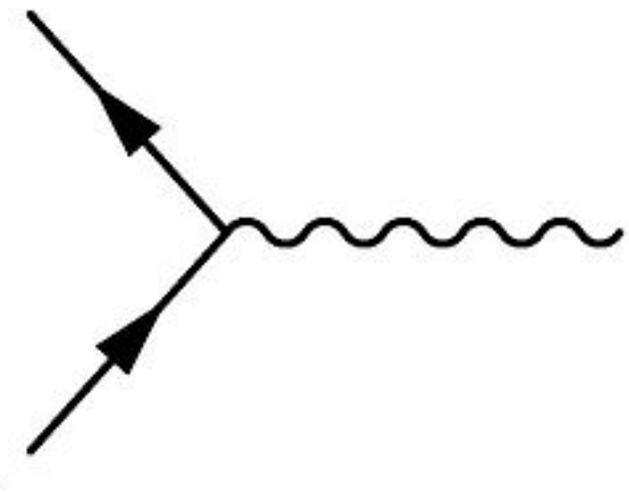
(*matter = boring standard model stuff)

Setting the stage

(the theory)

Quantum Electrodynamics (QED)

Describes charged fermions interacting with photons



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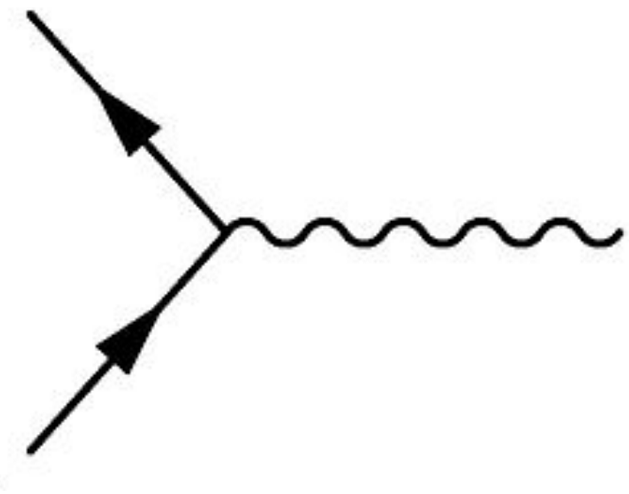

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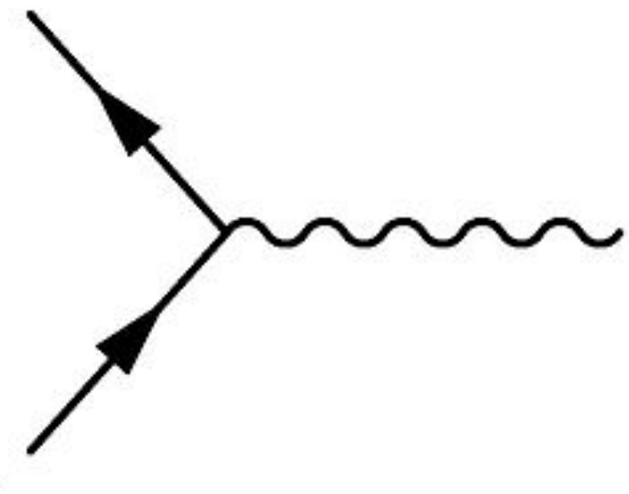

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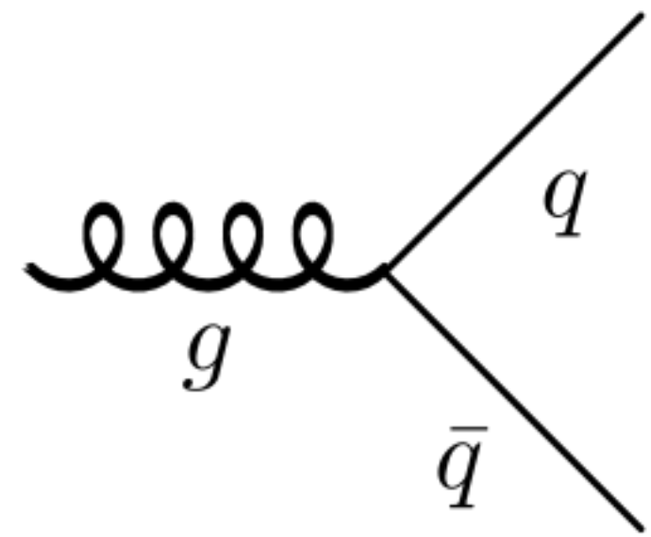
Quantum Chromodynamics (QCD)

Describes quarks interacting with gluons…



… and gluons interacting with gluons…

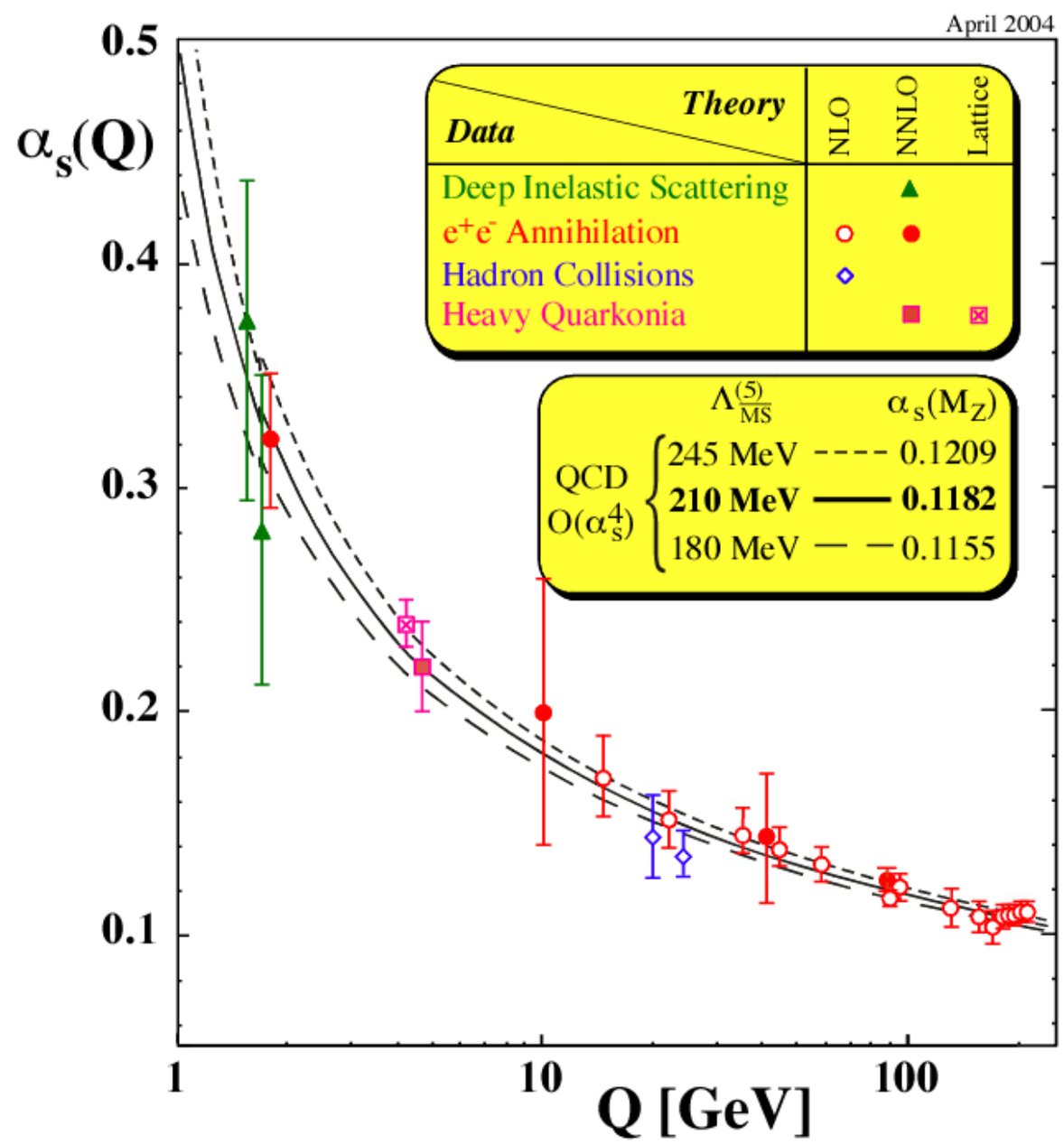
QCD Running coupling

Up to intermediate energies, the QCD coupling is large!

Dynamical development of a characteristic scale

$$\Lambda_{QCD} \sim 200 \mathrm{MeV}$$

At high energies, coupling becomes small:
asymptotic freedom



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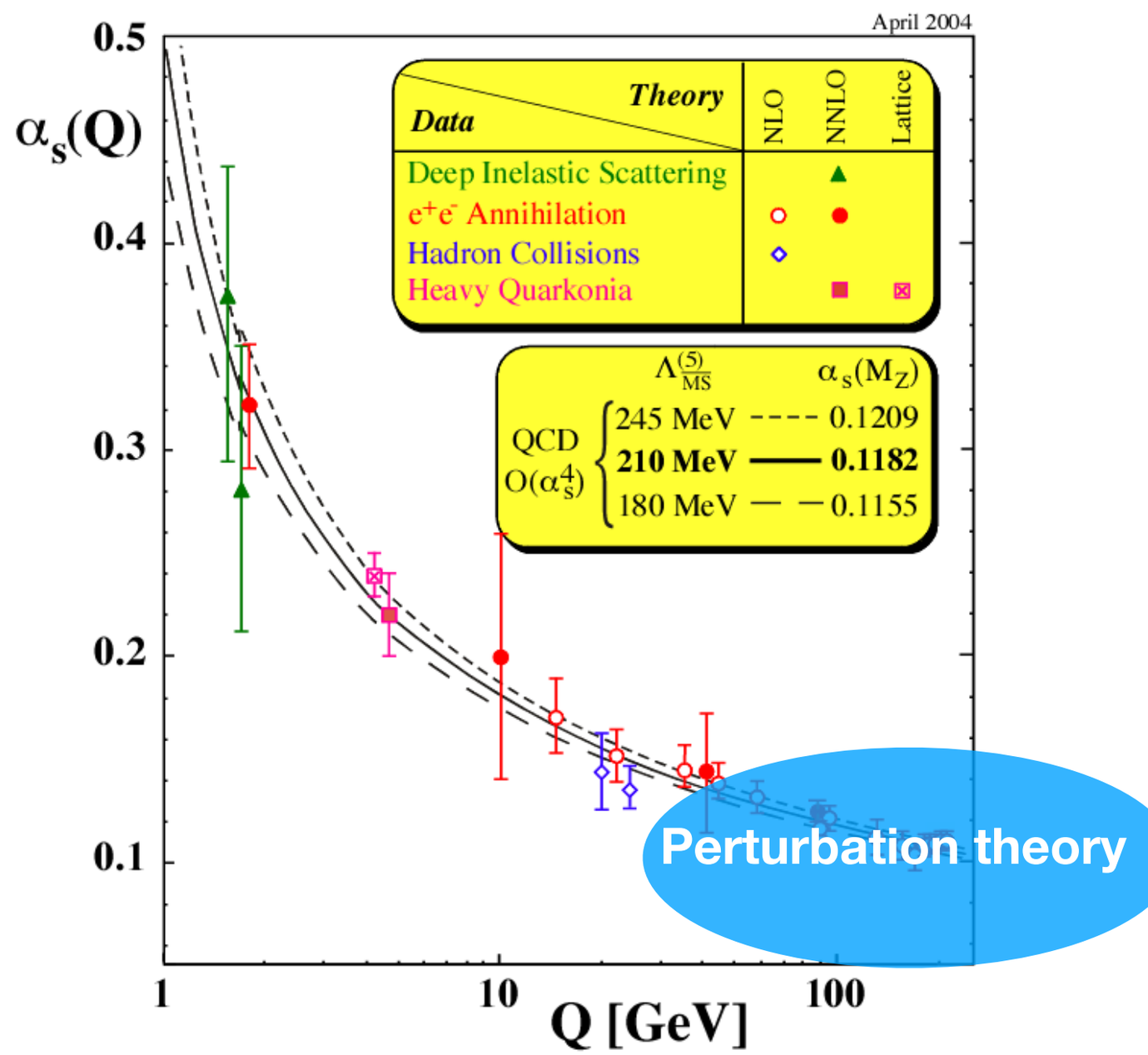

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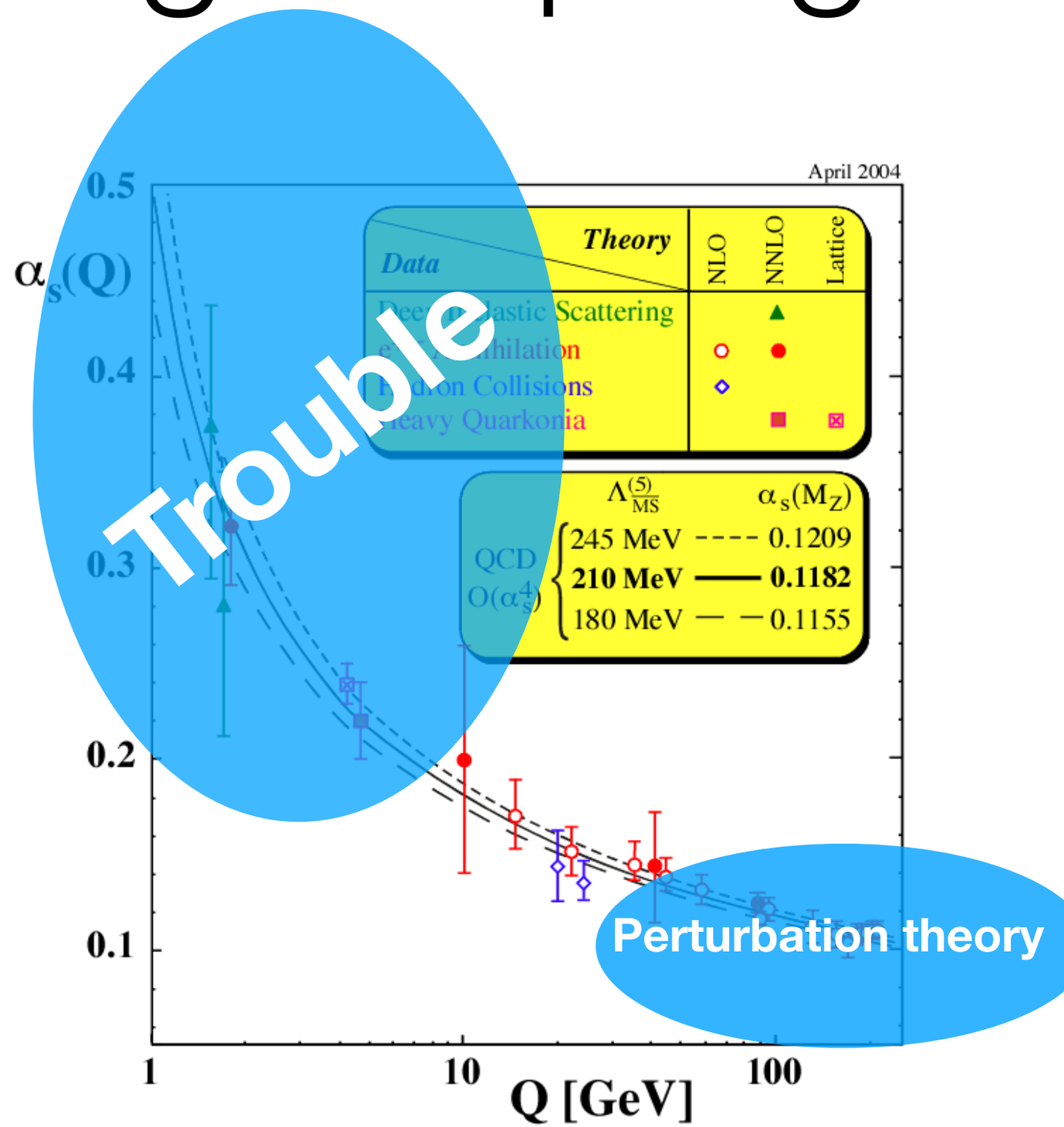


QCD at low energies

Consequences of a large coupling for the QCD vacuum:

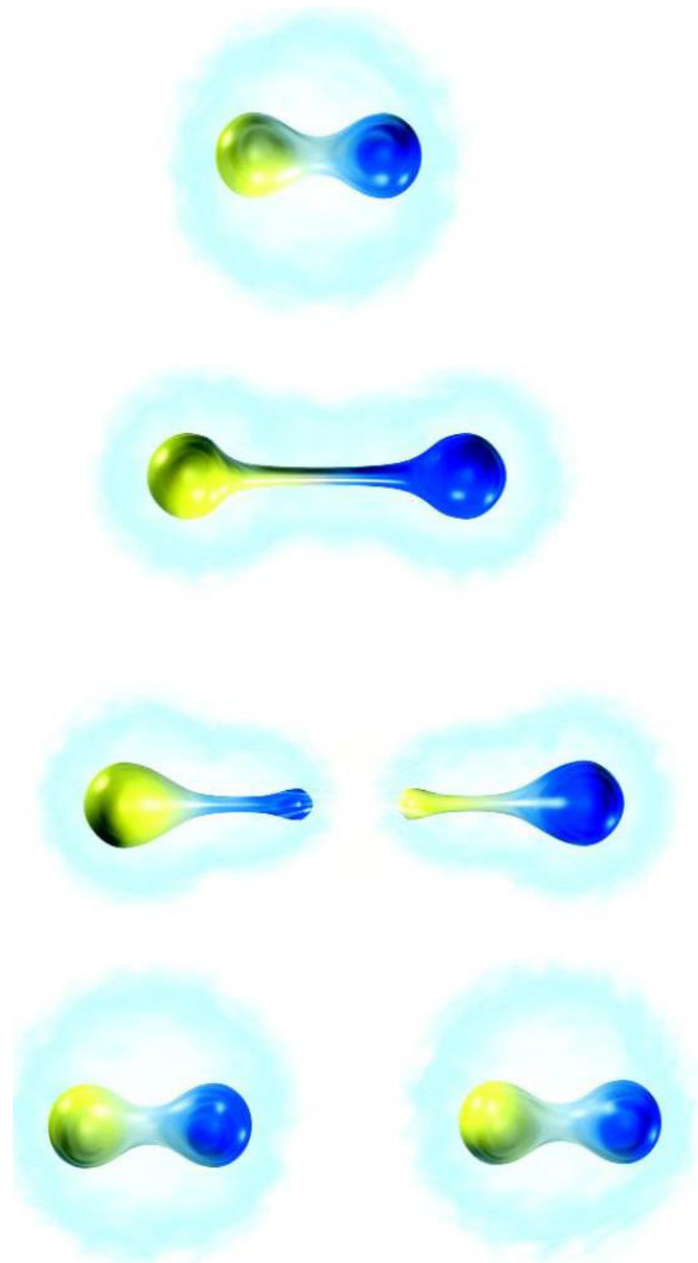
- Confinement
- Spontaneous chiral symmetry breaking

``Non-perturbative" effects

cannot be described using perturbation theory

Confinement

Try pulling apart two quarks
(or a quark-antiquark)…



Hadrons break up into other hadrons:

mesons $(\bar{q}q)$ and baryons (qqq)

-> No free quarks observed

Need to think carefully about
the relevant degrees of freedom!

Chiral symmetry

- For massless quarks, it's a symmetry of $\mathcal{L}_{QCD}$
- Formally (2f): $SU(2)_V \times SU(2)_A \equiv SU(2)_L \times SU_2(R)$

$$SU(2)_V : \qquad \psi \to e^{i\tau_a \theta^a} \psi$$

$$SU(2)_A : \qquad \psi \to e^{i\gamma^5 \tau_a \theta^a} \psi$$

Is it a good symmetry?

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- Spontaneously broken by the QCD vacuum!
- Non-degenerate chiral partners (sigma-pion, rho-a1..)
- Goldstone modes: pions

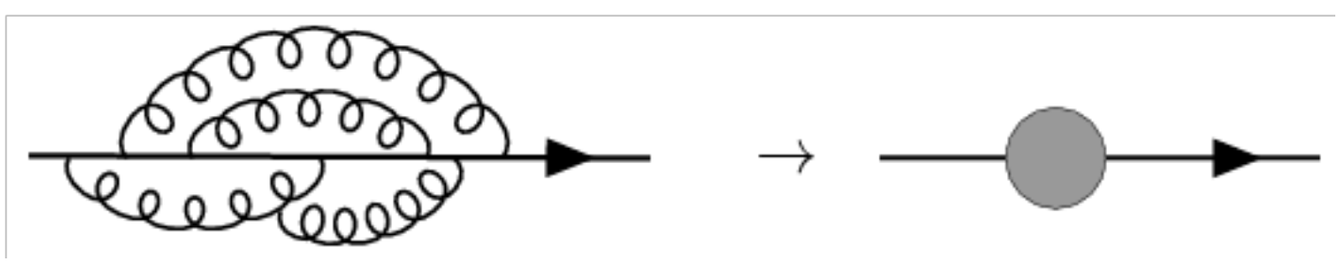
Dynamical mass generation

- Spontaneous breaking of chiral symmetry related to dynamical mass generation

$$\delta\mathcal{L}_{\chi SB} \sim M\bar{\psi}_L\psi_R$$

- Generation of a chiral condensate $\langle\bar{\psi}\psi\rangle$ in the QCD vacuum

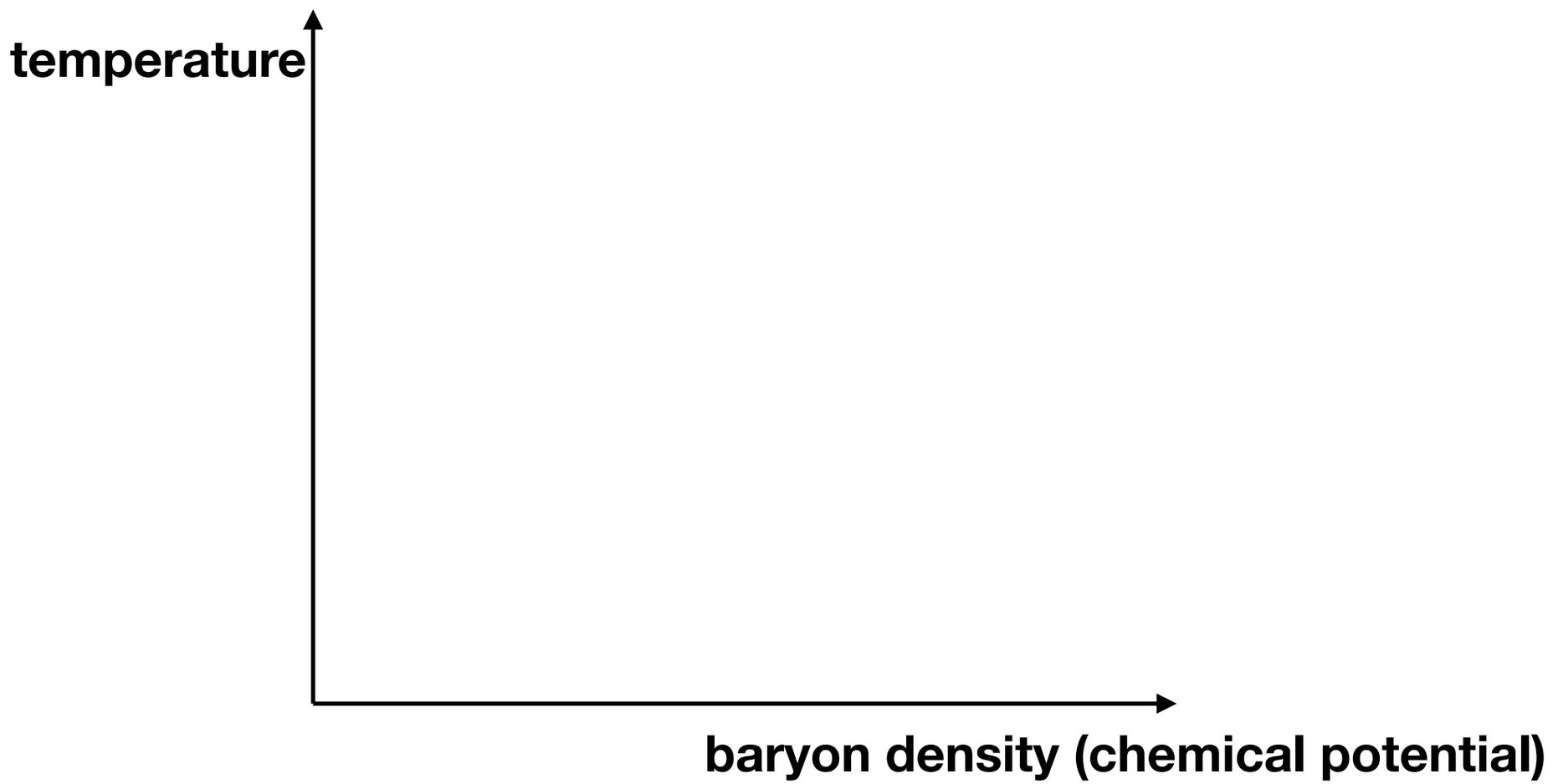
- Interpretation: Strong interactions ``dress" particles and the interaction energy generates constituent masses



Extreme conditions

(=crank up the temperature/density)

QCD phase diagram



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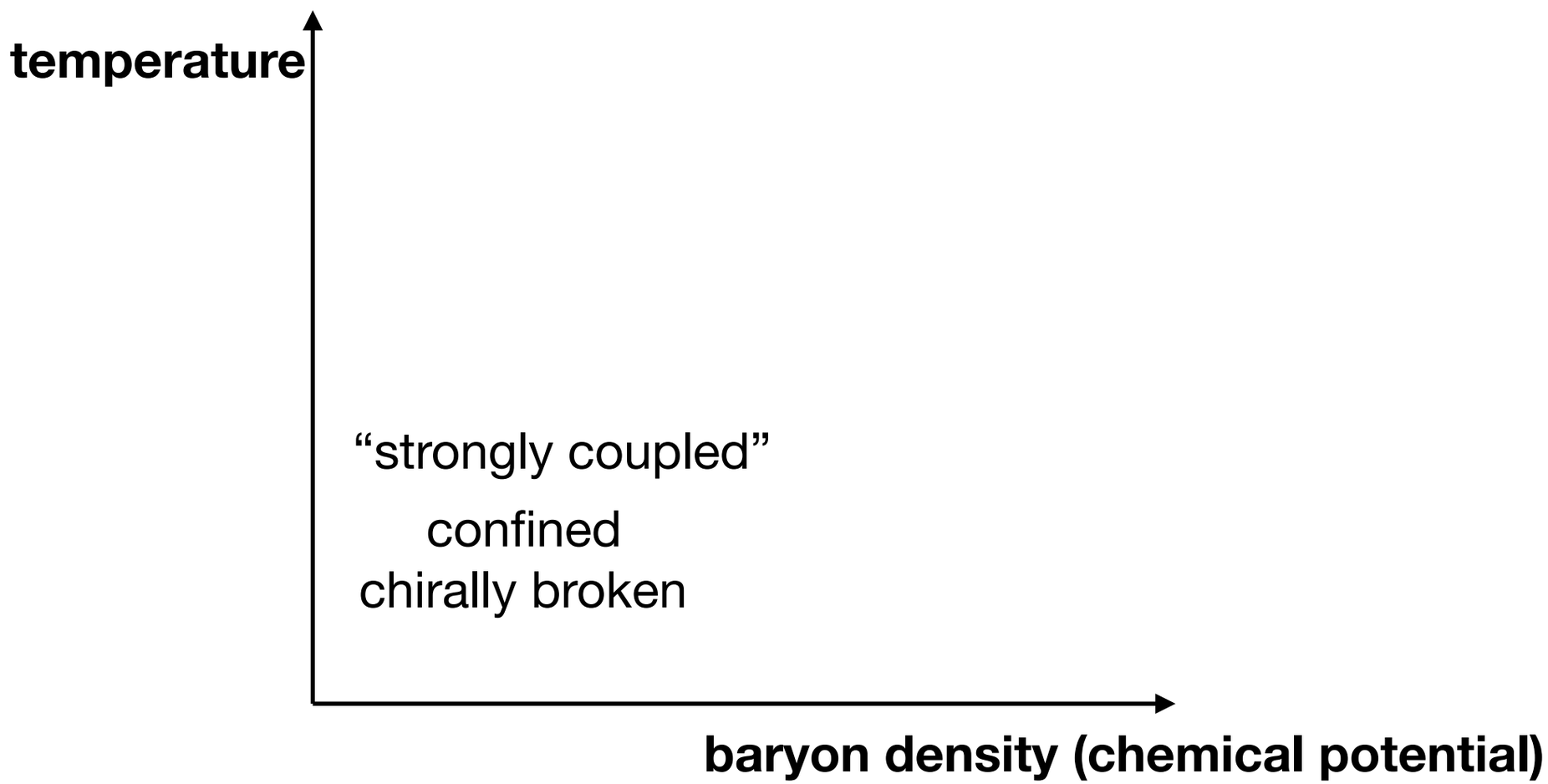

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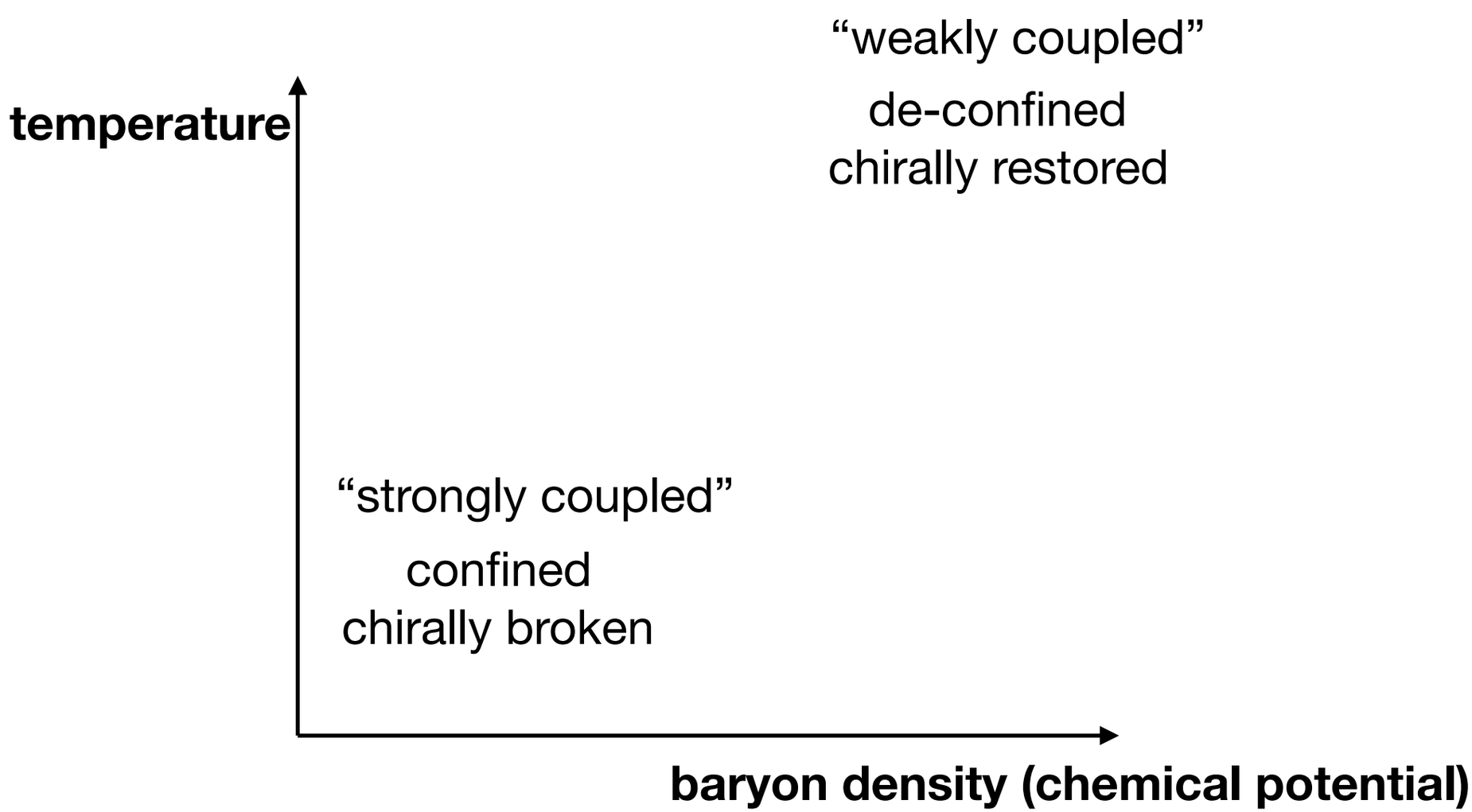

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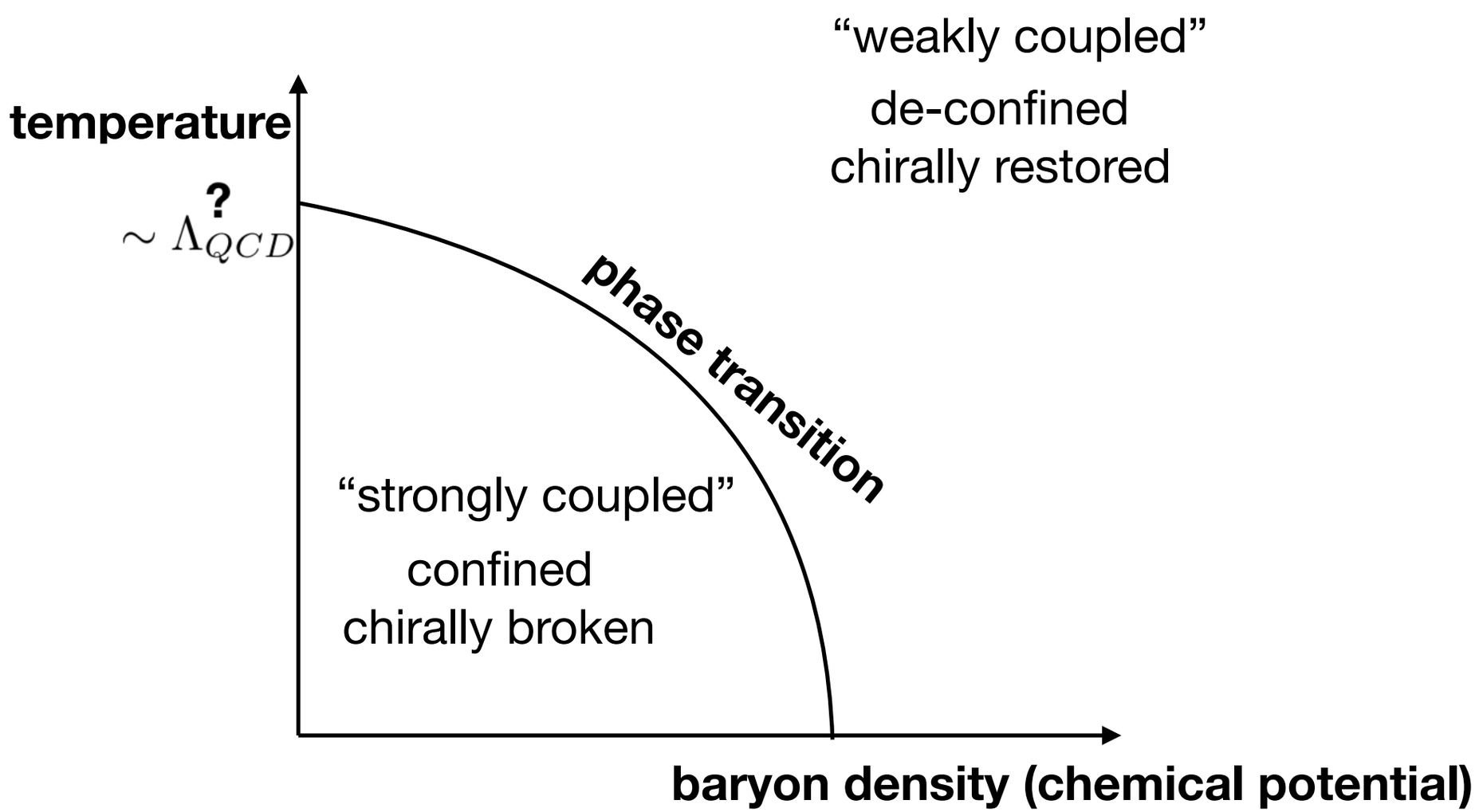

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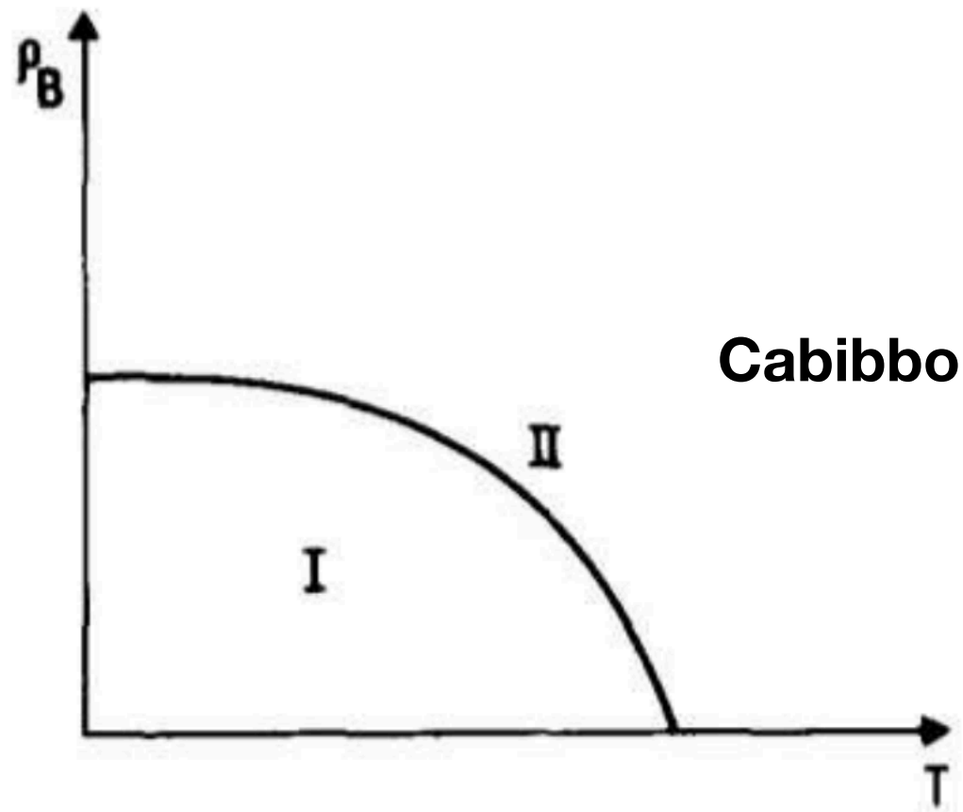
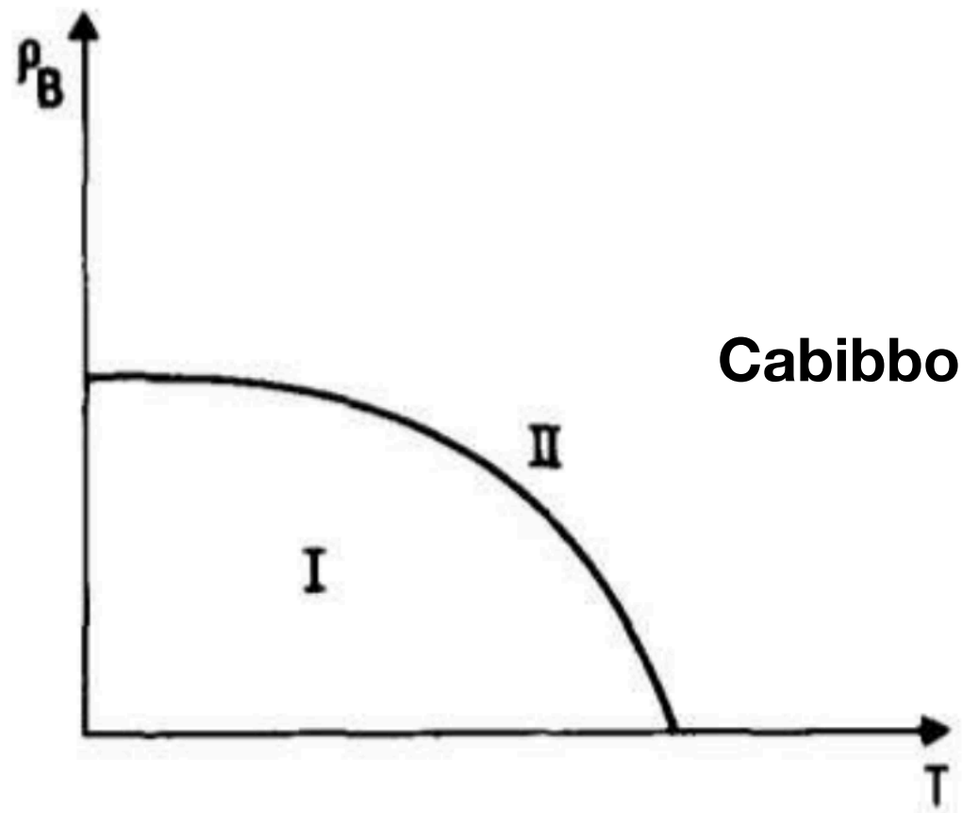


Cabibbo & Parisi, 1975

Fig. 1. Schematic phase diagram of hadronic matter. ρ_B is the density of baryonic number. Quarks are confined in phase I and unconfined in phase II.

QCD phase diagram



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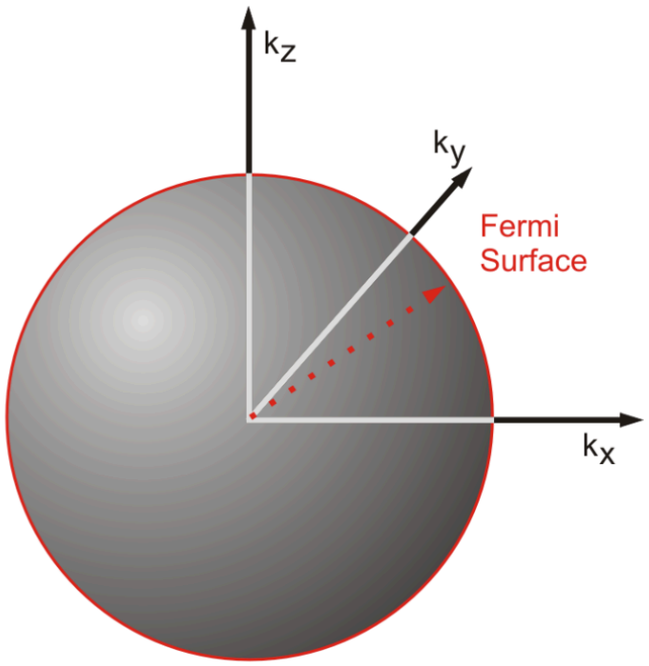
Is that all ?

Color superconductivity

- High density and low temperature: Fermi sphere of quarks



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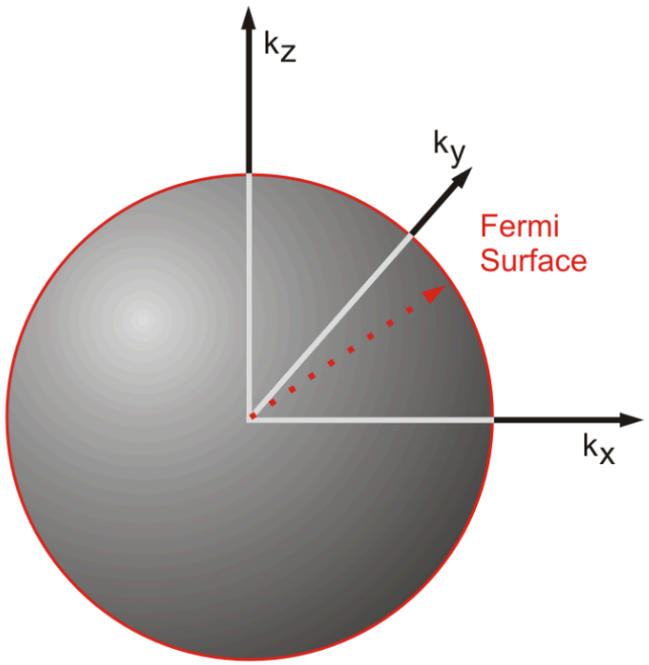

- Weak coupling: attractive channel in 1-gluon exchange

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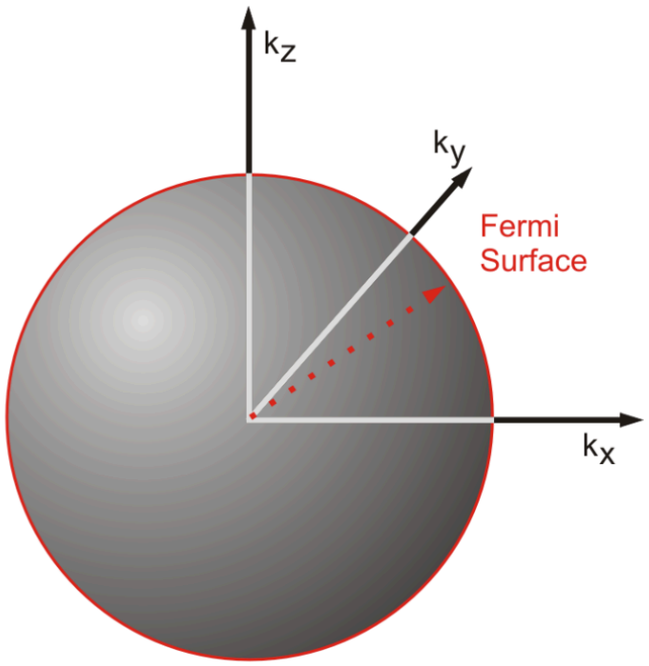
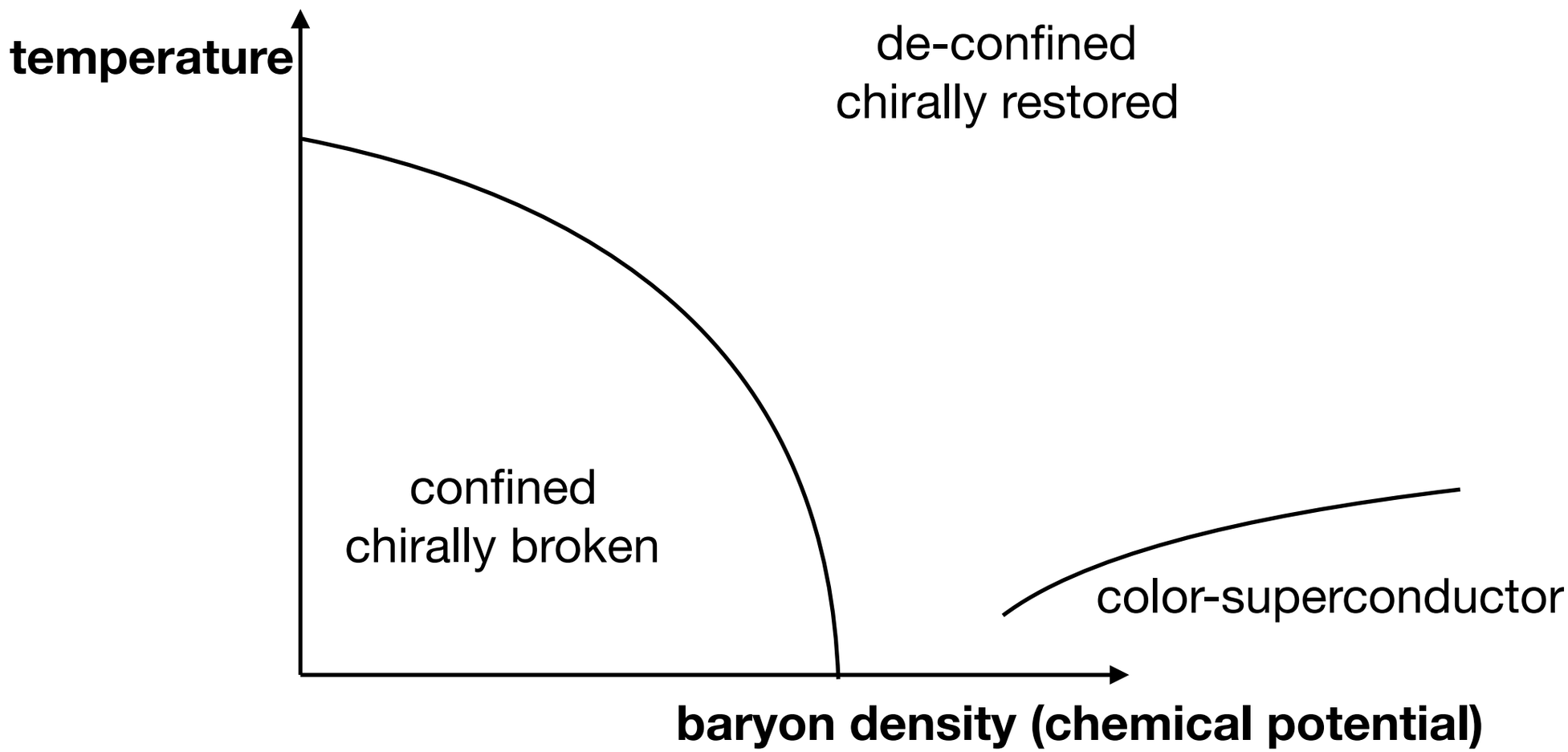


- Weak coupling: attractive channel in 1-gluon exchange

→ Cooper instability: $\langle qq \rangle \neq 0$

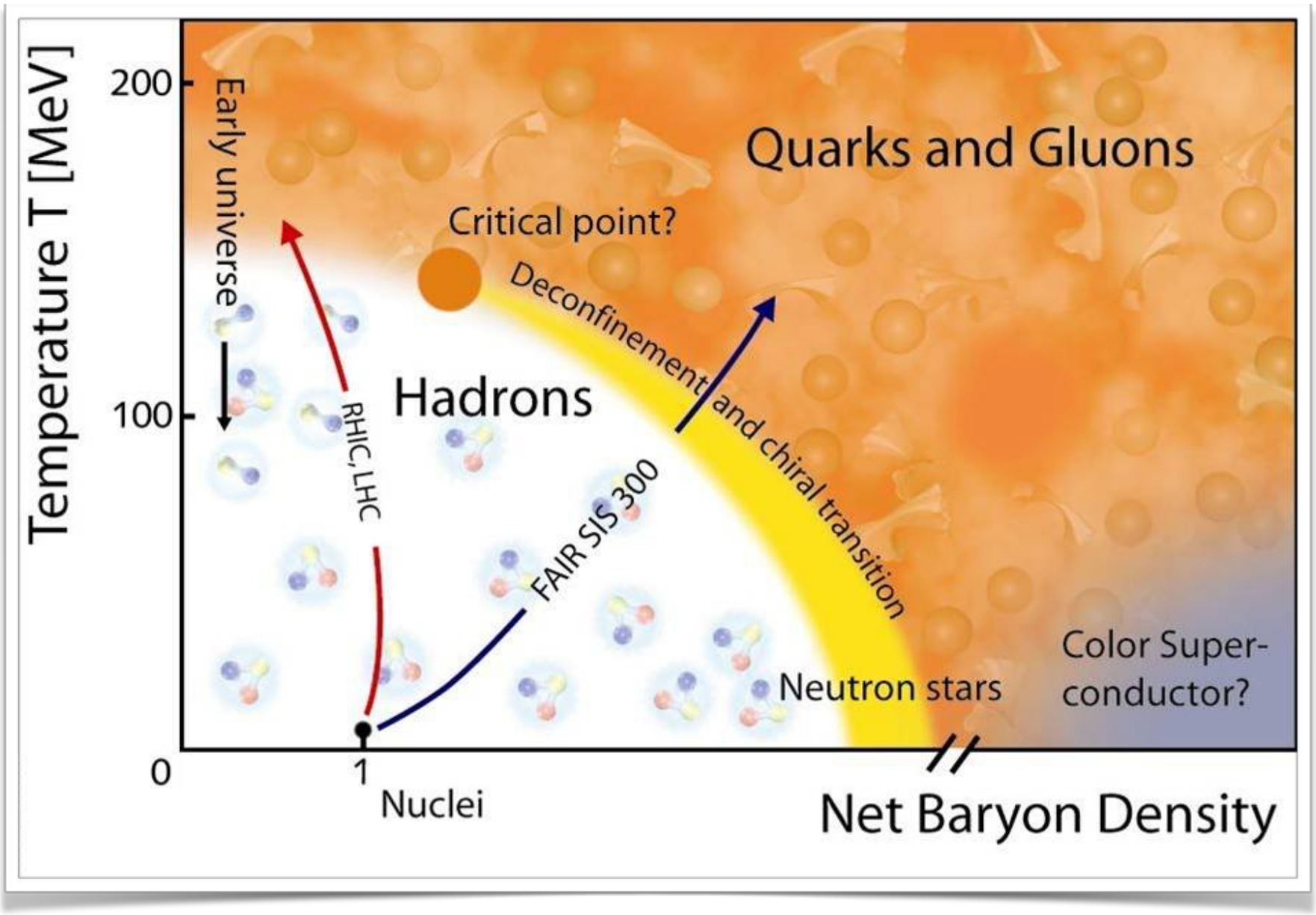
- Cold quark matter at **(very)** high density is a color superconductor!

QCD phase diagram



40something years later...

QCD phase diagram



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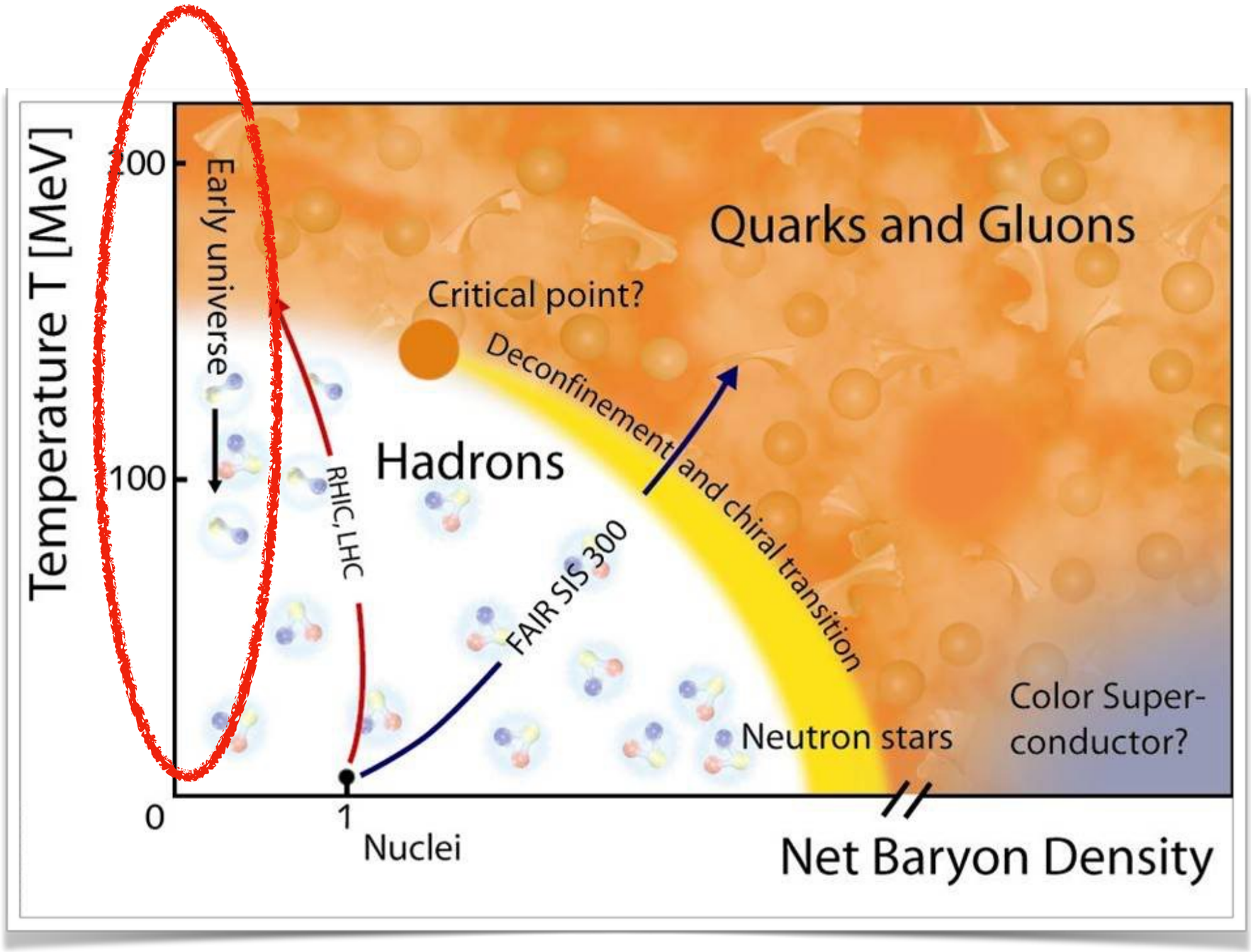

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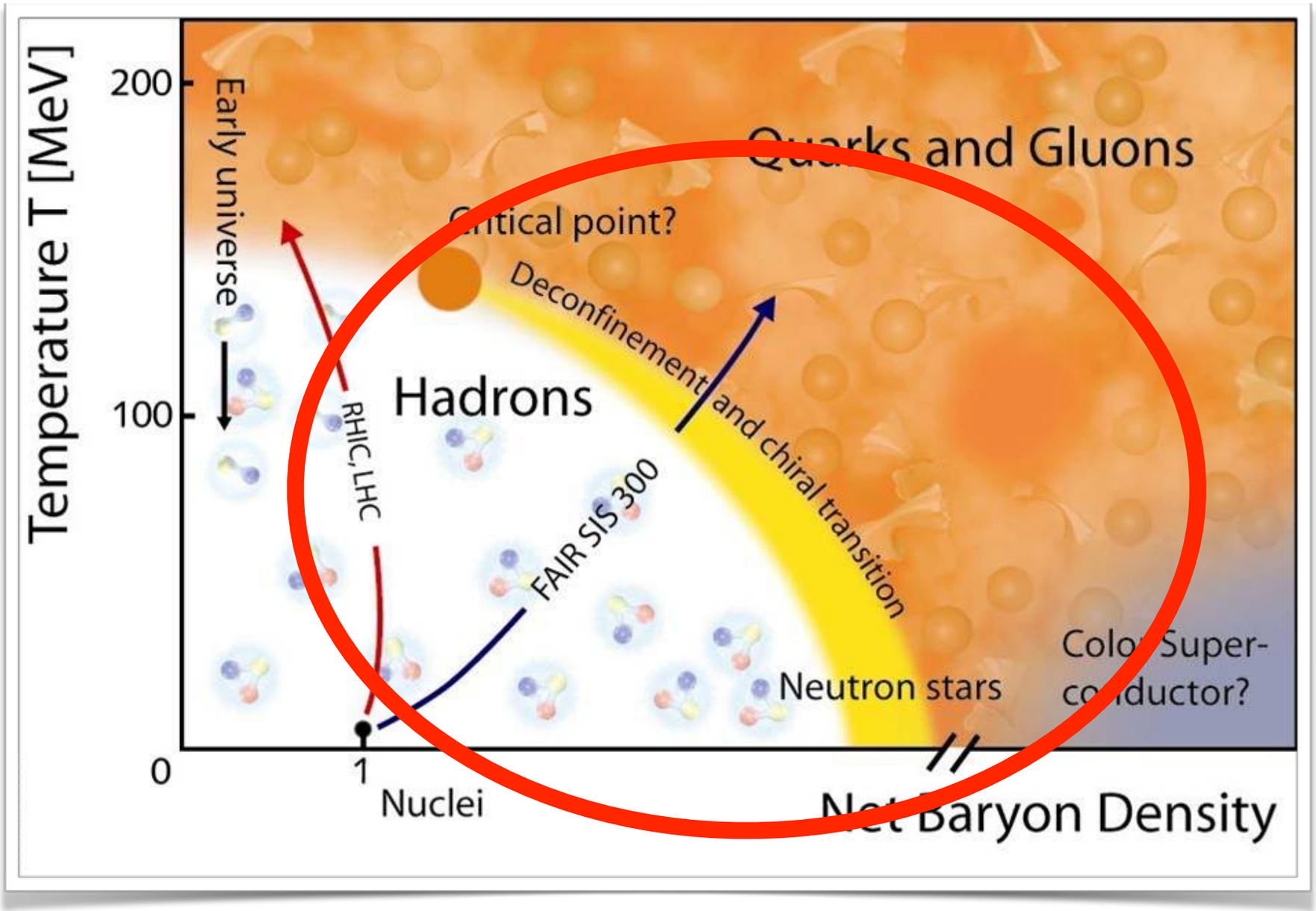

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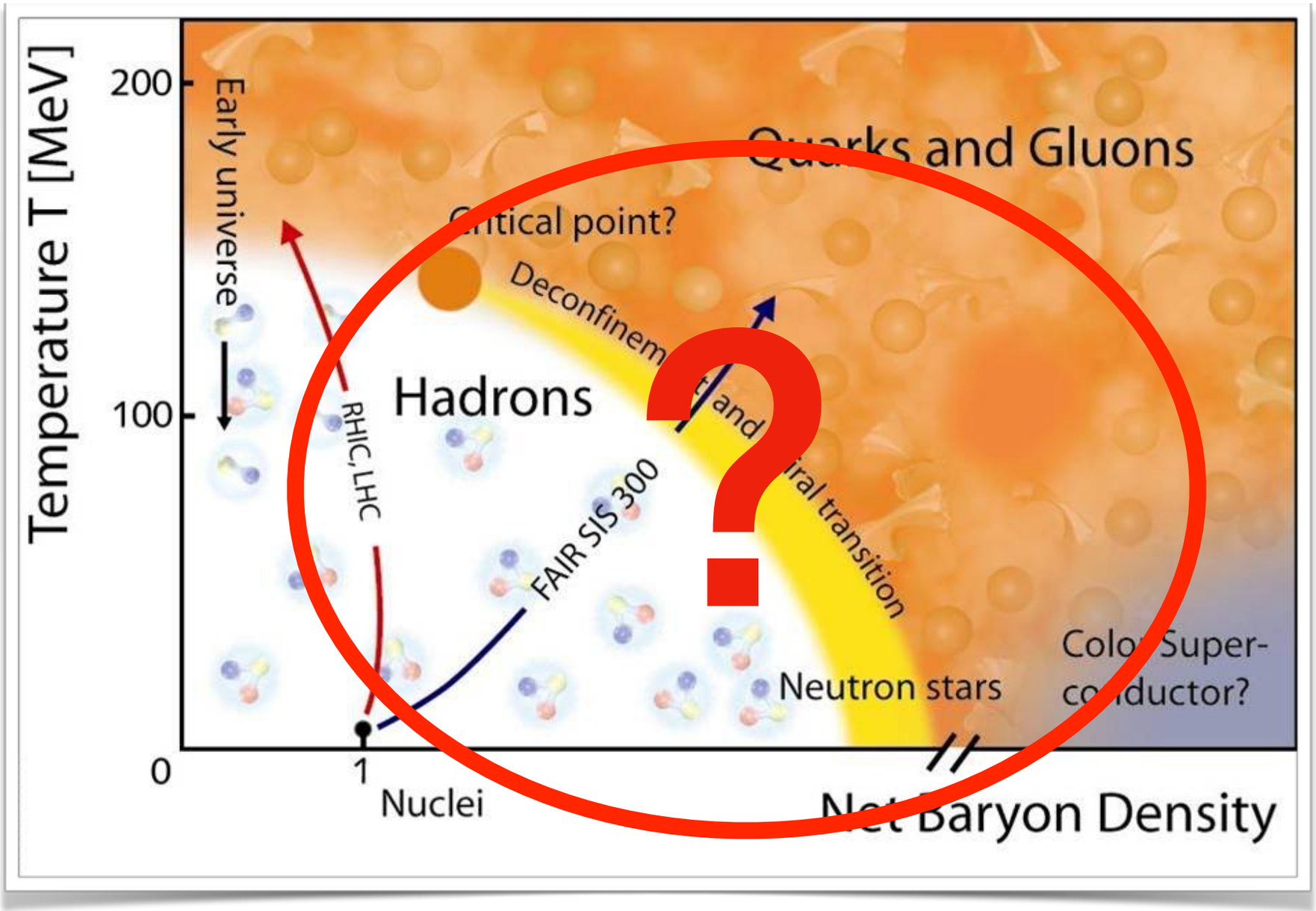
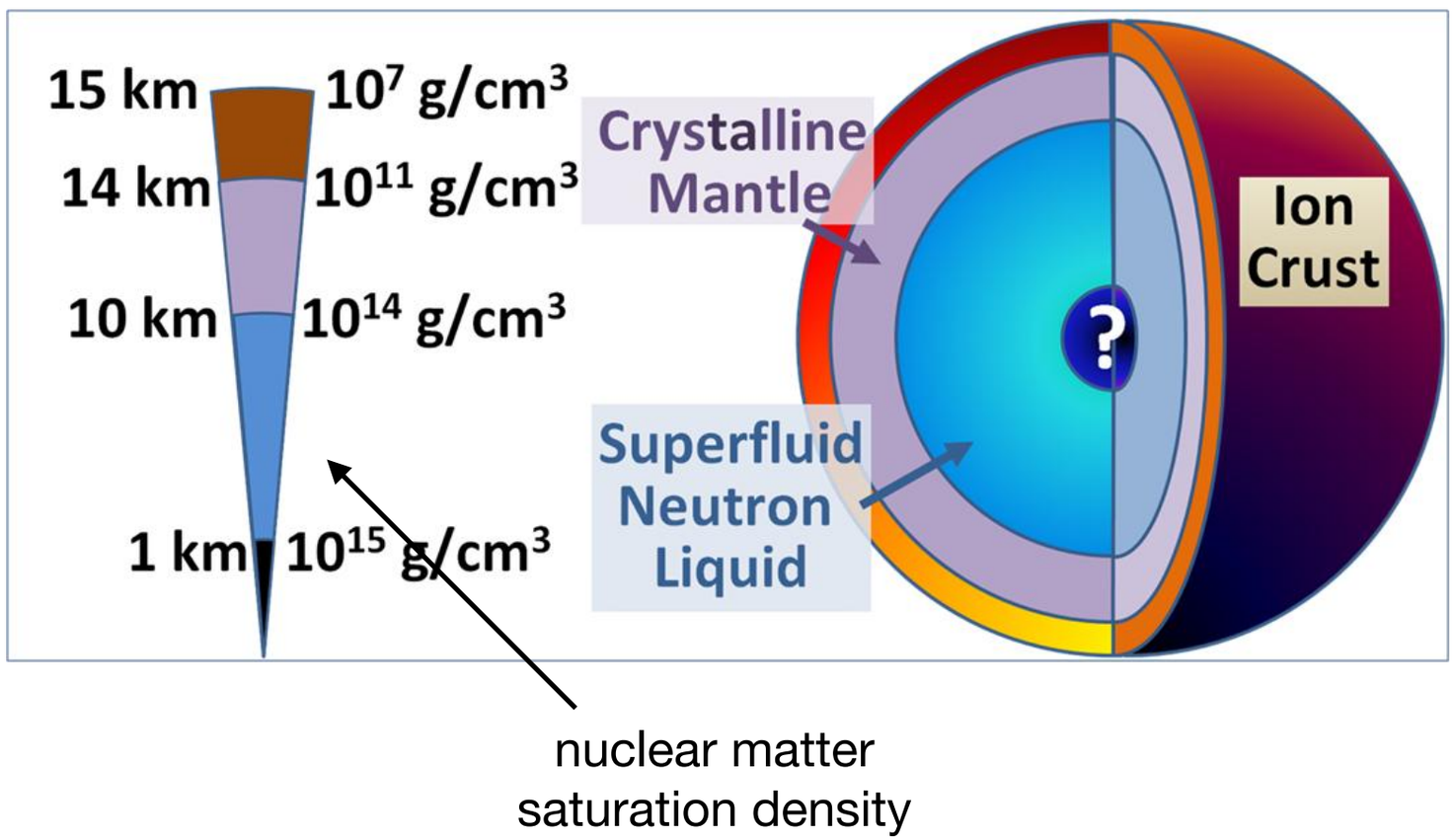


(some) Open questions

- Location of the phase transition(s?)
- Order of phase transition(s?)
- (How) are the chiral symmetry and (de)confinement transitions related ?
- Does color-superconductivity reach all the way down to intermediate densities?
- Can some exotic phase appear at finite density?

Studying dense matter could be relevant for

Physics of compact stars **and heavy-ion collisions!**



Intermediate densities

- From the theory side:
  no lattice QCD
  no perturbative expansions

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  no heavy-ion collisions there (yet!)
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first-order phase transition
separating hadrons/QGP
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OR

More exciting!
exotic phases,
several phase transitions..

Unlikely?

Who knows, think for example of water

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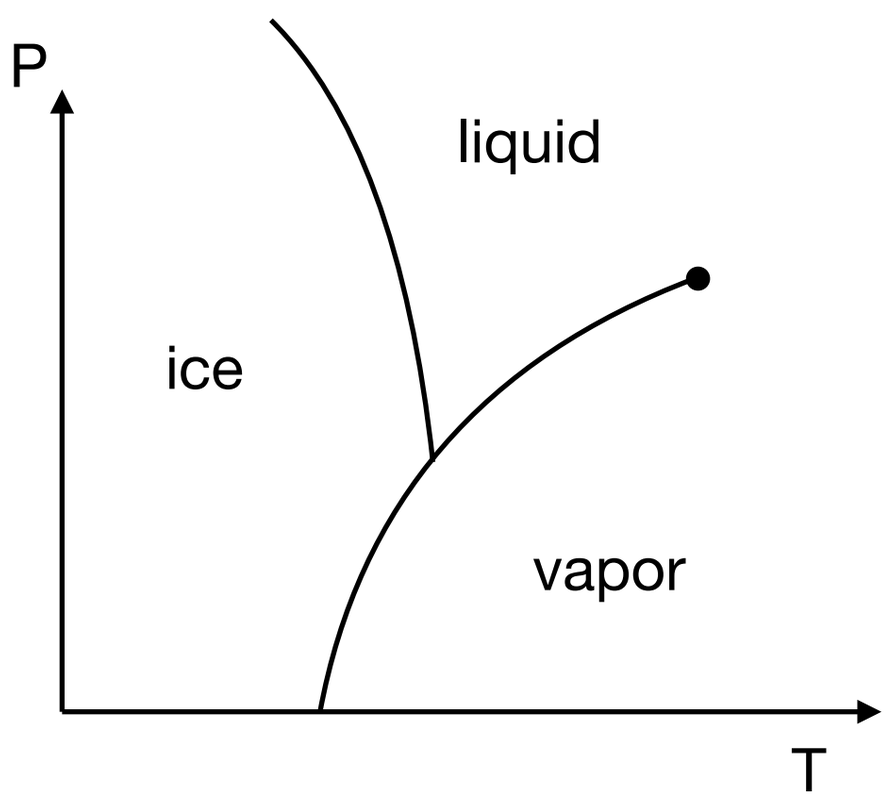
The picture I usually
have in my head



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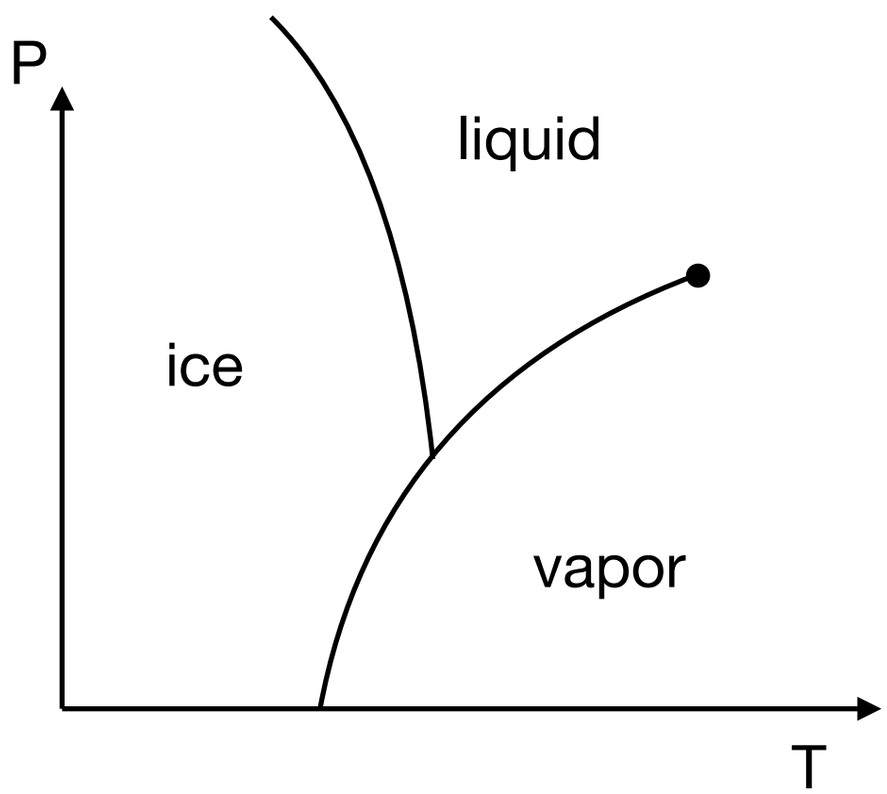
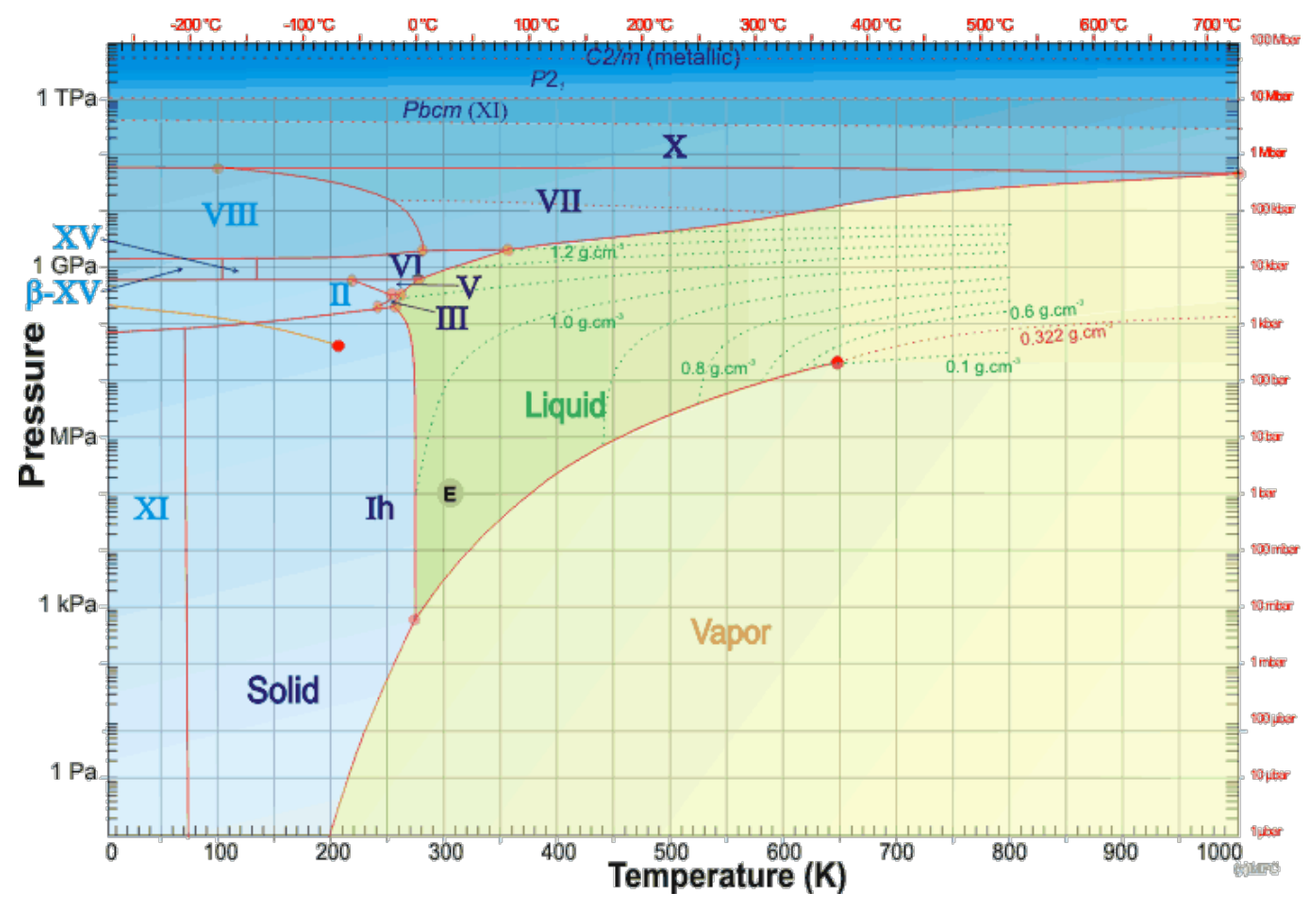
The picture I usually have in my head

vs.

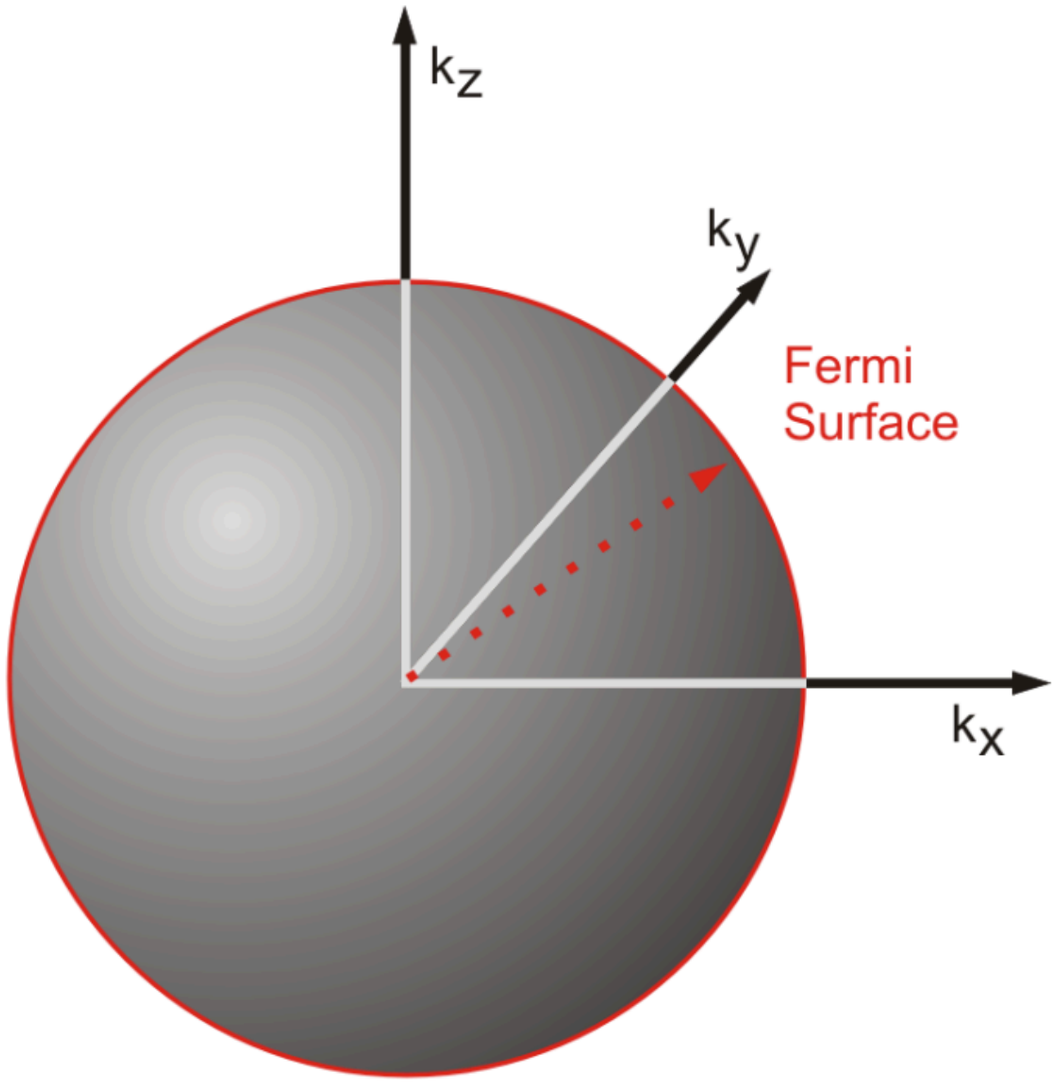
What it actually is like





Quarkyonic matter

- Very speculative/qualitative scenario for dense matter
- Perturbative physics in the bulk, nonperturbative excitations at the Fermi surface
- Large Nc limit, dimensional reduction at the Fermi surface
- Effective theory: QCD(1+1D): favored ground state for low T is a crystal! ("chiral spirals")



McLerran, Pisarski, Kojo, Hidaka…

How to tackle QCD at finite density?

(some) Theoretical methods for QCD at finite density

- Functional methods:

  Dyson-Schwinger equations

  Functional renormalization group

- Effective quark (and quark-hadron) models

  NJL, Quark-meson ..

- Effective field theories

  Chiral perturbation theory

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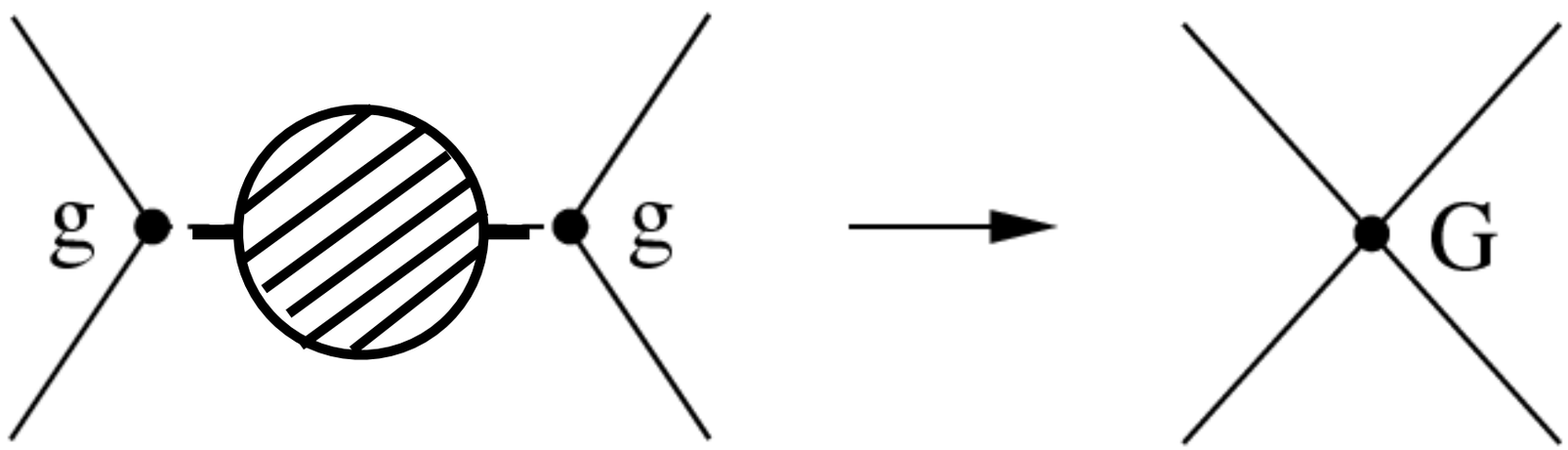
Advantages of effective models

- Built around symmetries of the full theory
- Relatively easy computations
- (Typically) Few parameters
  (typically) fitted to experimental data in vacuum
- Qualitatively reasonable results

Nambu—Jona-Lasinio (NJL) model

Complicated quark-gluon interaction replaced by effective four-fermion vertex with fixed coupling constant G



Simplest version: 2 flavor, scalar-pseudoscalar interaction

$$\mathcal{L}_{NJL} = \bar{\psi} i\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi + G[(\bar{\psi}\psi)^2 + (\bar{\psi} i\gamma^5 \tau^a \psi)^2]$$

Mean-field approximation

- Typical assumption: mean-field approximation $(\bar{\psi}\psi) \approx \langle\bar{\psi}\psi\rangle$

- A constant mean-field chiral condensate acts as constituent quark mass:

$$M_q = m - 2G\langle\bar{\psi}\psi\rangle$$

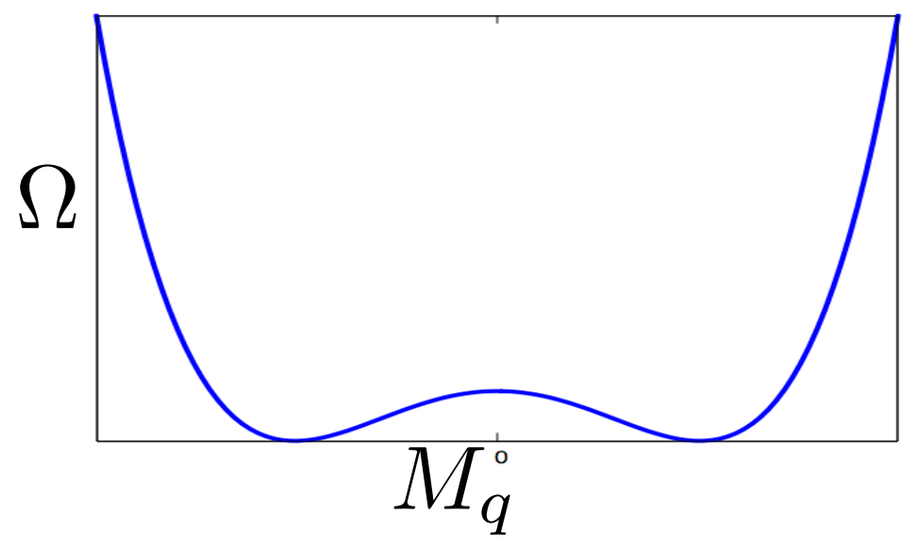
- Neglecting fluctuations, it is possible to obtain the free energy of the system as a trace over the inverse quark propagator:

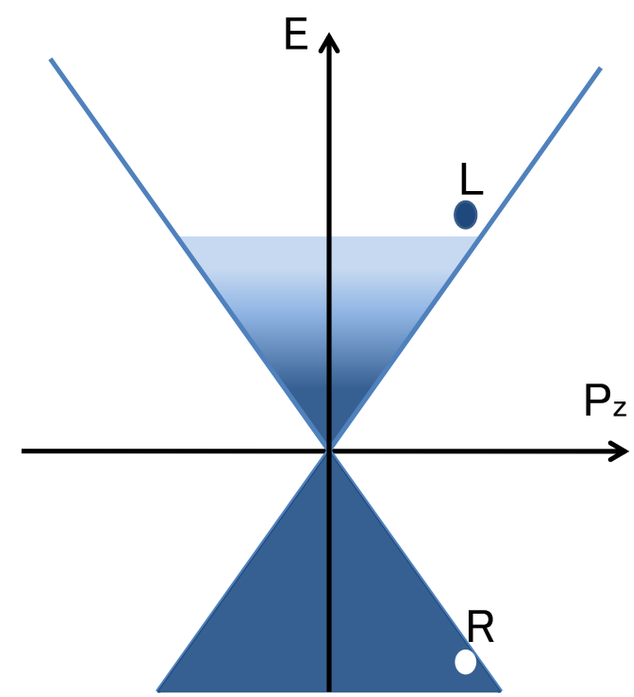
$$\Omega \sim \frac{T}{V}\mathrm{Tr}\log\left(\frac{S^{-1}(M_q)}{T}\right)$$

Chiral condensate

- Optimization problem: minimize the free energy wrt. M_q to find the ground state of the system

- In vacuum:





- Chiral symmetry is **spontaneously broken**: quark-antiquark chiral condensate forms

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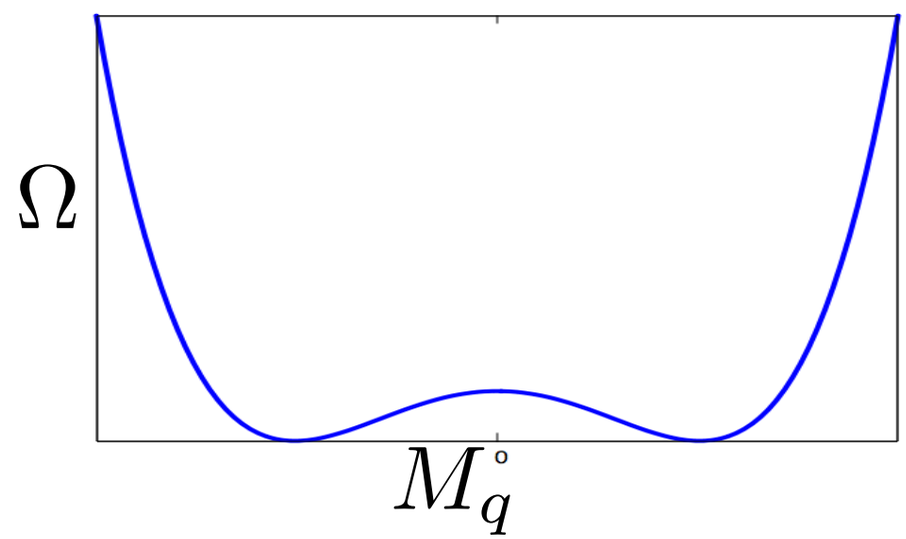

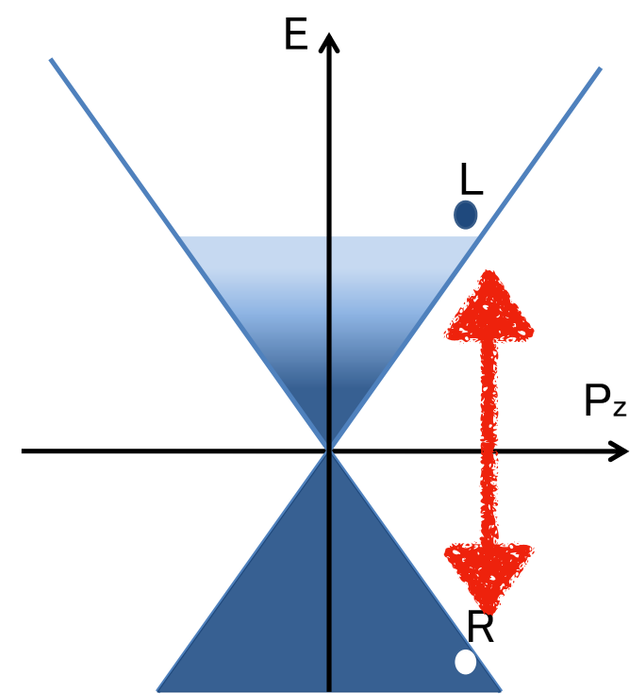

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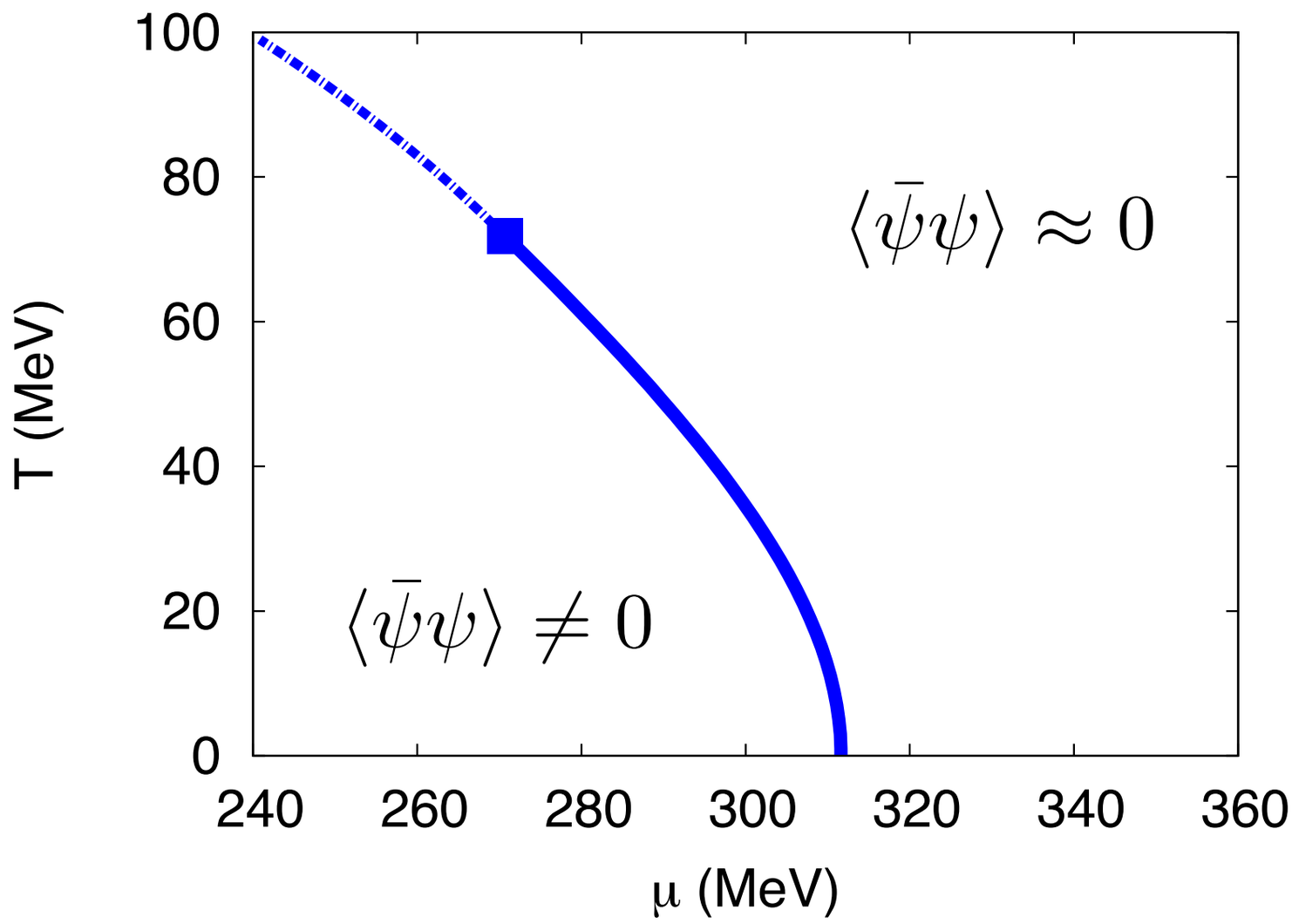
- As density increases, higher energy cost to form particle-antiparticle pairs -> chiral restoration

NJL phase diagram

Two-flavor, chiral transition only



Crystalline phases

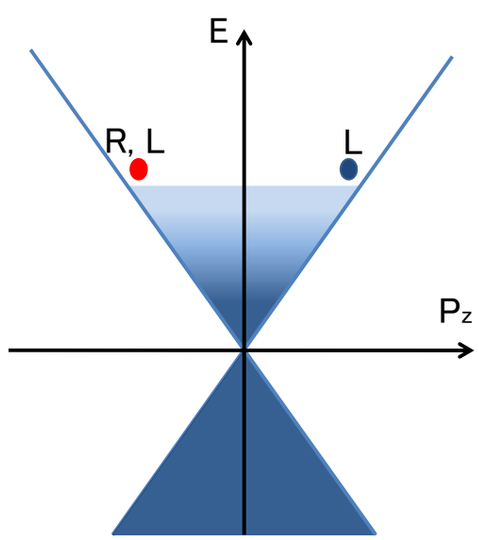
- What happens if we relax the approximation of spatially constant condensates?
  -> Inhomogeneous phases

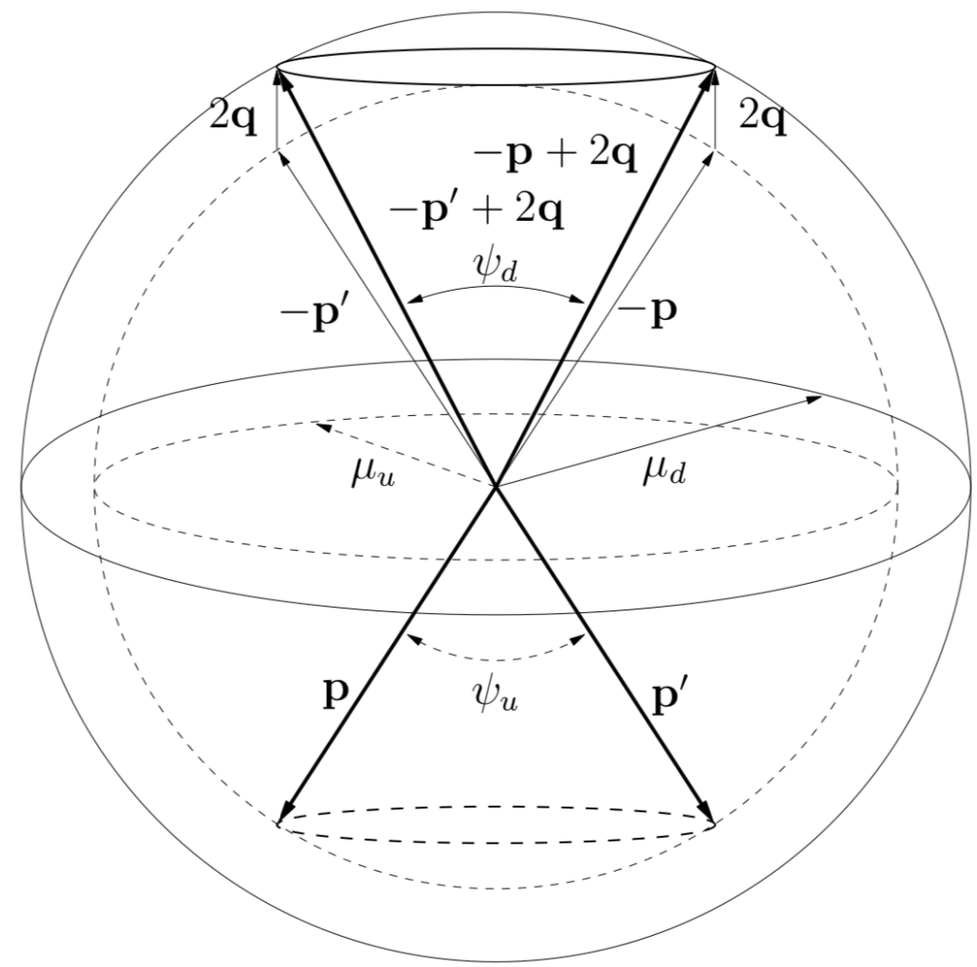
- Not a new idea: density waves in nuclear matter (1960s), p-wave pion condensation (1970s) …
  More recently: quarkyonic chiral spirals, hints from 1+1D models (Gross-Neveu, NJL_2 ..)

- For QCD at finite density?

Inhomogeneous color-superconductivity

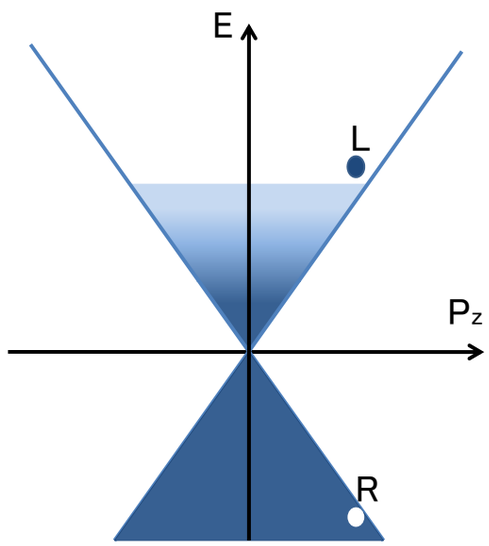
- In presence of isospin imbalance, Fermi surfaces for u,d quarks are unequal
- Favored to create Cooper pairs with nonzero total momentum
- Crystalline diquark condensate!





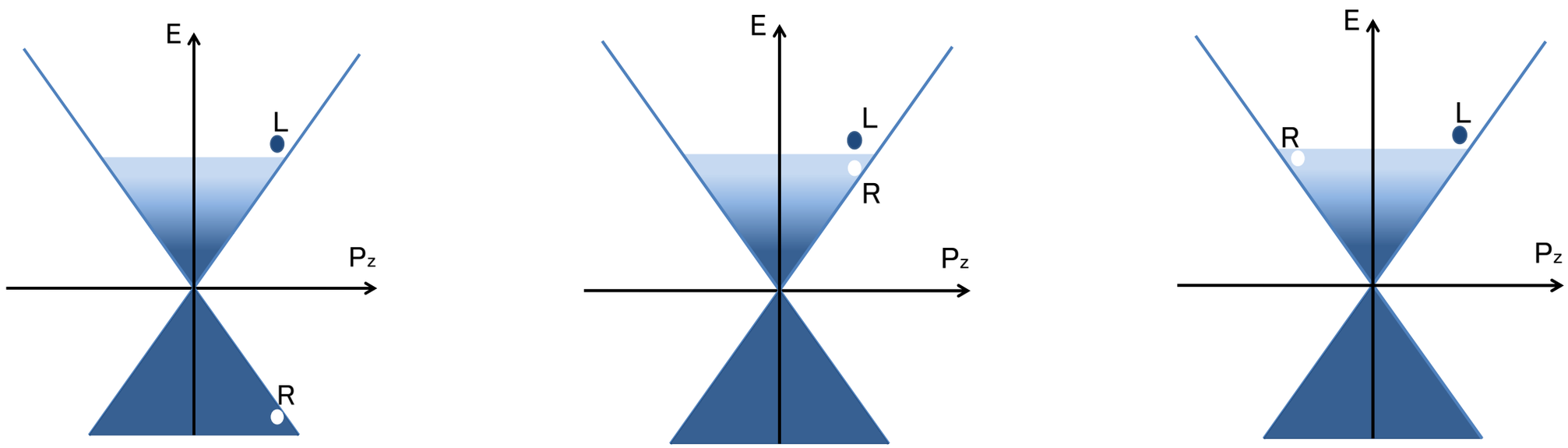
Inhomogeneous chiral condensates

Instead of the standard particle-antiparticle condensate…



Inhomogeneous chiral condensates

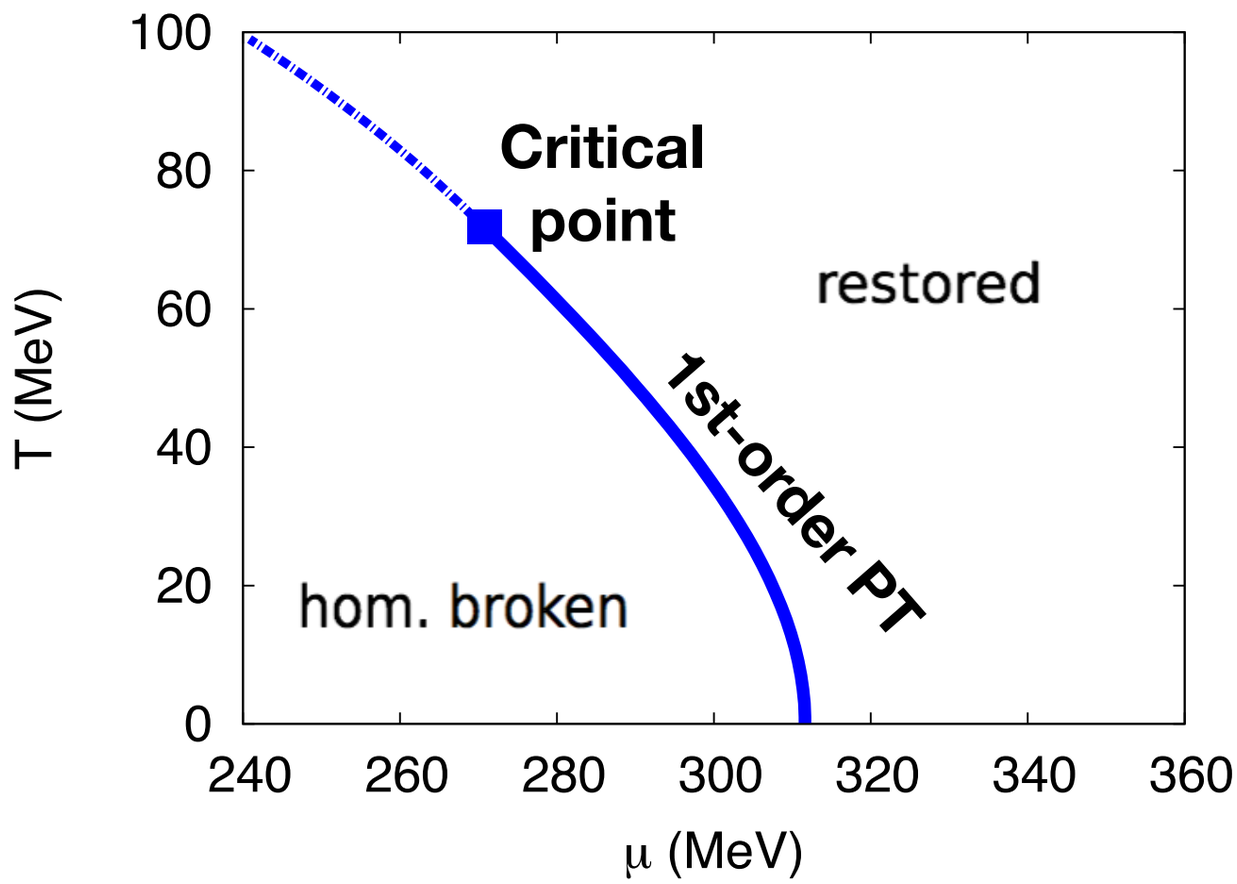
…particle-hole pairing at the Fermi surface (“density waves”)



- Can occur at finite density: could be relevant at intermediate densities, close to the chiral phase transition

NJL phase diagram

- Allowing for inhomogeneous phases, we go from this…



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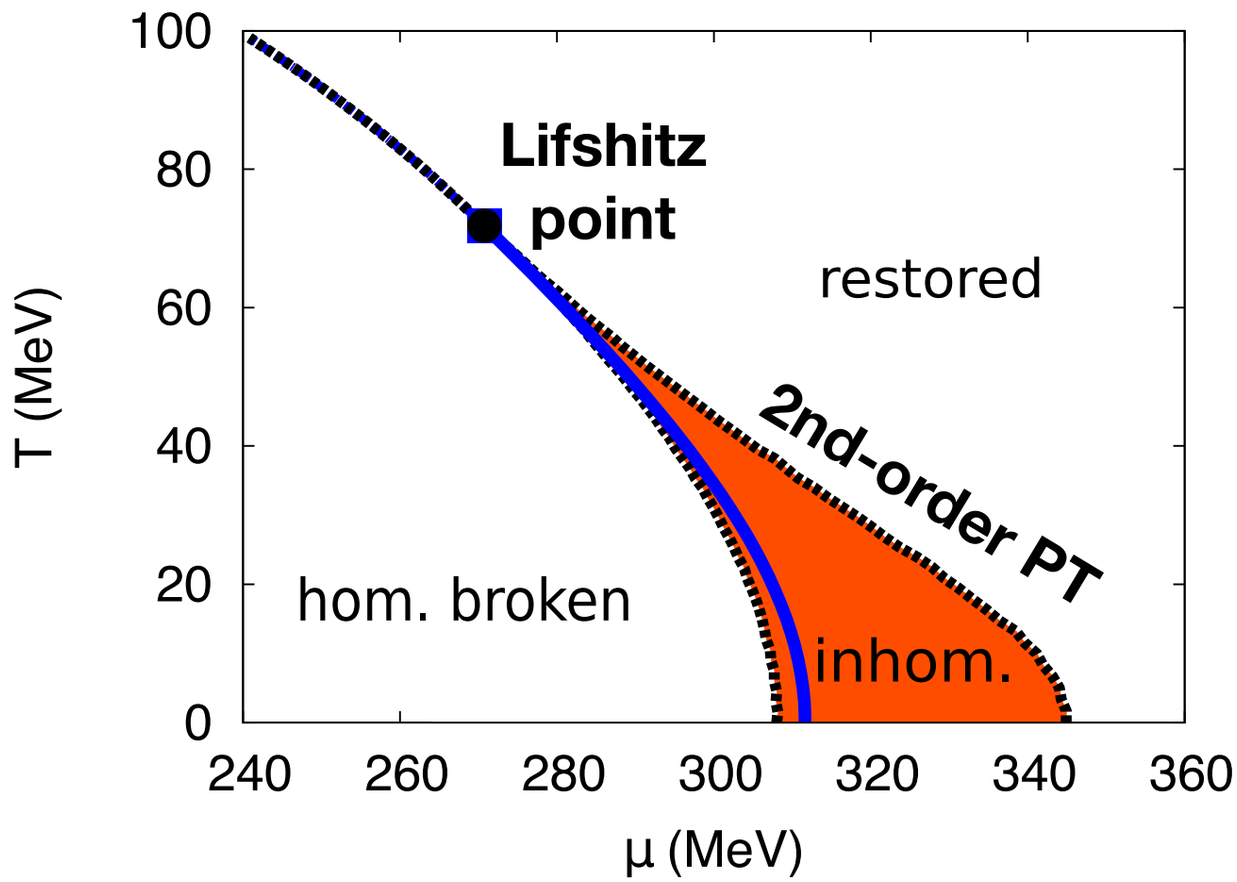
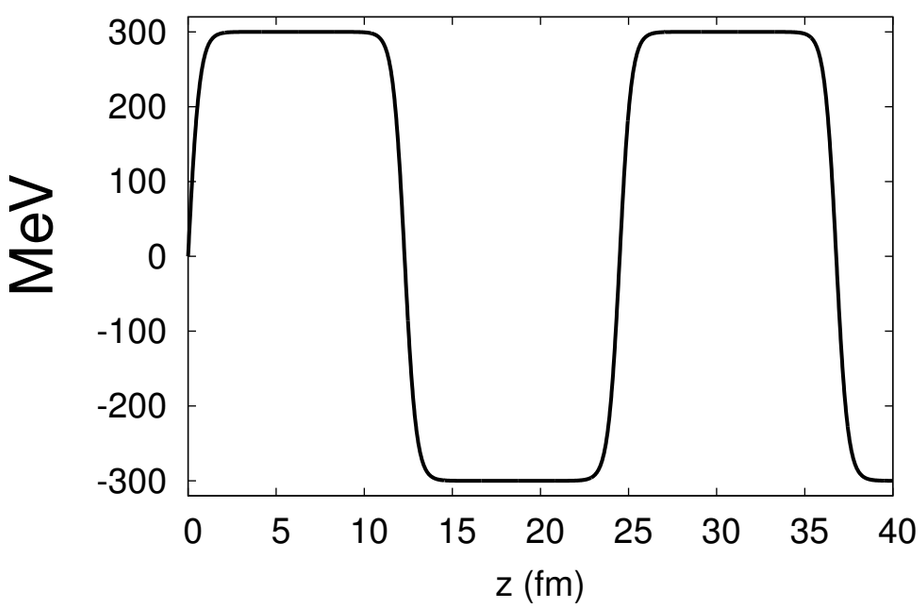


Condensate and density

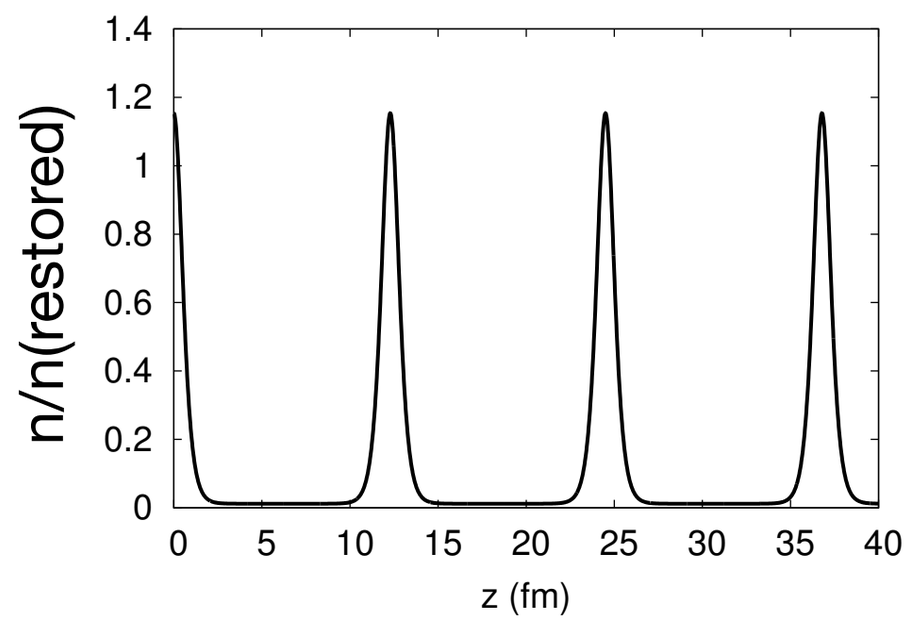
- If the chiral condensate is spatially modulated, the density of the system becomes inhomogeneous as well

- For the real kink crystal

$$M(z) \sim \langle \bar{\psi}\psi \rangle$$



$$n(z) \sim \langle \psi^\dagger \psi \rangle$$



$$\mu \sim 308 \mathrm{MeV}$$

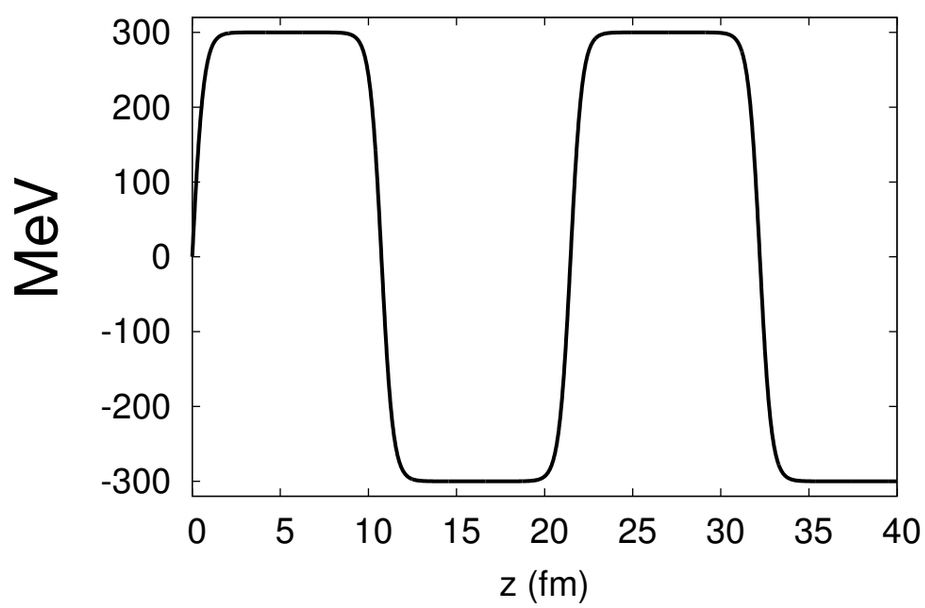
SC, D,Nickel and M.Buballa, Phys.Rev. D82 (2010) 054009
M.Buballa and SC, Phys.Rev. D87 (2013) no.5, 054004

Condensate and density

- If the chiral condensate is spatially modulated, the density of the system becomes inhomogeneous as well

- For the real kink crystal

$$M(z) \sim \langle \bar{\psi}\psi \rangle$$



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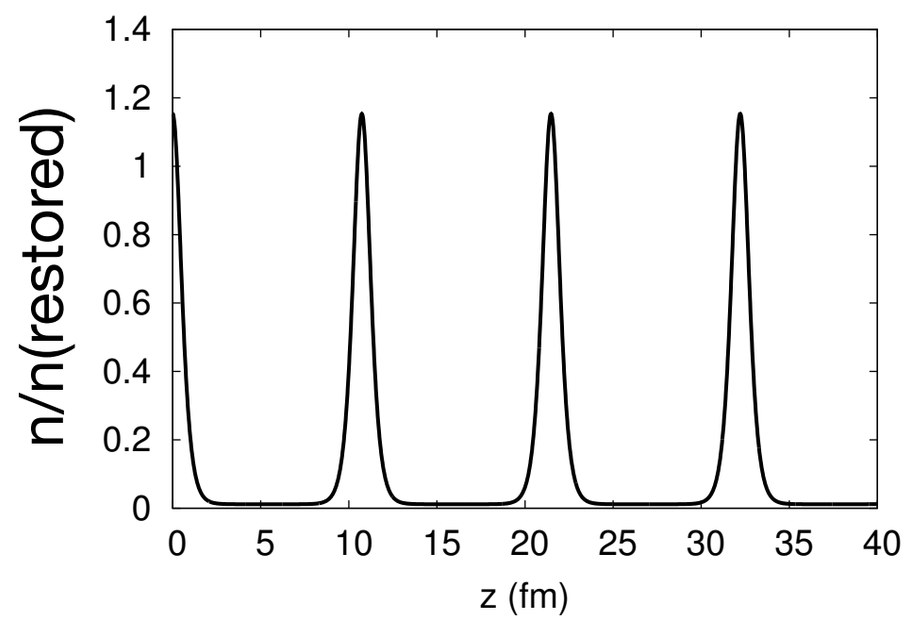

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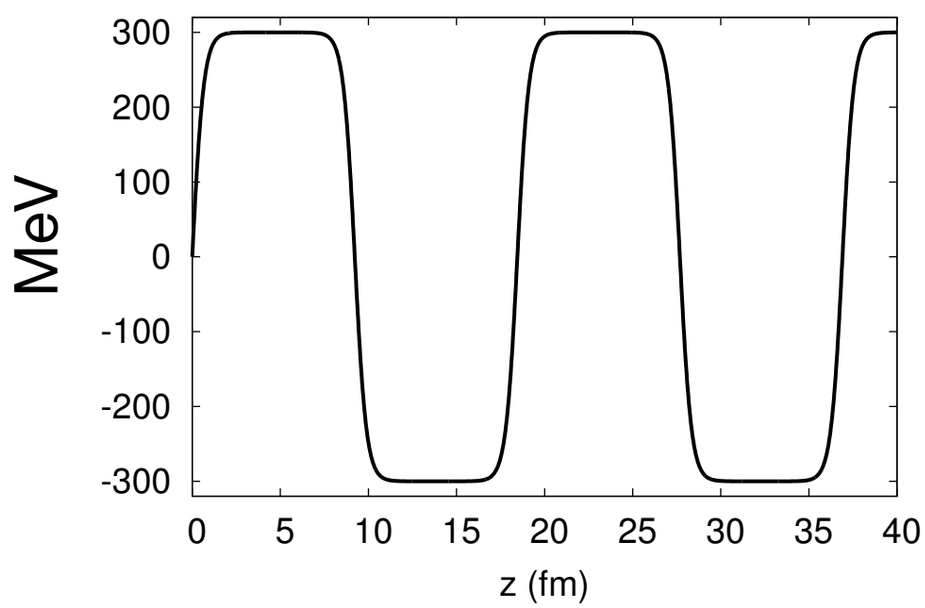

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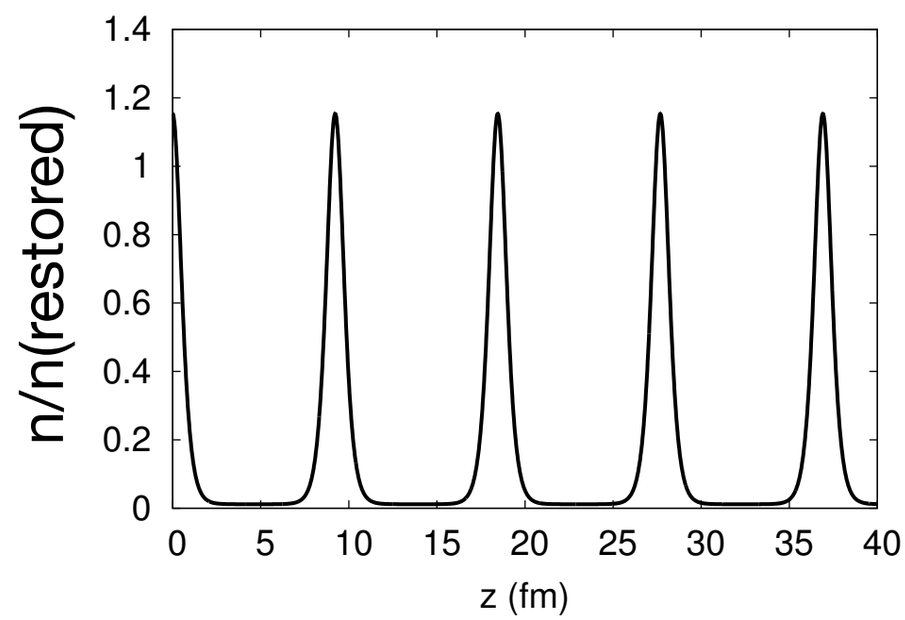

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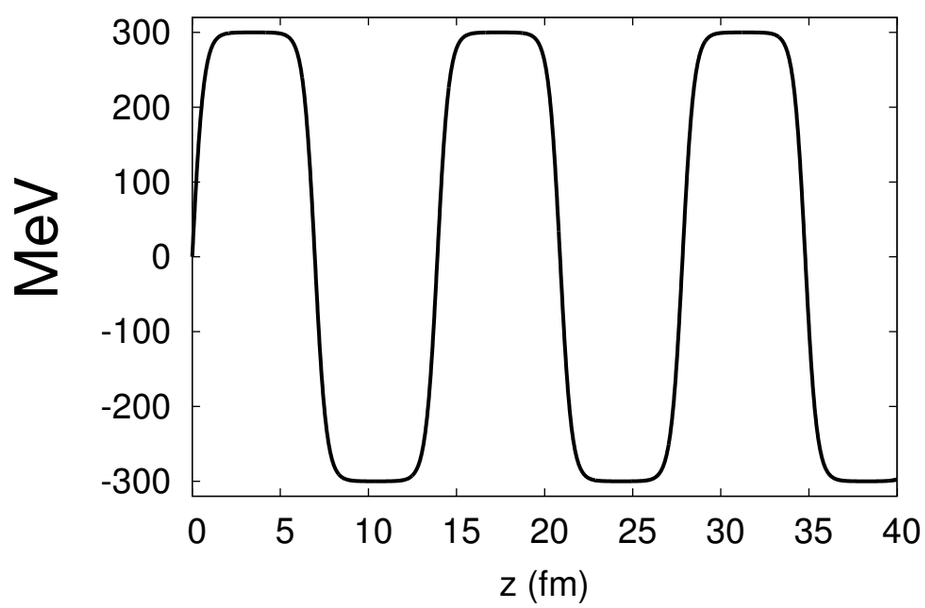

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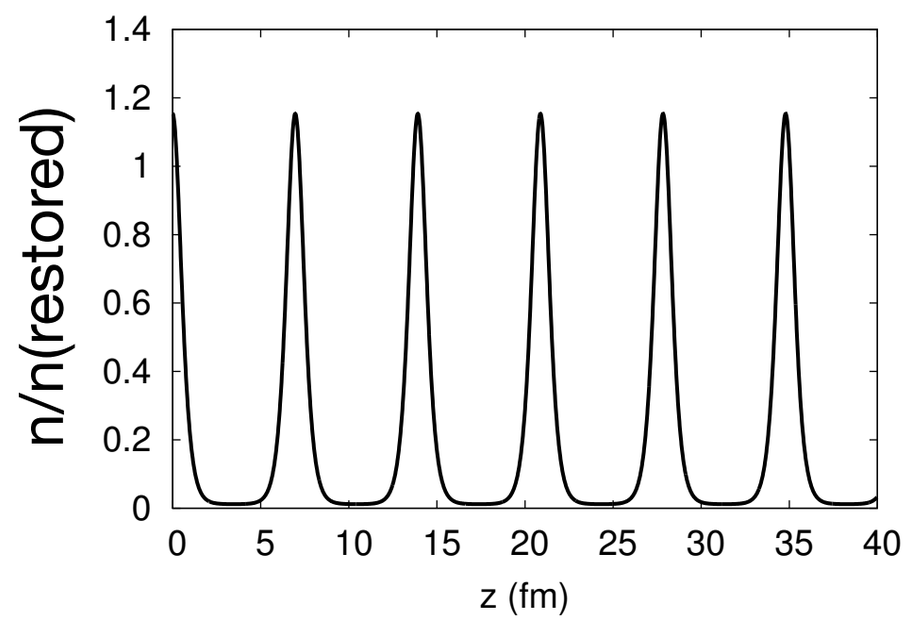

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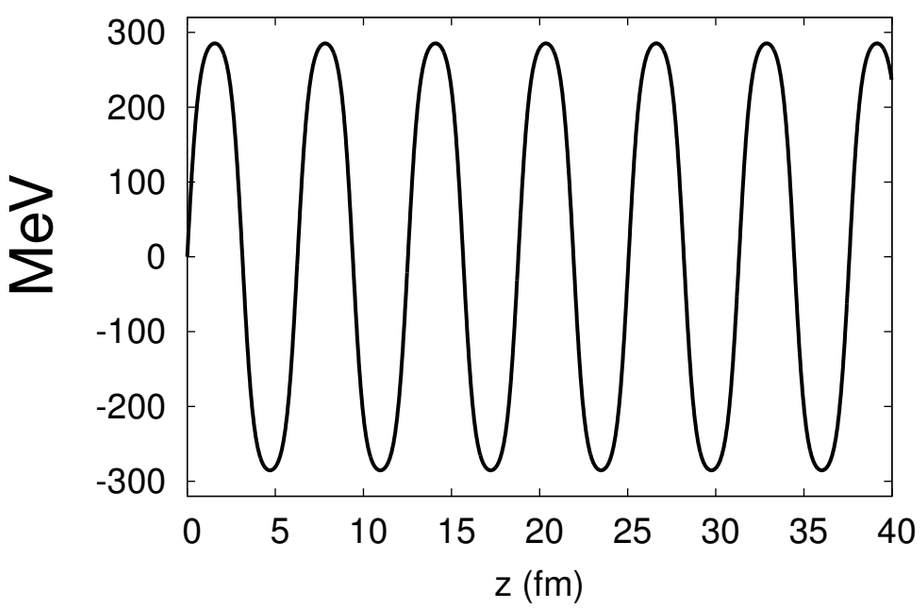

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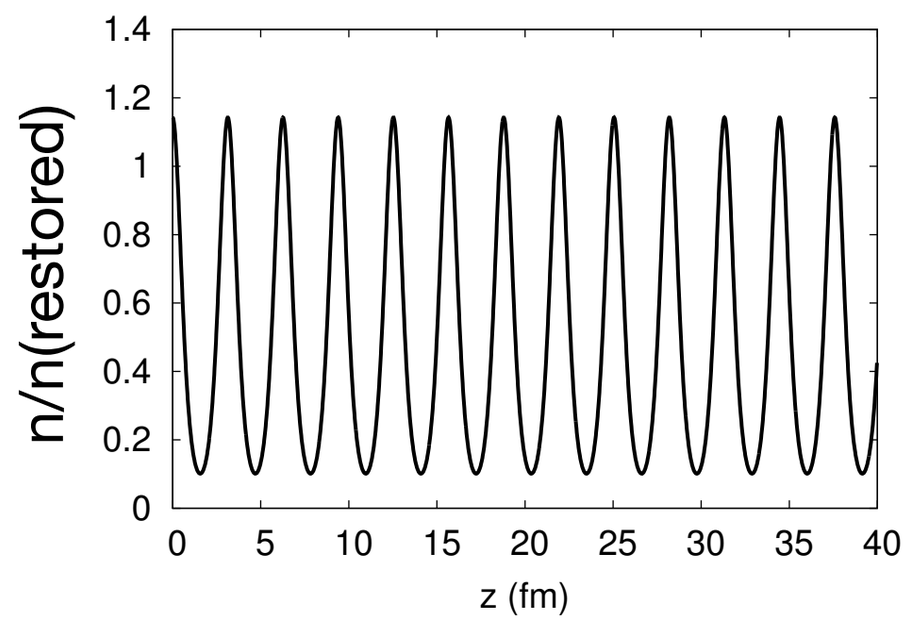

$$\mu \sim 310 \mathrm{MeV}$$

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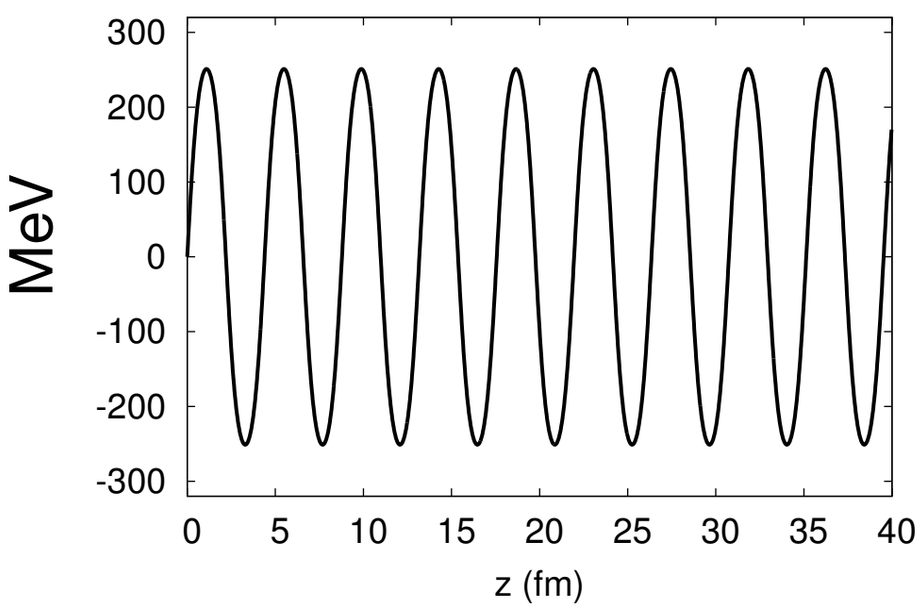

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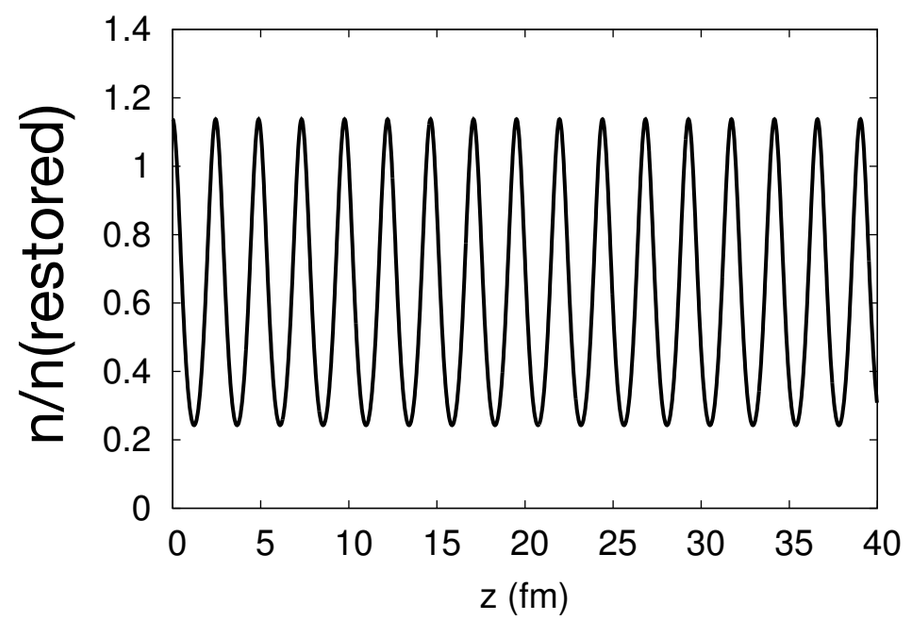

$$\mu \sim 315 \mathrm{MeV}$$

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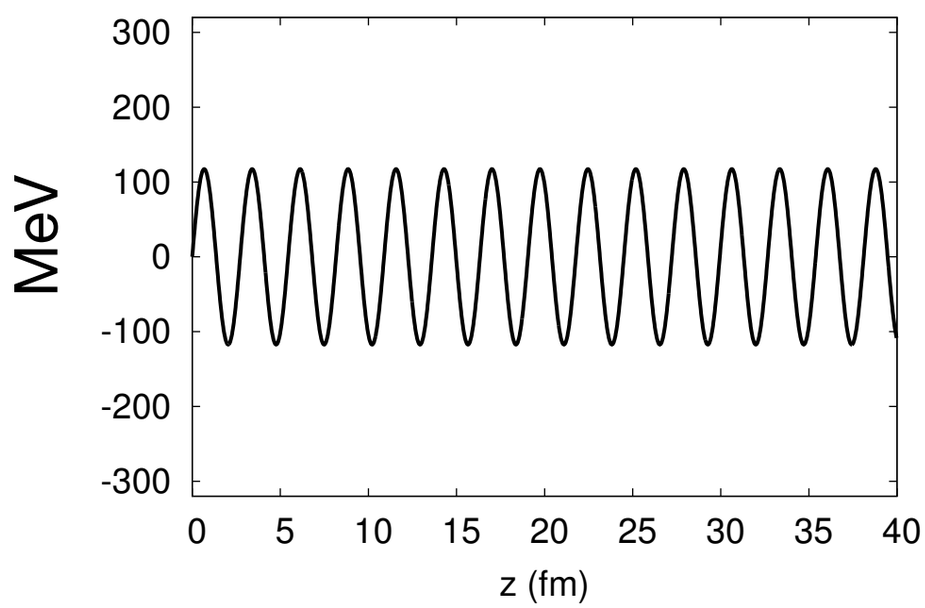

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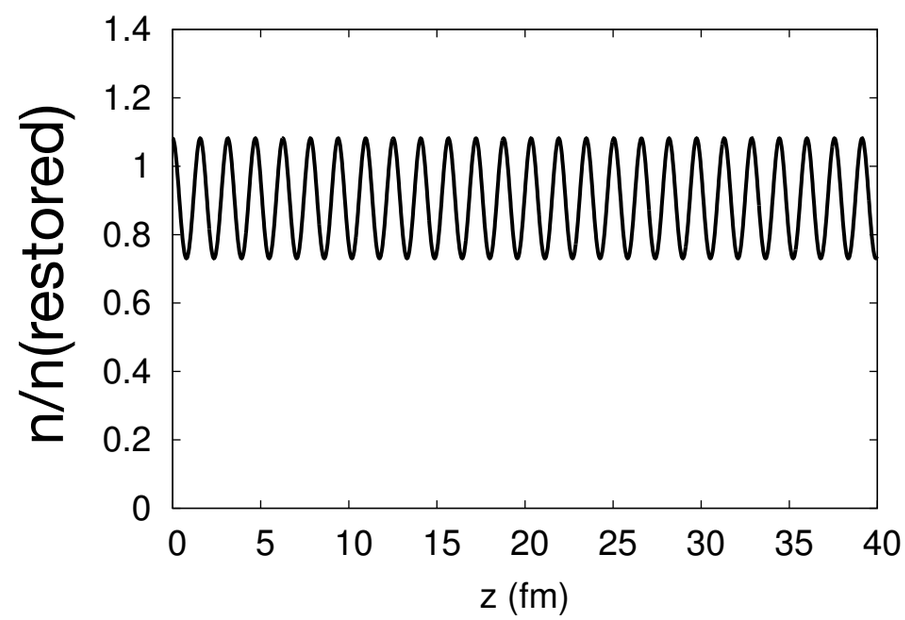

$$\mu \sim 320\mathrm{MeV}$$

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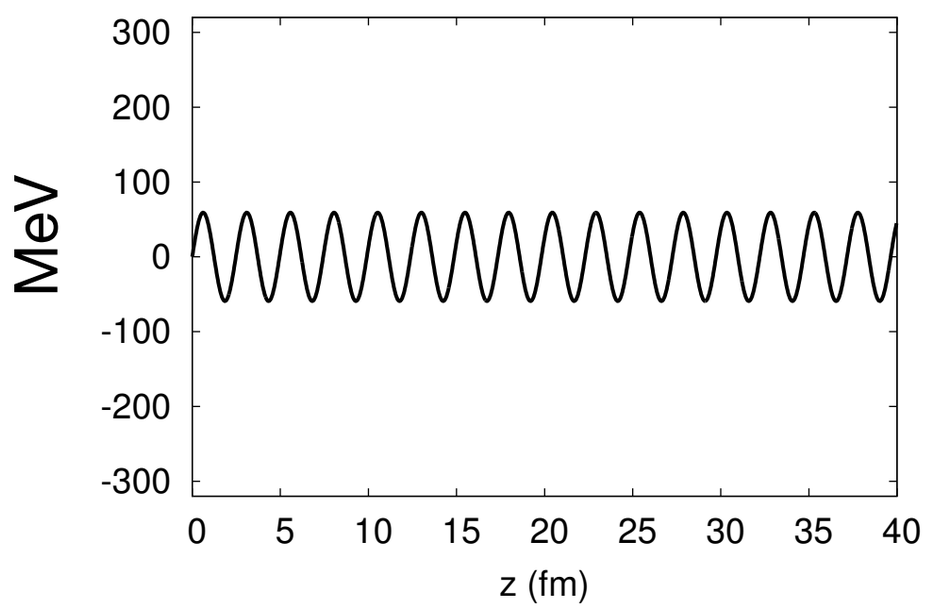

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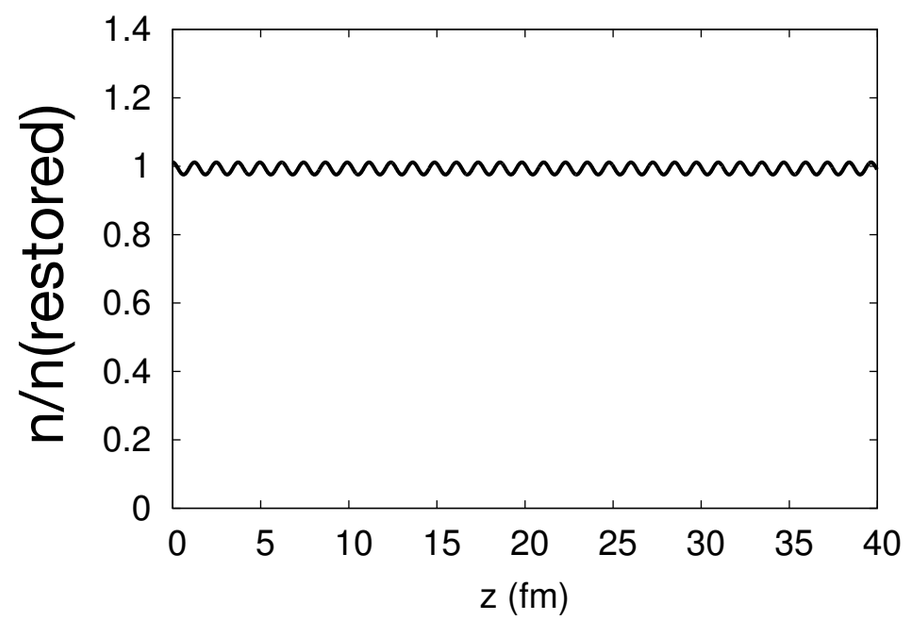

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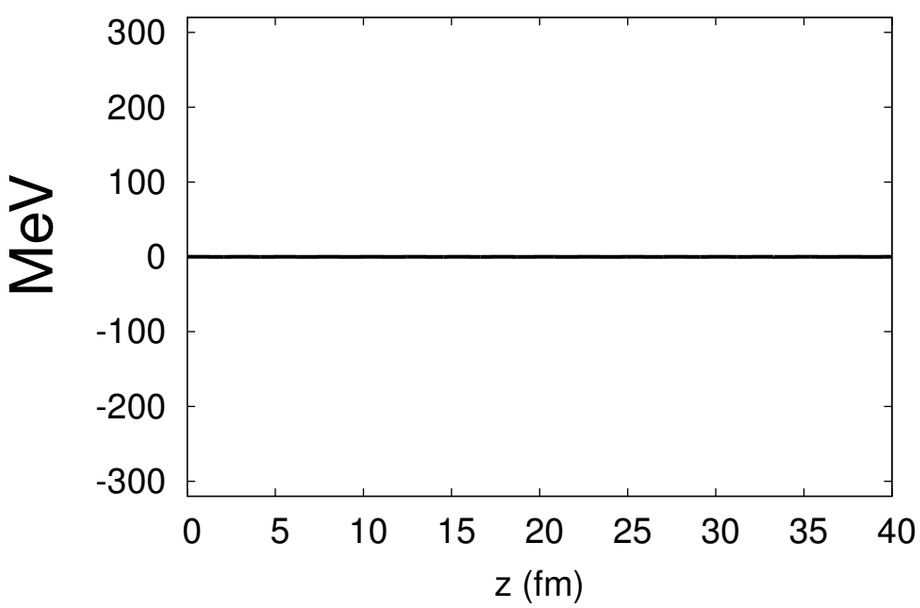

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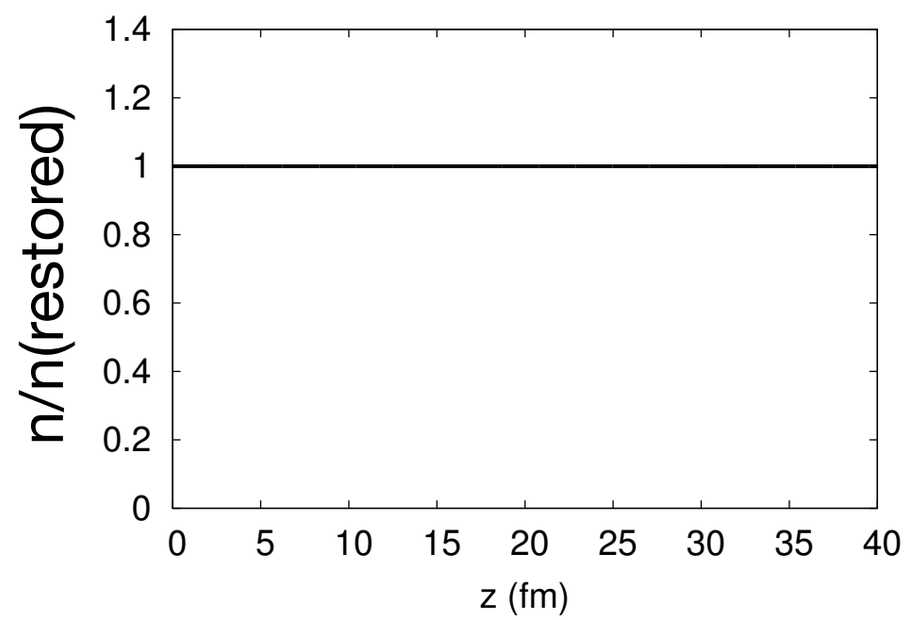

$$\mu \sim 350 \mathrm{MeV}$$

Model extensions and inhomogeneous phases

- Polyakov loop (PNJL) — SC, D,Nickel and M.Buballa, Phys.Rev. D82 (2010) 054009
- Magnetic fields — SC, E.Ferrer, V.Incera and L.Paulucci, Phys.Rev. D92 (2015) no.10, 105018
- Vector interactions — SC, M. Schramm and M.Buballa, Phys.Rev. D98 (2018) 014033
- Finite current masses — M. Buballa and SC, arXiv:1809.10066
- Strange quarks — SC and M.Buballa, WIP
- ….

One might wonder…

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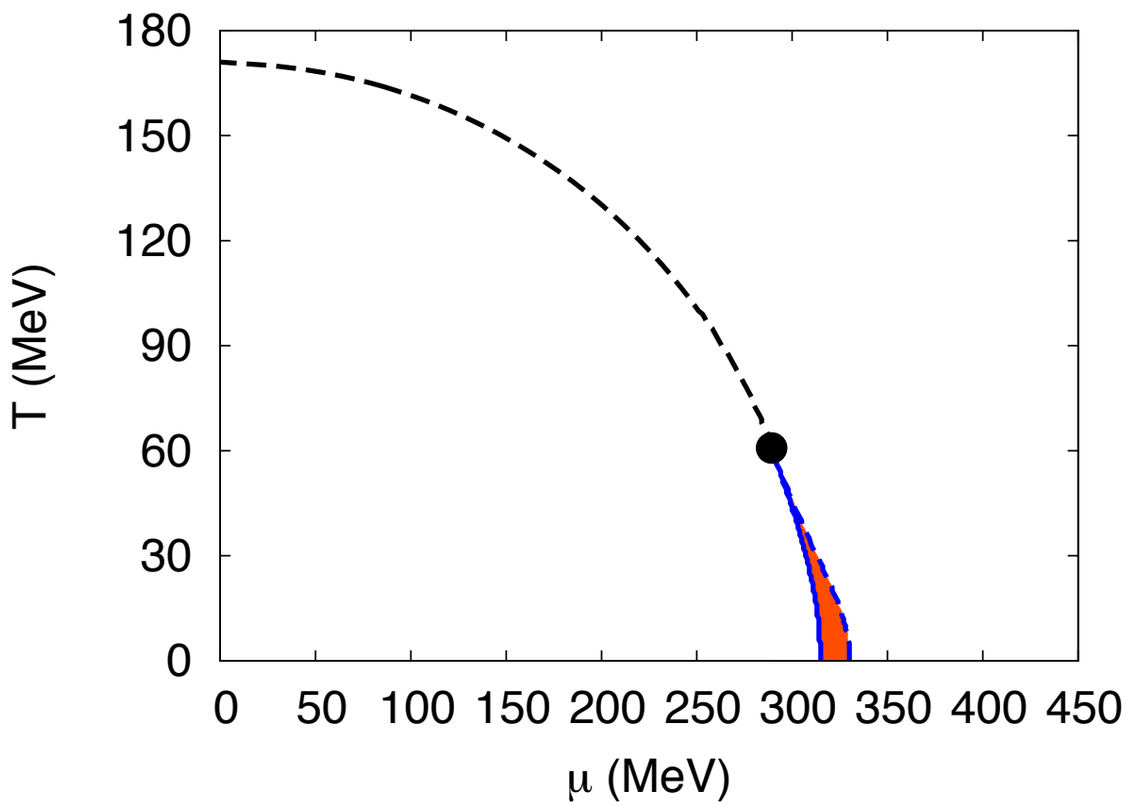
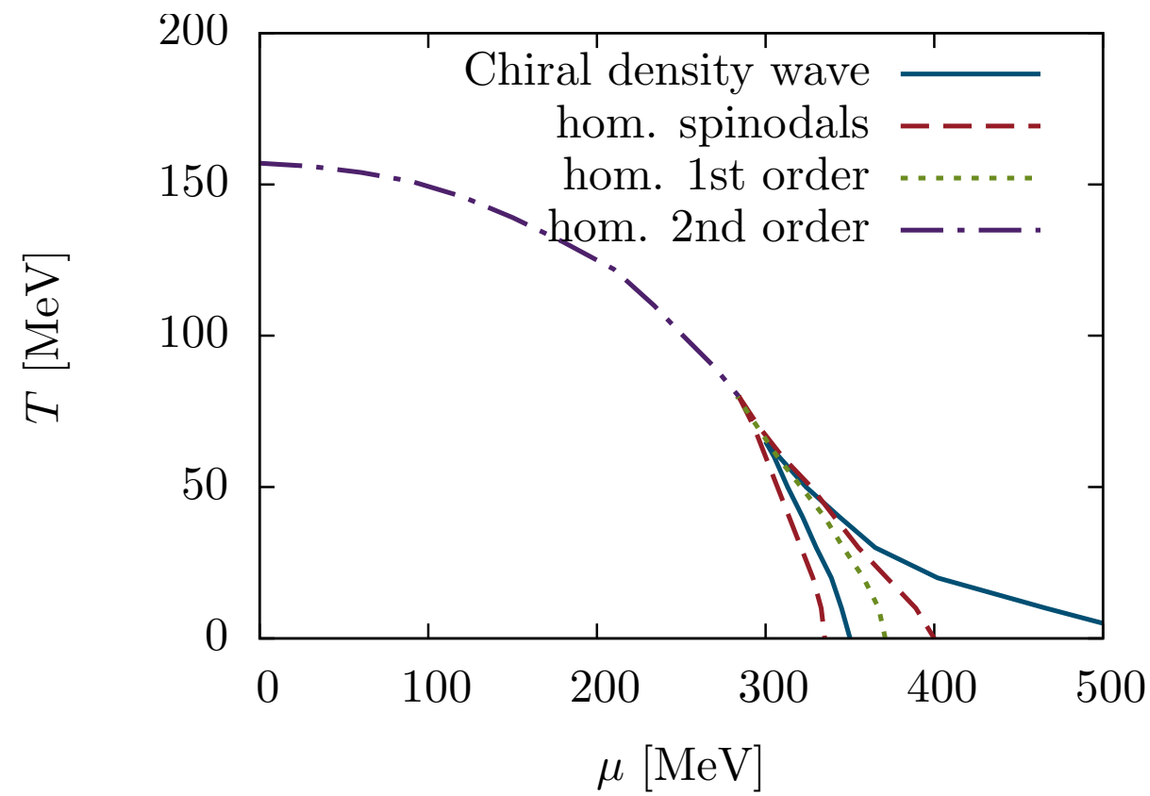
Quark-meson model



SC, M. Buballa and B.J. Schaefer, Phys. Rev. D 90 (2014)

Dyson-Schwinger



D.Müller, M.Buballa and J.Wambach, Phys.Lett. B727 (2013)

One might wonder…

- Could inhomogeneous phases be a NJL model feature - artifact ?! Unlikely, they also appear in Quark-meson model and DSE studies!

- So far: mean-field results. Fluctuations might play an important role, especially for lower-dimensional modulations!!

  -> Work in progress!

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- So far: mean-field results. Fluctuations might play an important role, especially for lower-dimensional modulations!!

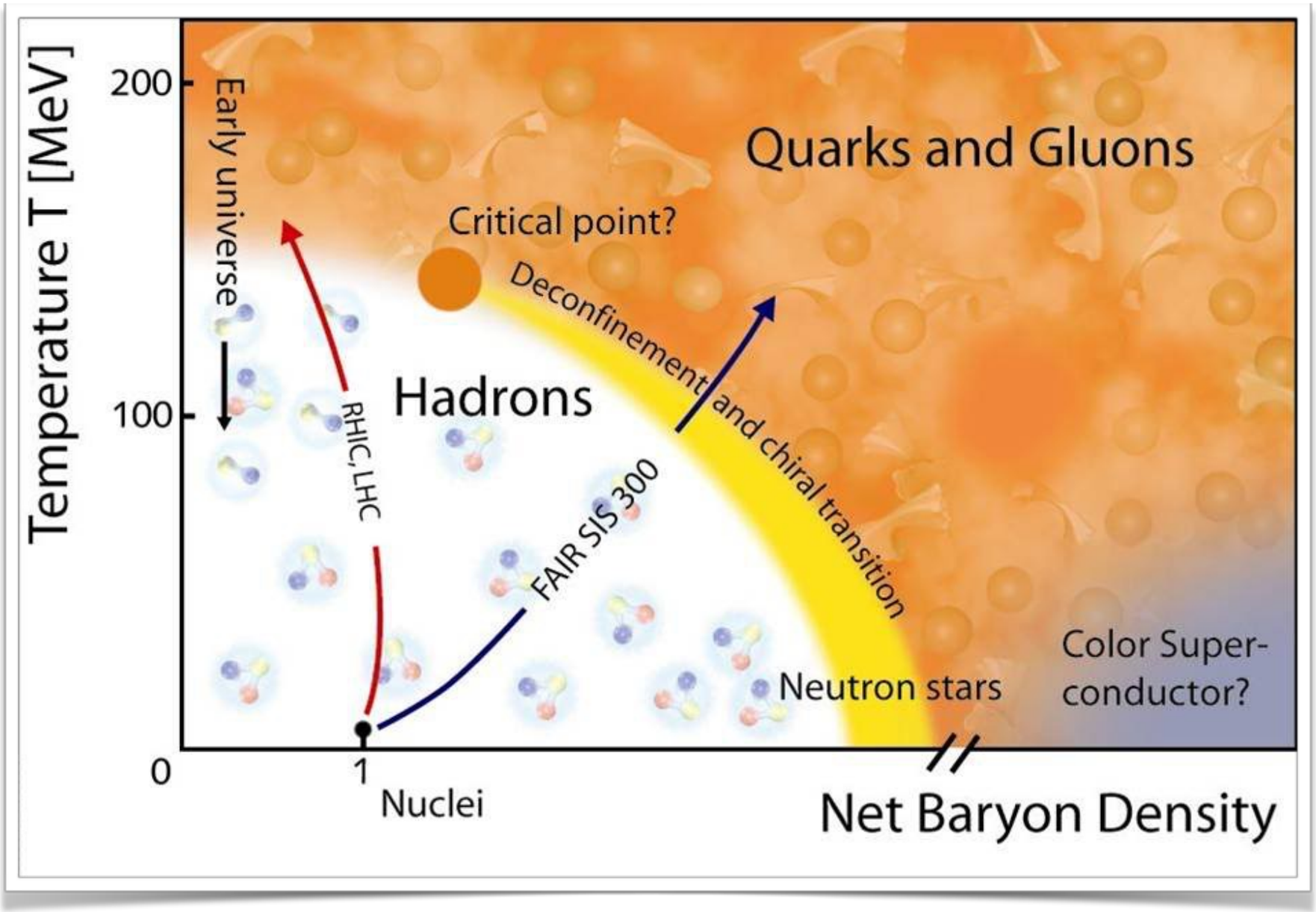
  -> Work in progress!

- Phenomenological relevance ?

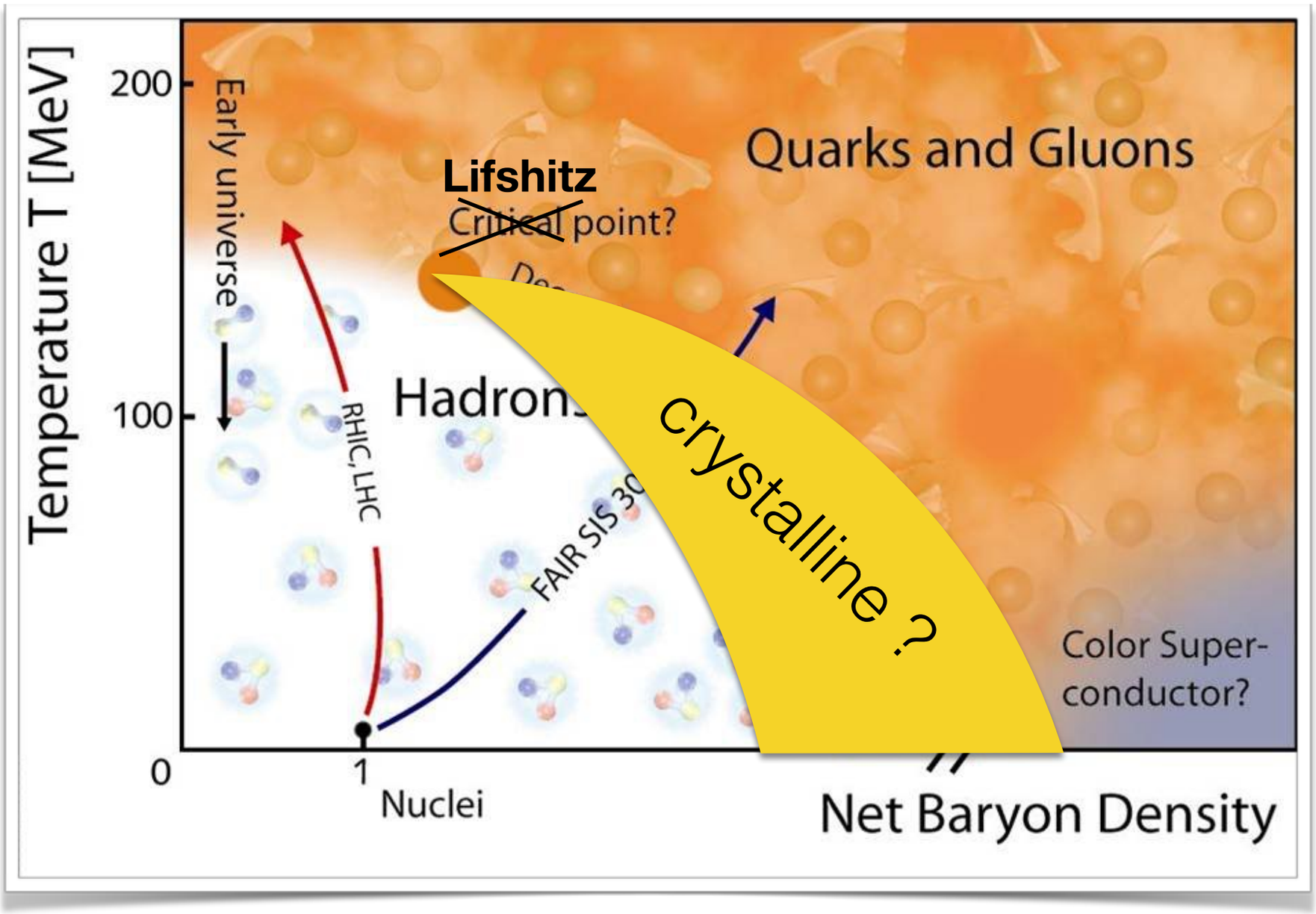
  -> For compact stars: SC, E.Ferrer, V.Incera and L.Paulucci, Phys.Rev. D92 (2015)
  M.Buballa and SC, Eur.Phys.J. A52 (2016)

  -> For heavy-ion collisions: SC, D.Nickel and M.Buballa, Phys.Rev. D82 (2010)

Take-home message #1



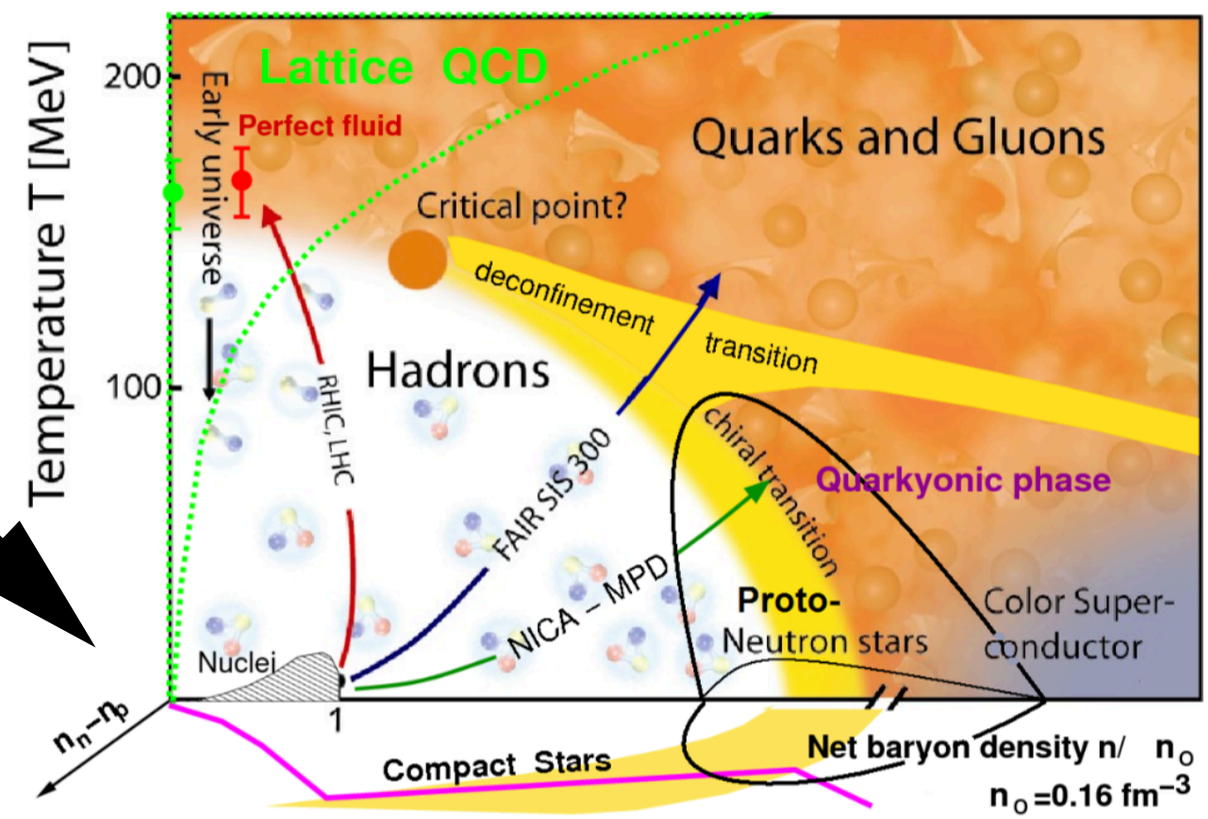
Take-home message #1



Matter at finite isospin density

Invitation: Charge neutrality

- Required for a realistic description of compact stars
- u,d quarks have different charges -> **isospin imbalance**
  - New axis on the phase diagram: μ_I
- Simultaneous description of baryon+isospin density can be tricky!



Try something simpler:

$$\mu_B = 0\,, \mu_I \neq 0$$

Phase structure at

$$\mu_B = 0\,, \mu_I \neq 0 \quad (T = 0)$$

Qualitative picture:

Phase structure at

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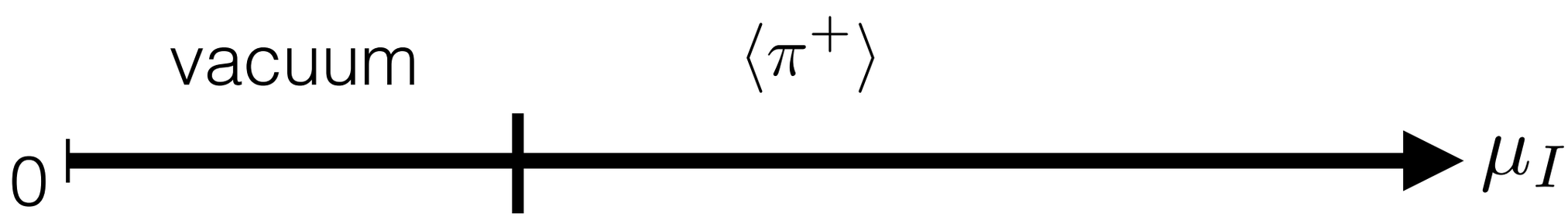
Qualitative picture:



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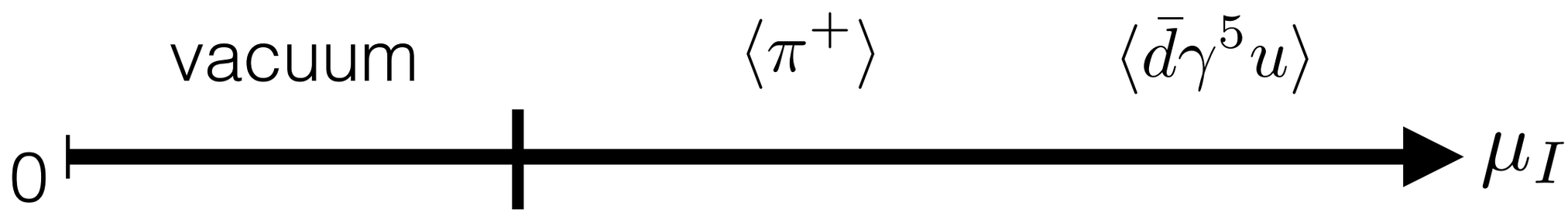

*not to scale

Phase structure at

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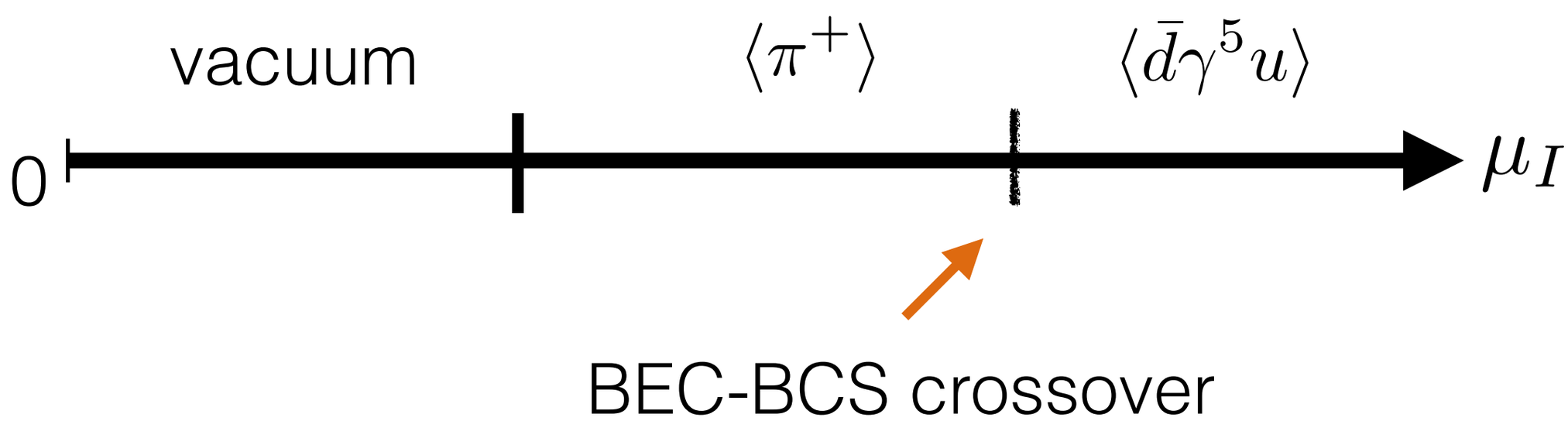
Qualitative picture:



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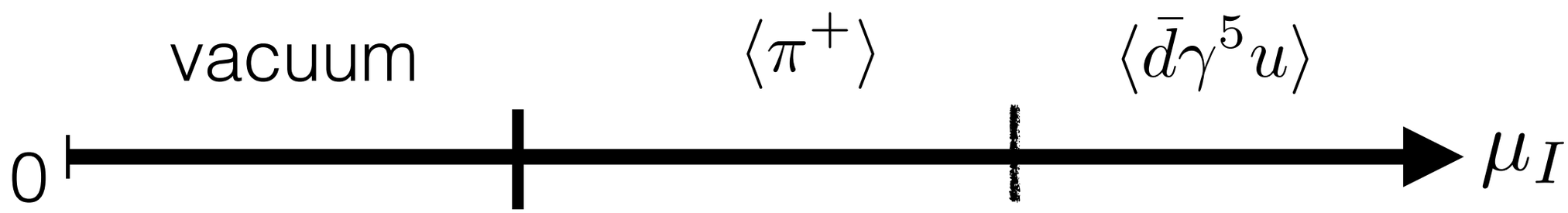


Quantitatively?

Meson chiral perturbation theory - SU(2)

$$\mathcal{L} = \frac{F_0^2}{4}\mathrm{Tr}[D_\mu U (D^\mu U)^\dagger] + \frac{F_0^2 m_\pi^2}{4}\mathrm{Tr}(U^\dagger + U)$$

Degrees of freedom: meson fields

$$U = e^{i\frac{\phi}{2F_0}} \sum e^{i\frac{\phi}{2F_0}} \qquad \phi = \tau_a \phi^a$$

$$SU(2) : \phi = \begin{pmatrix} \pi^0/\sqrt{2} & \pi^+ \\ \pi^- & -\pi^0/\sqrt{2} \end{pmatrix}$$

Two free parameters: F_0, m_π

External fields

Introduced through the covariant derivative

$$D_\mu U = \partial_\mu U - \frac{i}{2}\left[v_\mu, U\right] + \frac{i}{2}\left\{a_\mu, U\right\}$$

Chemical potentials too!

$$v^\mu = -2eQA^\mu - 2\mu\delta^{\mu 0}$$

$$\mu = diag\left(\frac{1}{3}\mu_B + \frac{1}{2}\mu_I, \frac{1}{3}\mu_B - \frac{1}{2}\mu_I, \frac{1}{3}\mu_B - \mu_S\right)$$

Limits of validity

Low-energy effective theory:

- Small momenta
- No baryons
- Lightest mesons only

$$\mu_B \lesssim 940\,\mathrm{MeV}$$
$$\mu_I \lesssim 770\,\mathrm{MeV} \approx 5\,m_\pi$$

Ground state - SU(2)

Ansatz: $$\Sigma = e^{i\alpha\cdot\sigma} = \cos\alpha + i(\mathbf{n}\cdot\sigma)\sin\alpha$$

$$\mathbf{n} = (\cos\theta, \sin\theta, 0)$$

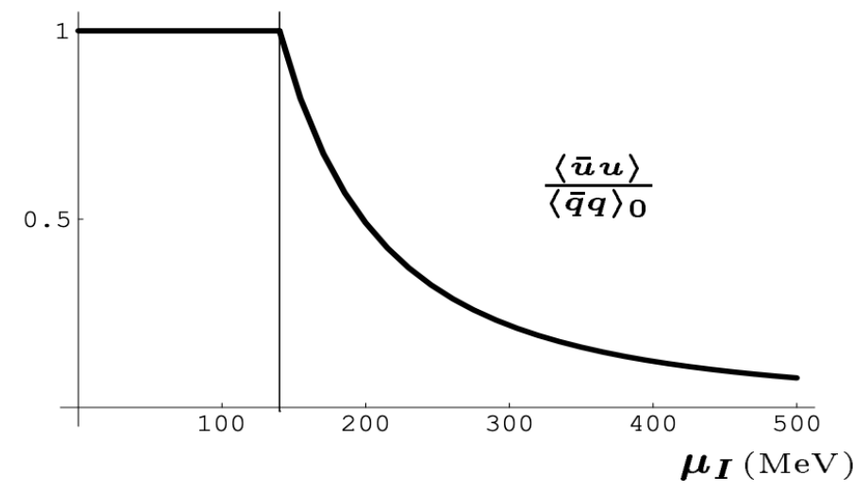
Ground state: maximize the static part of the Lagrangian
(“potential energy”)

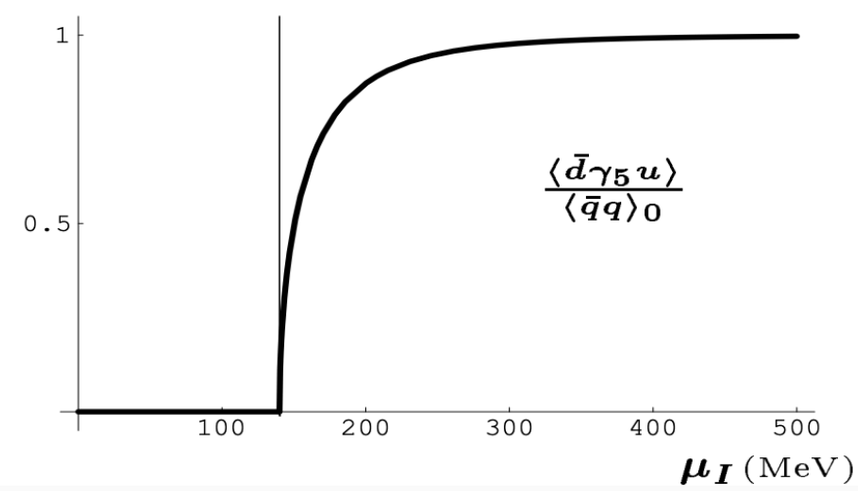
$$\mathcal{L}_{stat} = F_0^2 m_\pi^2 \cos\alpha + \frac{1}{2} F_0^2 \mu_I^2 \sin^2\alpha (n_1^2 + n_2^2)$$

Pion condensation

When μ_I reaches m_π
pion condensate forms!

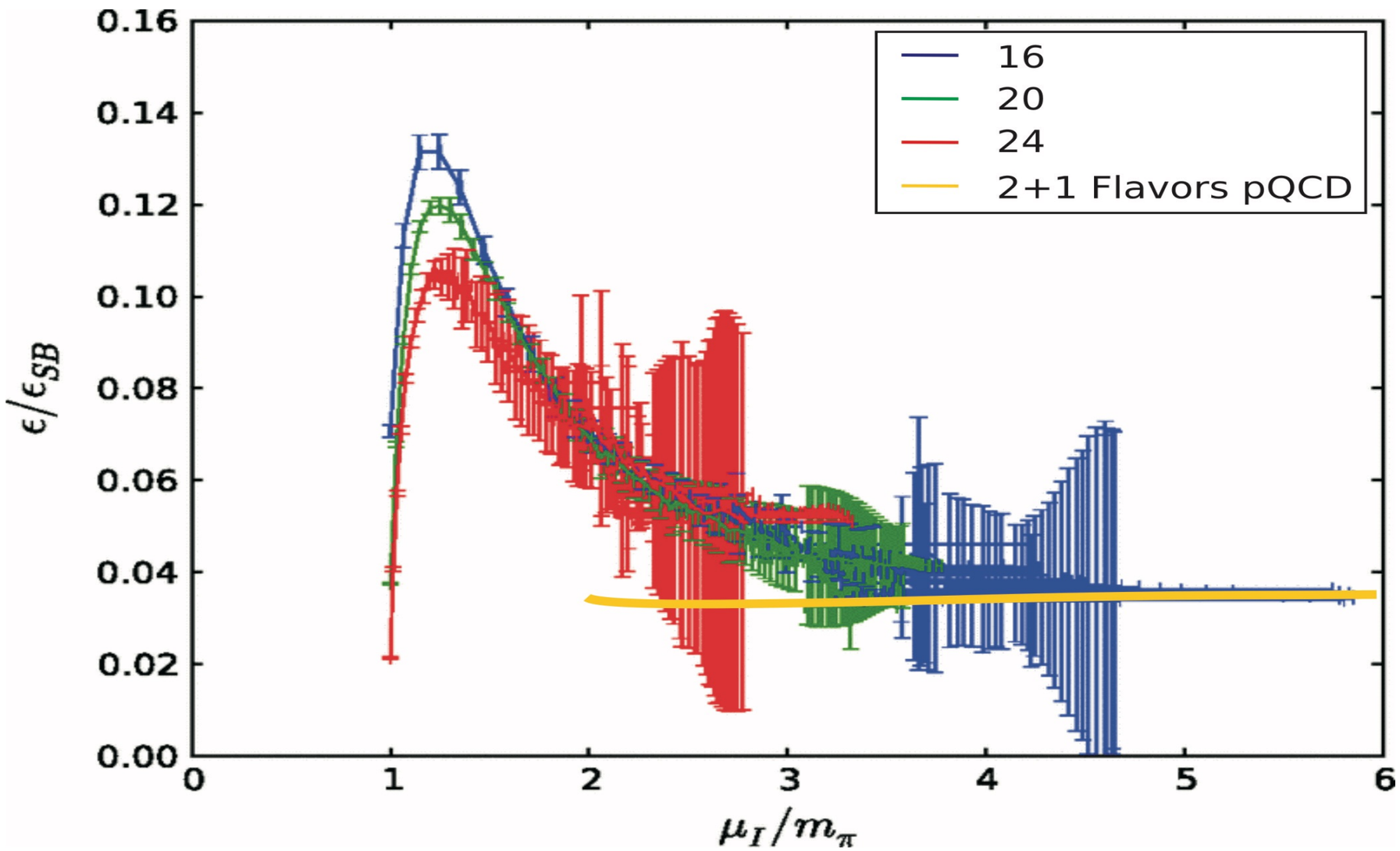
(and the
chiral condensate melts)





Son & Stephanov (2000), Kogut & Toublan (2001)

Energy density



W. Detmold et al, Phys.Rev. D86 (2012)
T. Graf et al, Phys.Rev. D93 (2016)

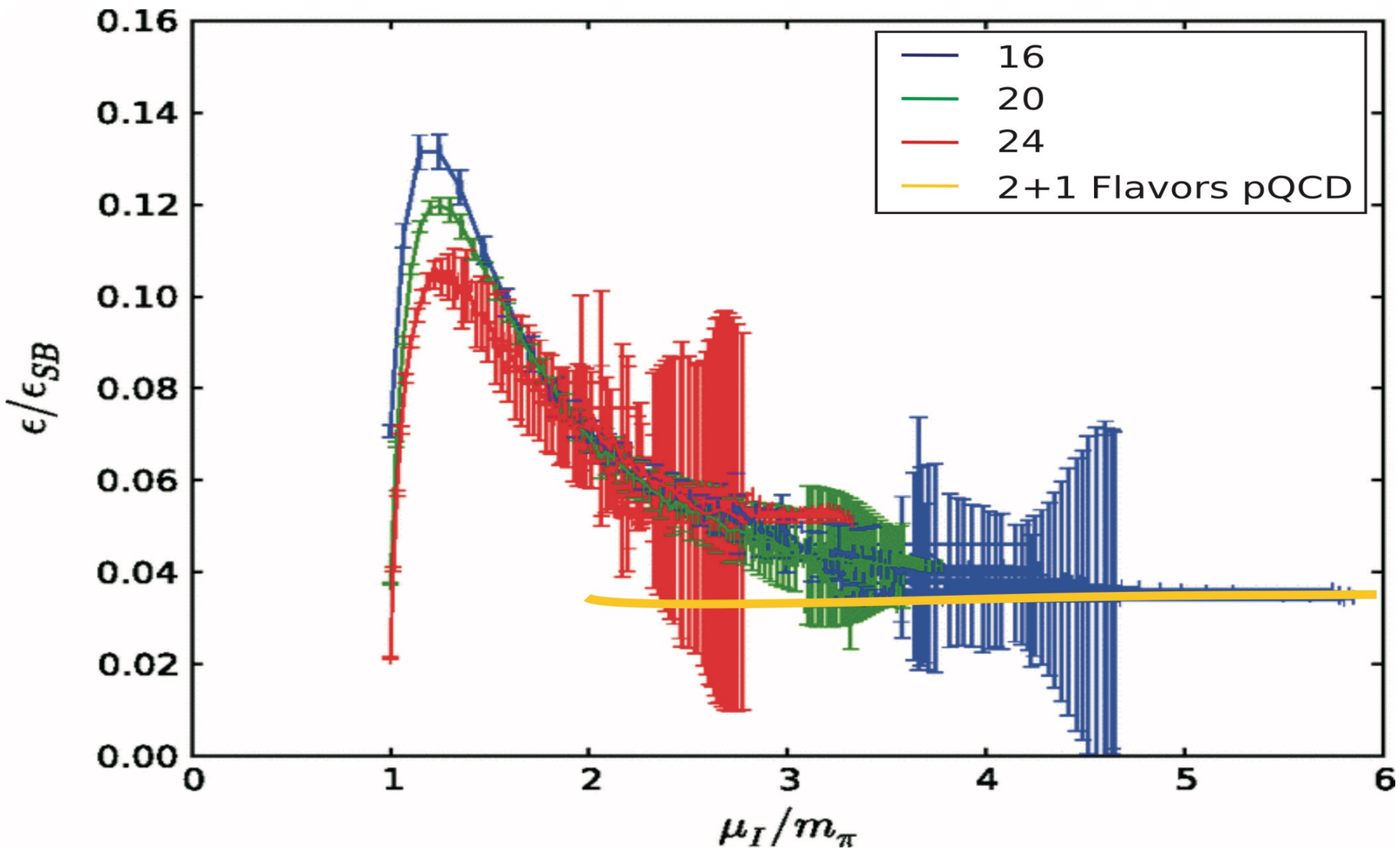
XPT Equation of state - πc phase

$$\epsilon^{\pi c} = \frac{f_\pi^2 \mu_I^2}{2}\left(1 + 2\frac{m_\pi^2}{\mu_I^2} - 3\frac{m_\pi^4}{\mu_I^4}\right)$$

$$\epsilon(P) = 2\sqrt{P(2F_0^2 m_\pi^2 + P)} - P$$

Comparison with lattice ?

Energy density



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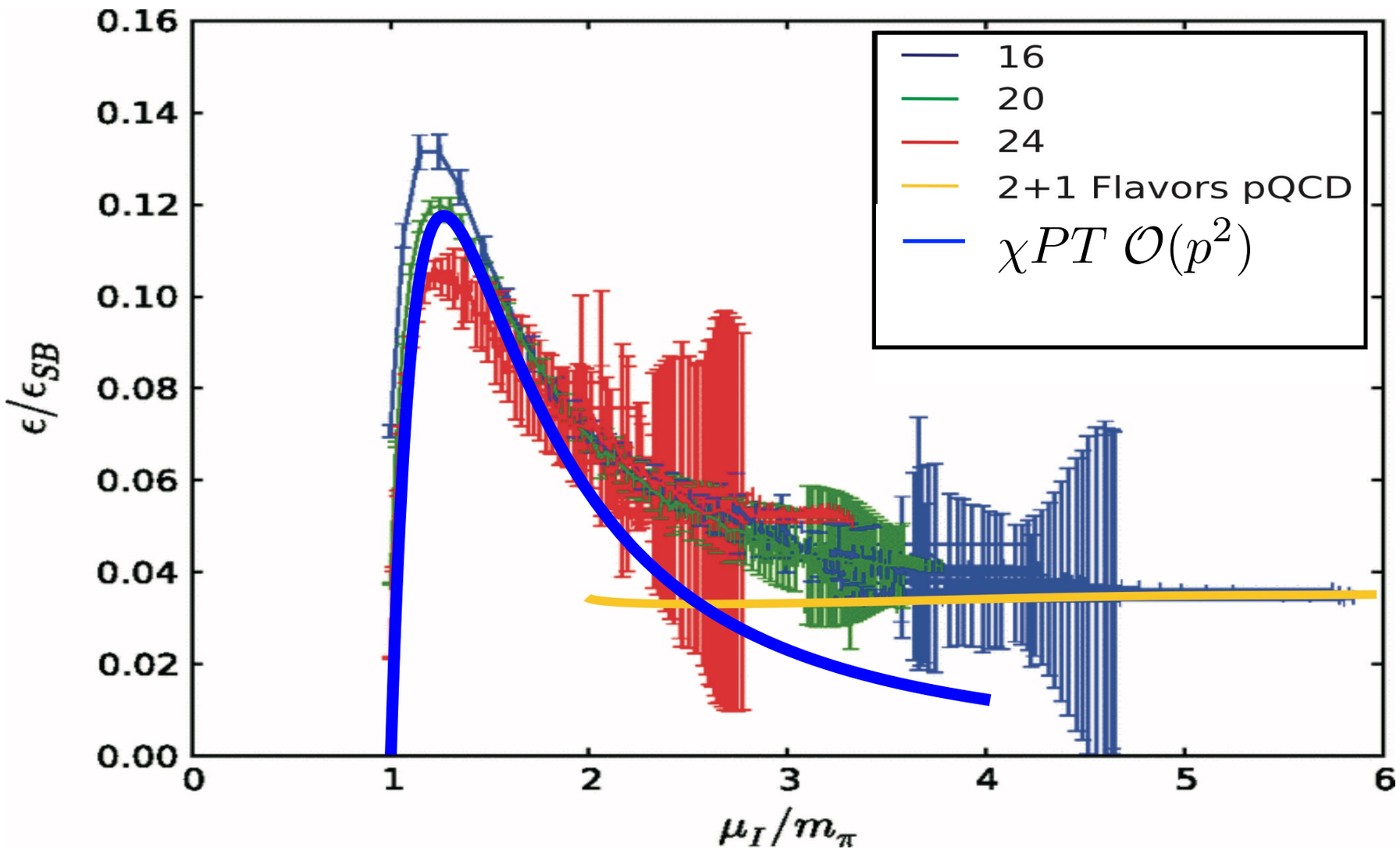


Lowest order XPT already gives surprisingly good agreement!

SC, A. Mammarella and M. Mannarelli, Phys. Rev. D 93 (2016)

Peak

Peak position

$$\mu_I^{peak} = \left(\sqrt{13} - 2\right)^{1/2} m_\pi \approx 1.28\, m_\pi$$

Parameter-independent!

Continuum-extrapolated lattice result:

$$\mu_I^{peak} = 1.3\, m_\pi$$

NLO Lagrangian

$$
\begin{aligned}
\mathcal{L} = & \frac{F_0^2}{4}\mathrm{Tr}[D_\mu U(D^\mu U^\dagger)] + \frac{F_0^2}{4}\mathrm{Tr}\left(\chi U^\dagger + U\chi^\dagger\right) \\
& + L_1 \left\{\mathrm{Tr}[D_\mu U(D^\mu U)^\dagger]\right\}^2 \\
& + L_2 \mathrm{Tr}\left[D_\mu U(D_\nu U)^\dagger\right] \mathrm{Tr}\left[D^\mu U(D^\nu U)^\dagger\right] \\
& + L_3 \mathrm{Tr}\left[D_\mu U(D^\mu U)^\dagger D_\nu U(D^\nu U)^\dagger\right] \\
& + L_4 \mathrm{Tr}\left[D_\mu U(D^\mu U)^\dagger\right] \mathrm{Tr}\left(\chi U^\dagger + U\chi^\dagger\right) \\
& + L_5 \mathrm{Tr}\left[D_\mu U(D^\mu U)^\dagger(\chi U^\dagger + U\chi^\dagger)\right] \\
& + L_6 \left[\mathrm{Tr}\left(\chi U^\dagger + U\chi^\dagger\right)\right]^2 \\
& + L_7 \left[\mathrm{Tr}\left(\chi U^\dagger - U\chi^\dagger\right)\right]^2 \\
& + L_8 \mathrm{Tr}\left(U\chi^\dagger U\chi^\dagger + \chi U^\dagger \chi U^\dagger\right) \\
& - iL_9 \mathrm{Tr}\left[f^R_{\mu\nu} D^\mu U(D^\nu U)^\dagger + f^L_{\mu\nu}(D^\mu U)^\dagger D^\nu U\right] \\
& + L_{10} \mathrm{Tr}\left(U f^L_{\mu\nu} U^\dagger f_R^{\mu\nu}\right)
\end{aligned}
$$

$\longleftarrow$ In principle: 2+10 free parameters

(“low-energy constants”)

NLO ground state

In practice:

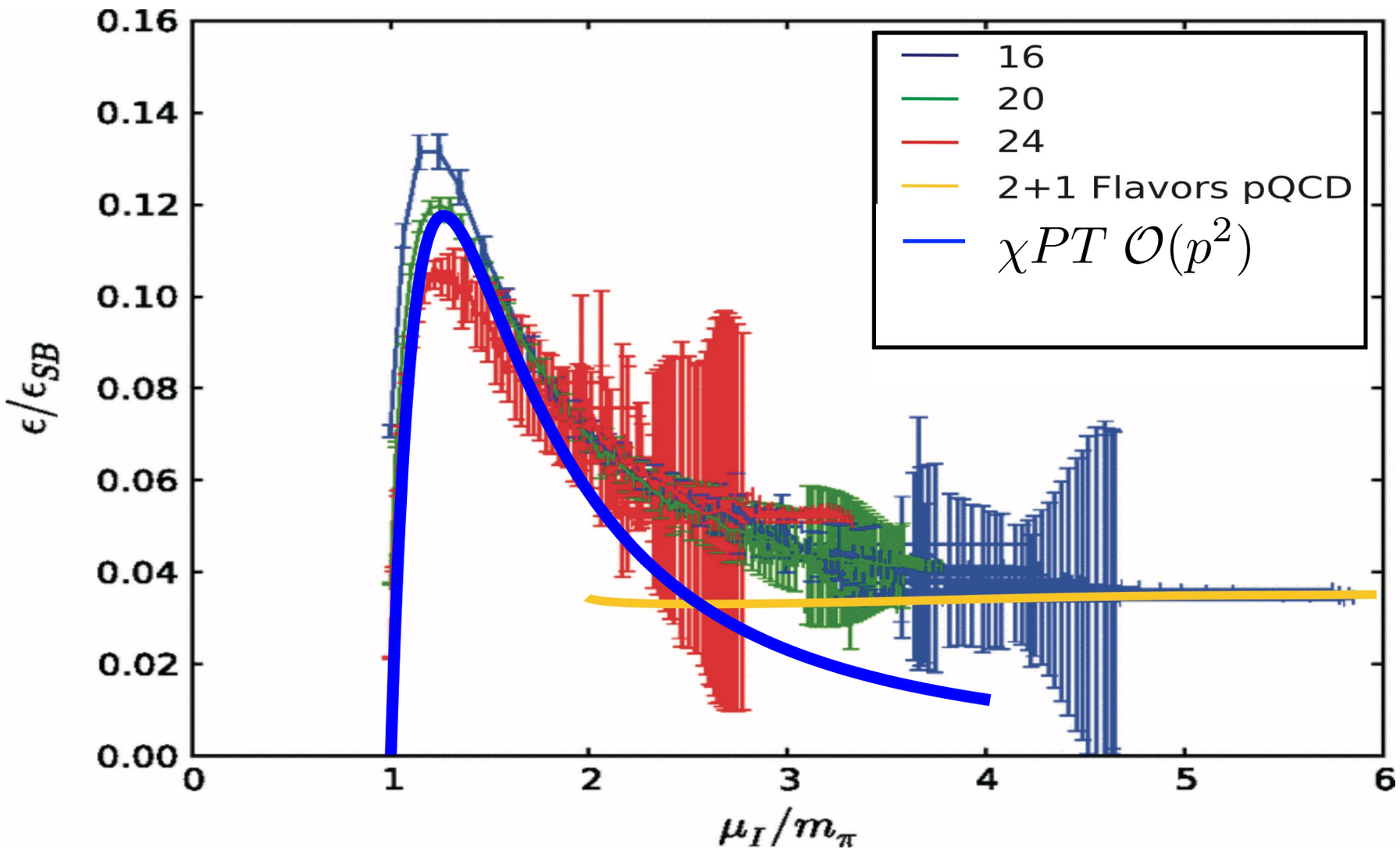
$$\begin{aligned}\mathcal{L} =& F_0^2 m_\pi^2 \cos\alpha + \frac{F_0^2}{2}\mu_I^2 \sin^2\alpha \\ &+ 2\mu_I^4 (2L_1 + 2L_2 + L_3)\sin^4\alpha + 4\mu_I^2 m_\pi^2 (2L_4 + L_5)\sin^2\alpha\cos\alpha \\ &+ 4m_\pi^4 \left[(4L_6 + L_8)\cos^2\alpha - L_8 \sin^2\alpha\right]\end{aligned}$$

Parameters only enter in given combinations!

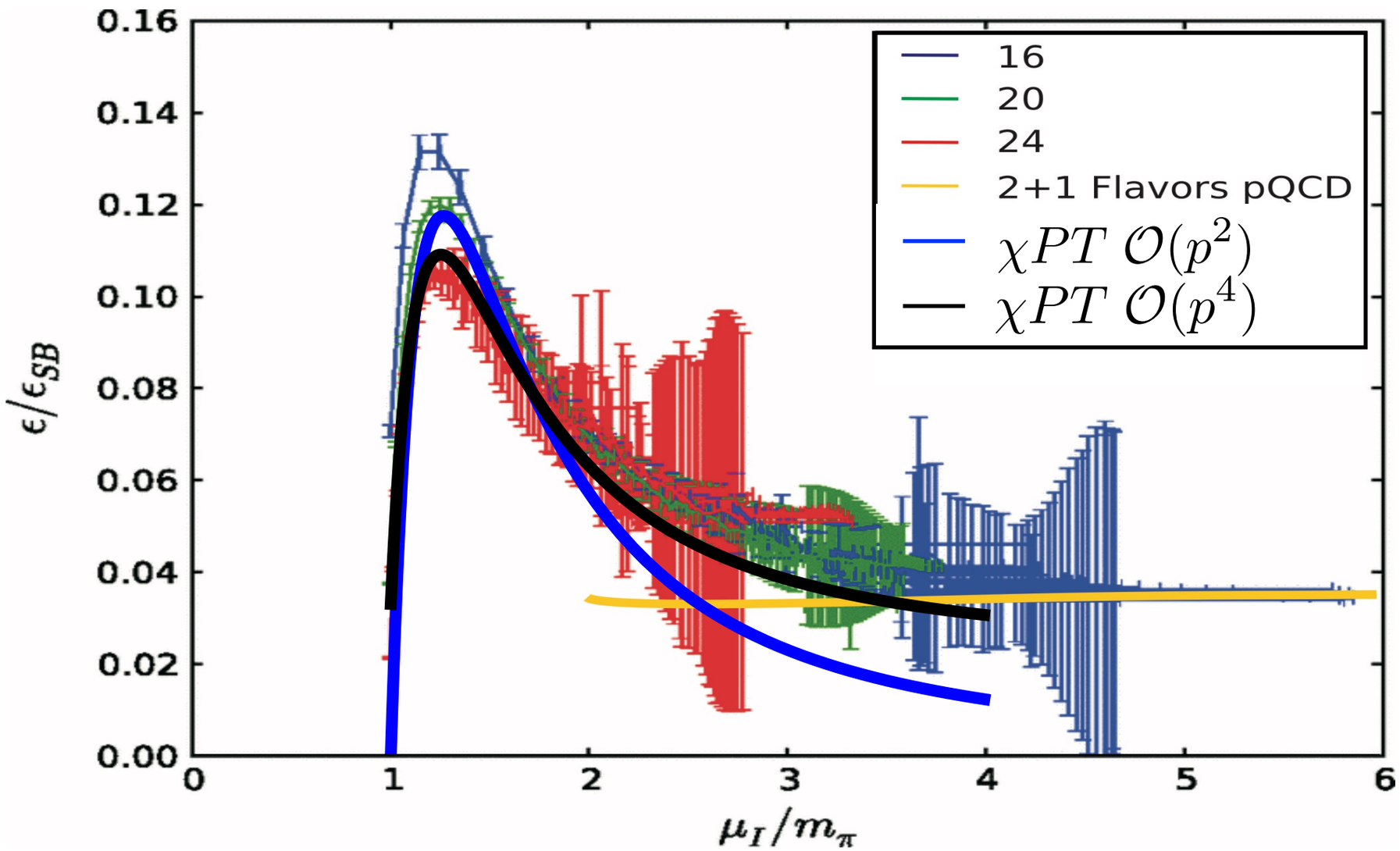
$$\begin{aligned}(2L_1 + 2L_2 + L_3) &= a_0\epsilon \\ (2L_4 + L_5) &= b_0\epsilon \\ (2L_6 + L_8) &= c_0\epsilon\end{aligned}$$

$$\epsilon = 10^{-3}$$

Energy density



Energy density



(Effectively a (2+1)-parameter fit -> can use lattice to fix LEC!)

SC. L.Lepori, A.Mammarella, M.Mannarelli and G.Pagliaroli, Eur.Phys.J. A53 (2017)

Take-home message #2

- Mesonic chiral perturbation theory is a powerful tool for describing strong interaction matter at finite isospin densities
- Peculiar shape of energy density curve very well reproduced by chiral perturbation theory
- Overlap of different methods - possibility to benchmark them against each other!
- Using LQCD to fix low-energy constants

Chiral plasmas

Chiral plasma

- System of massless fermions at finite temperature/density
- Could be relevant for
    - Heavy-ion collisions
    - Weyl-Dirac semimetals
    - Astrophysical scenarios
- Lots of new physics! Especially when chiral imbalanced (chiral magnetic effect, plasma instabilities…)

Theoretical framework

- Keep things simple: QED
- Introduce temperature/density: multi-scale problem (g, T..)
- Do perturbative computations behave well in thermal field theory?

Invitation: hard thermal loops

Separation of scales a fundamental concept in thermal FT

For a system at high temperature:

hard scale ~ T soft scale ~ gT $(g \ll 1)$

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Separation of scales a fundamental concept in thermal FT

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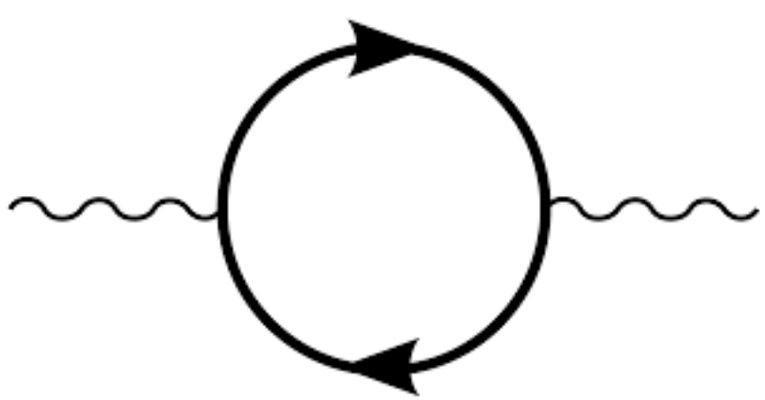
Now consider eg. photon self-energy



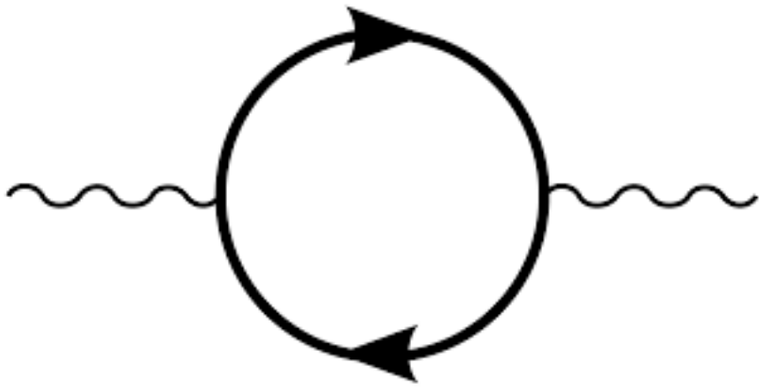
Leading contribution comes from hard loop momenta Q ~ T

$$\Pi_{HTL} \sim g^2 T^2$$

Invitation: hard thermal loops

For a soft photon momentum…

vs



$$D^{-1}_{\mu\nu}(L) \sim L^2 - \Pi$$

$$\sim g^2 T^2$$

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For a soft photon momentum…

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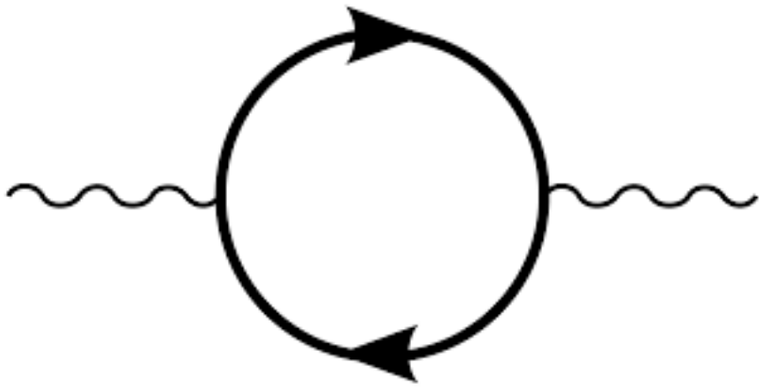

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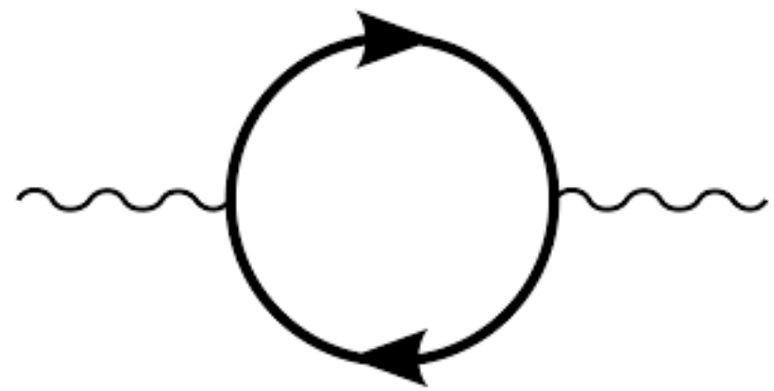


$$D^{-1}_{\mu\nu}(L) \sim L^2 - \Pi$$

$\sim g^2 T^2$ $\sim g^2 T^2$

Breakdown of perturbation theory: for soft external momenta, one-loop hard thermal corrections as relevant as tree amplitudes

→ Resummation is required !

Hard thermal loops

- HTL resummation:
  include (hard) correction into (soft) propagators

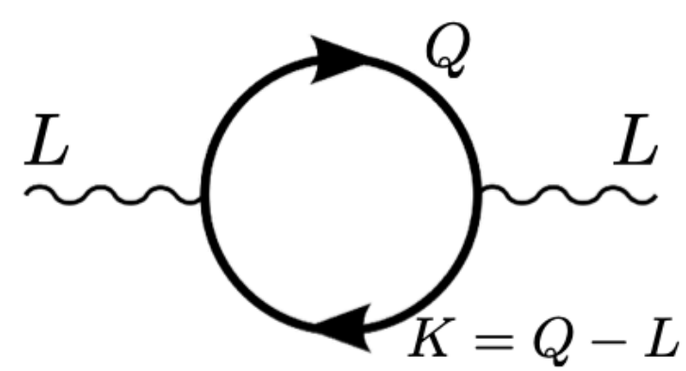
$$\Pi^L_{(1)} = -\frac{m_D^2}{2}\left(1 - \frac{l_0}{2l}\log\frac{l_0+l}{l_0-l}\right)$$
$$\Pi^T_{(1)} = -\frac{m_D^2}{2}\left[1 + \frac{L^2}{l^2}\left(1 - \frac{l_0}{2l}\log\frac{l_0+l}{l_0-l}\right)\right]$$

$m_D^2 \sim g^2 T^2$ Debye (“thermal”) mass

- Can we go beyond this? -> Power corrections

Deriving HTL corrections

Start from photon self-energy:



$$\Pi_R^{\mu\nu}(L) = -\frac{ie^2}{2}\int \frac{d^4q}{(2\pi)^4}\Big(\mathrm{Tr}[\gamma^\mu S_S(K)\gamma^\nu S_R(Q)] + \mathrm{Tr}[\gamma^\mu S_A(K)\gamma^\nu S_S(Q)]\Big)\ ,$$

$$S_{R/A}(Q) = \frac{\not{Q}}{Q^2 \pm i\,\mathrm{sgn}(q_0)\eta}\ , \qquad S_S(Q) = -2\pi i \not{Q}\,(1 - 2n_F(|q_0|))\delta(Q^2)$$

on-shell condition!

$$q^\mu = q v^\mu \qquad q^\mu = -q\tilde{v}^\mu$$

$$\vec{v} = \vec{q}/q$$
$$(q = |\vec{q}|)$$

HTL computation

One arrives at

occupation number encoding (T, μ) dependence

$$\Pi_R^{\mu\nu}(L) = e^2 \int \frac{d^3q}{(2\pi)^3} \frac{1 - 2n_F(q)}{q} \left(\frac{2qv^\mu v^\nu - (v^\mu L^\nu + v^\nu L^\mu) + g^{\mu\nu} \quad v \cdot L}{v \cdot L - \frac{L^2}{2q} + i\,\mathrm{sgn}(q - l_0)\eta} - \frac{2q\tilde{v}^\mu \tilde{v}^\nu - (\tilde{v}^\mu L^\nu + \tilde{v}^\nu L^\mu) + g^{\mu\nu}\, \tilde{v} \cdot L}{\tilde{v} \cdot L + \frac{L^2}{2q} + i\,\mathrm{sgn}(q + l_0)\eta} \right)$$

Leading contribution from hard (q ~ T) on-shell particles

HTL computation

One arrives at

$$\Pi_R^{\mu\nu}(L) = e^2 \int \frac{d^3q}{(2\pi)^3} \frac{1-2n_F(q)}{q} \left(\frac{2qv^\mu v^\nu - (v^\mu L^\nu + v^\nu L^\mu) + g^{\mu\nu} \quad v\cdot L}{v\cdot L - \frac{L^2}{2q} + i\,\mathrm{sgn}(q-l_0)\eta} - \frac{2q\tilde{v}^\mu \tilde{v}^\nu - (\tilde{v}^\mu L^\nu + \tilde{v}^\nu L^\mu) + g^{\mu\nu}\, \tilde{v}\cdot L}{\tilde{v}\cdot L + \frac{L^2}{2q} + i\,\mathrm{sgn}(q+l_0)\eta} \right)$$

now expand for large q the integrand..

QED photon self-energy

One finds

$$\Pi^L_{(1)} = -\frac{m_D^2}{2}\left(1 - \frac{l_0}{2l}\log\frac{l_0+l}{l_0-l}\right)$$

$$\Pi^T_{(1)} = -\frac{m_D^2}{2}\left[1 + \frac{L^2}{l^2}\left(1 - \frac{l_0}{2l}\log\frac{l_0+l}{l_0-l}\right)\right]$$

$$\Pi^L_{(3)} = \frac{\alpha}{3\pi}\left[\frac{l^2}{\epsilon} + 2l^2\left(\ln\frac{\sqrt{\pi}Te^{-\gamma_E/2}}{2\nu} - 1\right) + \left(2l^2 - L^2\right)\left(1 - \frac{l_0}{2l}\ln\frac{l_0+l}{l_0-l}\right)\right]$$

$$\Pi^T_{(3)} = \frac{2\alpha L^2}{3\pi}\left[\frac{1}{2\epsilon} + \left(\ln\frac{\sqrt{\pi}Te^{-\gamma_E/2}}{2\nu} - 1\right) + \frac{1}{4} + \left(1 + \frac{L^2}{4l^2}\right)\left(1 - \frac{l_0}{2l}\ln\frac{l_0+l}{l_0-l}\right)\right]$$

SC, Manuel & Soto, PLB780 (2018)

Missing stuff

We computed one power correction $\sim e^2$ of the HTL result

Is it the full story?

No! two-loop diagrams equally important

Π_γ	LO	e^2T^2 [1]
	1-loop soft	e^5T^2
	2-loop hard	e^4T^2
	1-loop hard correction	e^4T^2

Mirza & Carrington, PRD 87

….so not the full story

(but still one necessary piece!)

All in all a nasty computation…

… can we figure out a way to make our lives easier ?

OSEFT

The hint:

For many quantities in thermal field theory the relevant degrees of freedom are on-shell (quasi)particles

OSEFT

The idea:

An EFT to describe physical phenomena dominated by (almost) on-shell degrees of freedom

-> On-shell effective field theory

OSEFT

The idea:

An EFT to describe physical phenomena dominated by (almost) on-shell degrees of freedom

For massless fermions: $Q^2 \approx 0$

Particles: $$q^\mu = p v^\mu + k^\mu$$

Antiparticles: $$q^\mu = -p \tilde{v}^\mu + k^\mu$$

$$v^\mu = (1, \vec{v})$$
$$\tilde{v}^\mu = (1, -\vec{v})$$
$$v^2 = \tilde{v}^2 = 0$$
$$k << p$$

Manuel & Torres-Rincon, PRD90 (2014)

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Manuel & Torres-Rincon, PRD90 (2014)

OSEFT Lagrangian

1) Split fermion field

(almost) on-shell antiparticle

$$\psi_{\mathbf{v}} = e^{-ipv\cdot x}\left(P_v\chi_v(x) + P_{\tilde{v}}H^{(1)}_{\tilde{v}}(x)\right) + e^{ip\tilde{v}\cdot x}\left(P_{\tilde{v}}\xi_{\tilde{v}}(x) + P_v H^{(2)}_v(x)\right)$$

(almost) on-shell particle

$$P_v = \frac{1}{2}\gamma\cdot v\,\gamma_0\,,$$

$$P_{\tilde{v}} = \frac{1}{2}\gamma\cdot\tilde{v}\,\gamma_0$$

OSEFT Lagrangian

1) Split fermion field

$$\psi_{\mathbf{v}} = e^{-ipv\cdot x}\left(P_v\chi_v(x) + P_{\tilde{v}}H^{(1)}_{\tilde{v}}(x)\right) + e^{ip\tilde{v}\cdot x}\left(P_{\tilde{v}}\xi_{\tilde{v}}(x) + P_v H^{(2)}_v(x)\right)$$

2) Integrate out the H fields

OSEFT and other EFT

$$\mathcal{L}_{p,\mathbf{v}} = \chi_v^\dagger(x)\left(i\,v\cdot D + i\not{D}_\perp \frac{1}{2p + i\tilde{v}\cdot D}\, i\not{D}_\perp\right)\chi_v(x)$$
$$+\xi_{\tilde{v}}^\dagger(x)\left(i\,\tilde{v}\cdot D + i\not{D}_\perp \frac{1}{-2p + iv\cdot D}\, i\not{D}_\perp\right)\xi_{\tilde{v}}(x)$$

OSEFT and other EFT

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Compare with High-density effective field theory (HDET)..

$$\mathcal{L}_{\mathrm{HDET}} = \psi^\dagger(x)\left(i\,v\cdot D + i\not{D}_\perp \frac{1}{2\mu + i\tilde{v}\cdot D}\, i\not{D}_\perp\right)\psi(x)$$

..or heavy-quark effective field theory (HQET)..

$$\mathcal{L}_v = \bar{Q}_v\left\{iv\cdot D + i\not{D}_\perp \frac{1}{2m + iv\cdot D} i\not{D}_\perp\right\}Q_v$$

OSEFT and other EFT

$$\mathcal{L}_{p,\mathbf{v}} = \chi_v^\dagger(x)\left(i\,v\cdot D + i\not{D}_\perp \frac{1}{+2p + i\tilde{v}\cdot D}\, i\not{D}_\perp\right)\chi_v(x)$$
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Order by order..

In practice: expand our Lagrangian in powers of 1/p …

$$\mathcal{L}_{p,\mathbf{v}} = \chi_v^\dagger(x)\left(i\,v\cdot D + i\not{D}_\perp \frac{1}{2p + i\tilde{v}\cdot D}\, i\not{D}_\perp\right)\chi_v(x)$$

$$= \sum_n \mathcal{L}_{p,v}^{(n)}$$

Order by order..

In practice: expand our Lagrangian in powers of 1/p …

$$\mathcal{L}_{p,v}^{(0)} = \chi_v^\dagger \, (i \, v \cdot D \,) \, \chi_v$$

Order by order..

In practice: expand our Lagrangian in powers of 1/p …

$$\mathcal{L}_{p,v}^{(0)} = \chi_v^\dagger \, (i\, v \cdot D\,)\, \chi_v$$

$$\mathcal{L}_{p,v}^{(1)} - \frac{1}{2p} \chi_v^\dagger \left(D_\perp^2 - \frac{e}{2} \sigma_\perp^{\mu\nu} F_{\mu\nu} \right) \chi_v$$

Order by order..

In practice: expand our Lagrangian in powers of 1/p …

$$\mathcal{L}^{(0)}_{p,v} = \chi_v^\dagger \, (i\, v \cdot D\,)\, \chi_v$$

$$\mathcal{L}^{(1)}_{p,v} - \frac{1}{2p}\chi_v^\dagger \left(D_\perp^2 - \frac{e}{2}\sigma_\perp^{\mu\nu} F_{\mu\nu} \right) \chi_v$$

$$\mathcal{L}^{(2)}_{p,v} = \frac{1}{8p^2}\chi_v'^\dagger \Big([\not{D}_\perp\,,\, [i\tilde{v}\cdot D\,,\, \not{D}_\perp]] - \big\{ (\not{D}_\perp)^2,\, (iv\cdot D - i\tilde{v}\cdot D) \big\} \Big) \chi_v'$$

ugliness intensifies..

Propagators and vertices

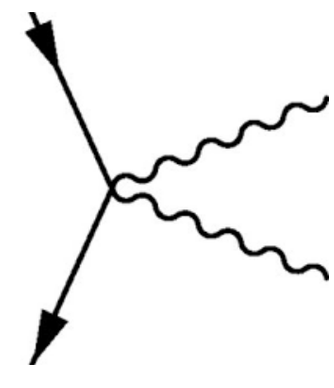
Real-time (Keldysh) propagators in OSEFT:

$$S^{R/A}(k) = \frac{P_v \gamma_0}{k_0 \pm i\epsilon - f(\mathbf{k})},$$

$$S^S(k) = P_v \gamma_0 \left(-2\pi i \delta(k_0 - f(\mathbf{k})) \left(1 - 2n_f(p + k_0)\right)\right)$$

dispersion relation (order (2)): $f^{(2)}(\mathbf{k}) = k_\parallel + \frac{\mathbf{k}_\perp^2}{2p} - \frac{k_\parallel \mathbf{k}_\perp^2}{2p^2}$

Additional vertex from O(1/p):



Applications

- Derivation of power corrections to Hard Thermal Loops

-> C. Manuel, J. Soto and S. Stetina, Phys.Rev. D94 (2016)

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- Kinetic theory

Chiral kinetic theory

- Kinetic theory for massless fermions
  describing evolution of quasi-particle distribution functions

- Chiral anomaly has macroscopic effects on transport
  (chiral magnetic effect, chiral vortical effect…)

- What is the correct kinetic equation
  to describe these effects ?

Chiral kinetic theory and OSEFT

Transport equations describe propagation of on-shell quasiparticles

OSEFT works with on-shell degrees of freedom

Chiral kinetic theory and OSEFT

Transport equations describe propagation of on-shell quasiparticles

OSEFT works with on-shell degrees of freedom

Anomaly is a quantum effect: can be formulated in terms of Berry curvature/phase…

OSEFT provides a systematic way to include quantum corrections!

Chiral kinetic theory and OSEFT

Transport equations describe propagation of on-shell quasiparticles → OSEFT works with on-shell degrees of freedom

Anomaly is a quantum effect: can be formulated in terms of Berry curvature/phase… → OSEFT provides a systematic way to include quantum corrections!

OSEFT seems to be the perfect tool to derive the equations of chiral kinetic theory!

The recipe

- Build order-by-order equations of motions for the two-point function starting from the OSEFT Lagrangian formulated in a general covariant frame

- Perform a Wigner transform, add link operators to make results gauge-invariant

- Make sure you've had enough coffee because equations are lengthy and it's easy to miss factors of 2

- Go back from the EFT variables to full momenta

Chiral kinetic theory

Covariant form of the transport equation

$$\left(v^q_\mu - \frac{e}{2E_q^2} S^{\mu\nu}_\chi F_{\nu\rho}\left(2u^\rho - v^\rho_q\right)\right)\Delta_\mu f(X,q)\delta_+(Q) = 0$$

with $\quad v^q_\mu = \dfrac{q^\mu}{E_q} \qquad\qquad E_q = q\cdot u$

$$\Delta^\mu \equiv \partial^\mu_X - eF^{\mu\nu}(X)\partial_{q,\nu}$$

SC, Manuel & Torres-Rincon, PRD 98 (2018)

CKT and beyond

With OSEFT one can

Derive kinetic equations for a chiral plasma

Check Lorentz invariance (= reparametrization invariance) of the results

Derive the current and the anomalous current

Recover the chiral anomaly equation

…..

SC, Manuel & Torres-Rincon, PRD 98 (2018)

Chiral plasma instabilities

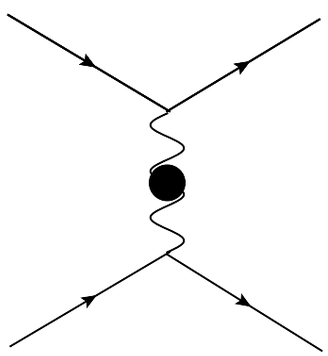
- A chirally imbalanced plasma could develop instabilities leading to the reduction of the chiral charge ~ n_5
- It has been argued that this could result in generation of axial magnetic fields in the early universe and in proto-neutron stars
- Can we see this instability by inspecting the lifetime of a fermion in such a system?

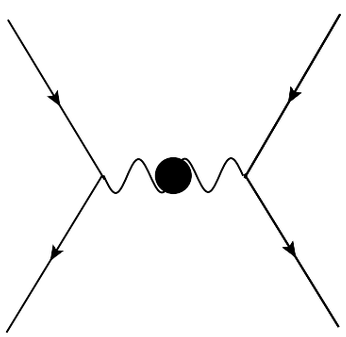
Fermion damping

Fermion propagating through a hot/dense plasma

For simplicity: consider QED

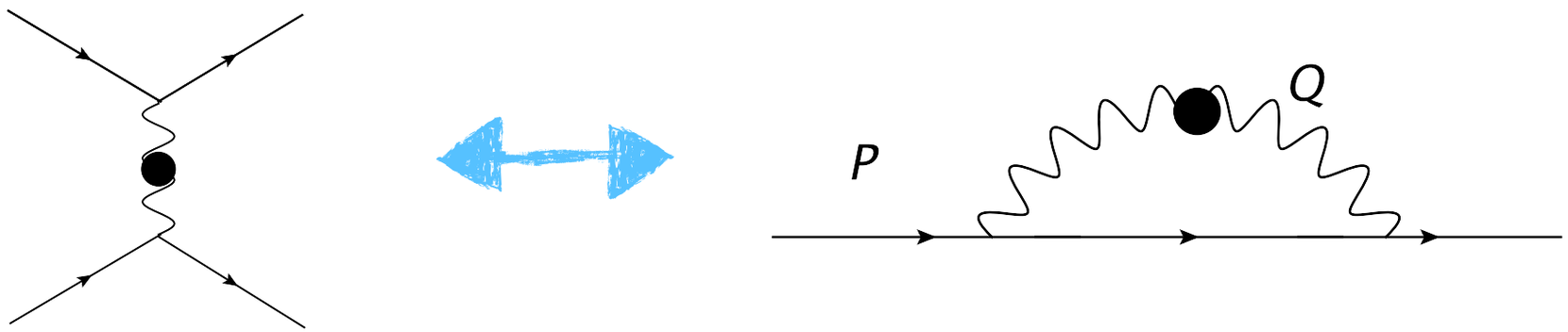
Interactions with the medium: decay (damping)



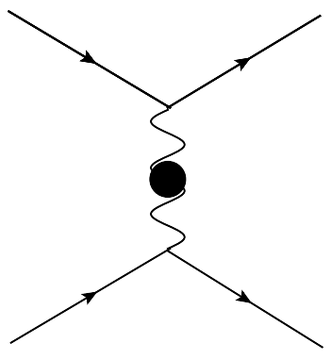


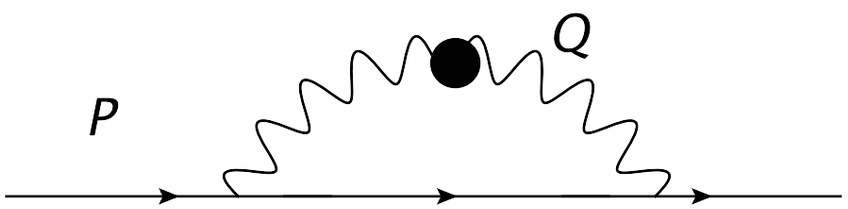
(only at finite temperature)

Fermion damping



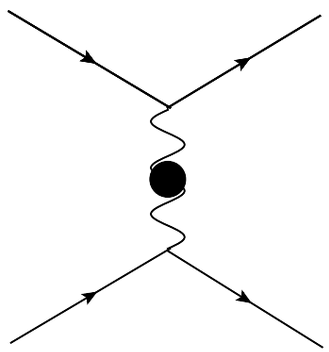
Fermion damping

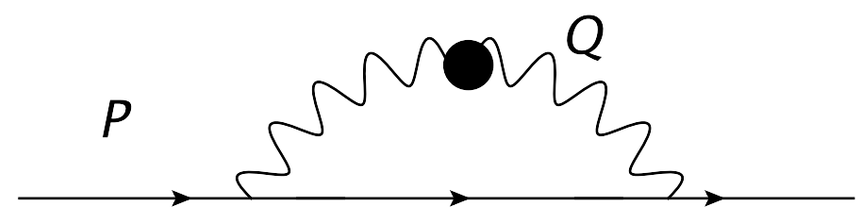




$$\gamma_\chi(E) = -\frac{1}{2E}\mathrm{Tr}\Big[\mathcal{P}_\chi \not{P}\,\mathrm{Im}\Sigma(p_0 + i\eta, \mathbf{p})\Big]\Big|_{p^0=E}$$

Fermion damping





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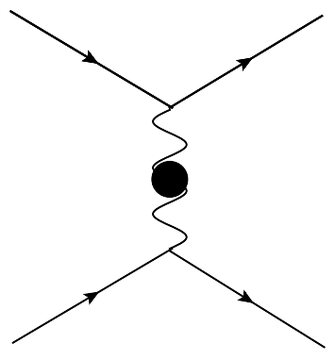
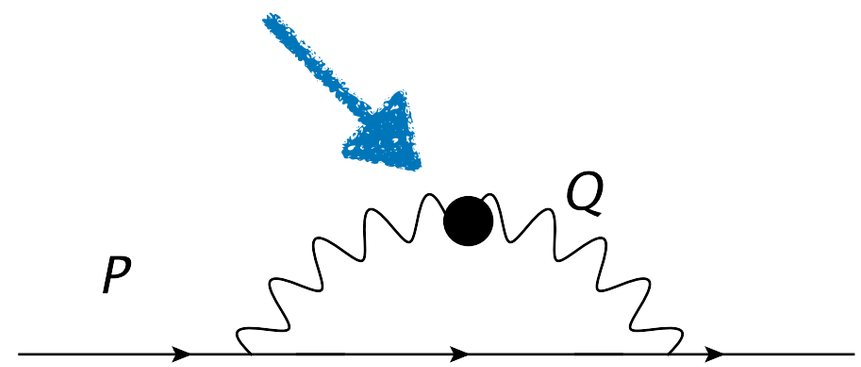
Chirality projector

Particle projector

$$\mathcal{P}_\chi = \frac{1}{2}\left(1 + \chi\gamma_5\right)$$

Fermion damping





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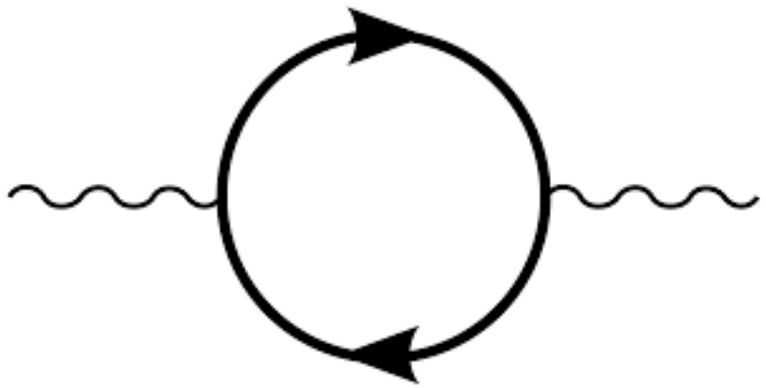
Leading contribution from soft photon momenta:
need HTL resummed propagator!

Photon self-energy with chiral imbalance



$$\mu_R = \mu_V + \mu_5 \neq \mu_L = \mu_V - \mu_5$$

Fermion loop with asymmetric contributions from different chiralities generates parity-violating contribution to the self-energy

-> Different propagation of transverse modes!

Photon self-energy with chiral imbalance

$$\Delta_{\mu\nu}(q_0, \mathbf{q}) = \delta_{\mu 0}\delta_{\nu 0}\Delta_L(q_0, q) + \sum_{h=\pm} \mathcal{P}^{T,h}_{\mu\nu}\Delta^h_T(q_0, q)$$

$$\mathcal{P}^{T,h}_{\mu\nu} = \frac{1}{2}\left(\delta^{ij} - \hat{q}^i\hat{q}^j - ih\epsilon^{ijk}\hat{q}^k\right)\delta_{\mu i}\delta_{\nu j}$$

$h = \pm$ circular polarized modes

$$\Delta_L = \frac{1}{q^2 + \Pi_L}\,, \qquad \Delta^h_T = \frac{1}{q_0^2 - q^2 - \Pi_T - h\Pi_P}$$

anomalous HDL

$$\Pi_P(q_0, q) = -\frac{e^2\mu_5}{2\pi^2}\frac{q_0^2 - q^2}{q}\left[1 - \frac{q_0}{2q}\ln\left(\frac{q_0 + q}{q_0 - q}\right)\right]$$

Fermion damping in chiral imbalanced system

$$\gamma_\chi(E) = -\frac{e^2}{2E}\,\mathrm{Im}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\int_{-\infty}^{\infty}\frac{\mathrm{d}k_0}{2\pi}\rho_f(k_0,k)$$
$$\times\frac{1+f(q_0)-\tilde{f}(k_0-\mu_\chi)}{p_0-k_0-q_0+i\eta}\mathrm{Tr}\left(\mathcal{P}_\chi\not{p}\gamma^\mu\not{k}\gamma^\nu\right)$$
$$\times\left[\delta_{\mu0}\delta_{\nu0}\rho_L(q_0,\mathbf{q})+\sum_{h=\pm}\mathcal{P}^{T,h}_{\mu\nu}\rho^h_T(q_0,\mathbf{q})\right]$$

$$\mathcal{P}^{T,h}_{\mu\nu} = \frac{1}{2}\left(\delta^{ij}-\hat{q}^i\hat{q}^j-ih\epsilon^{ijk}\hat{q}^k\right)\delta_{\mu i}\delta_{\nu j}$$

Fermion damping in chiral imbalanced system

After some massaging and expanding…

$$\gamma_\chi^{\rm soft} \simeq \frac{e^2}{2}\int \frac{d^3q}{(2\pi)^3}\Big(\Theta(q_0)-\Theta(\mu_\chi - E + q_0)\Big)\Theta(q^* - q)$$
$$\times \left\{\rho_L(Q)\Big(1-\frac{q_0}{E}\Big)+\frac{1}{2}(1-\cos^2\theta)\right.$$
$$\left.\times \sum_{h=\pm}\Big[\Big(1-\chi h\frac{q}{E}\Big)\rho_T^h(Q)\Big]\right\}\Bigg|_{q_0=q\cos\theta}$$

SC and C.Manuel, arXiv:1811.06394

Fermion damping in chiral imbalanced system

Leading chemical potential dependence ($\mu_\chi = \mu_V + \chi\,\mu_5$)

$$\gamma_\chi^{\rm soft} \simeq \frac{e^2}{2}\int \frac{d^3q}{(2\pi)^3}\Big(\Theta(q_0) - \Theta(\mu_\chi - E + q_0)\Big)\Theta(q^* - q)$$
$$\times \left\{\rho_L(Q)\Big(1 - \frac{q_0}{E}\Big) + \frac{1}{2}(1 - \cos^2\theta)\right.$$
$$\left.\times \sum_{h=\pm}\Big[\Big(1 - \chi h\frac{q}{E}\Big)\rho_T^h(Q)\Big]\right\}\Bigg|_{q_0 = q\cos\theta}$$

SC and C.Manuel, arXiv:1811.06394

Fermion damping in chiral imbalanced system

At Fermi surface damping is zero!

$$\gamma_\chi^{\rm soft} \simeq \frac{e^2}{2}\int \frac{d^3q}{(2\pi)^3}\Big(\Theta(q_0)-\Theta(\mu_\chi - E + q_0)\Big)\Theta(q^*-q)$$
$$\times\left\{\rho_L(Q)\left(1-\frac{q_0}{E}\right)+\frac{1}{2}(1-\cos^2\theta)\right.$$
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Fermions couple differently with transverse photons depending on their chirality

Fermion damping

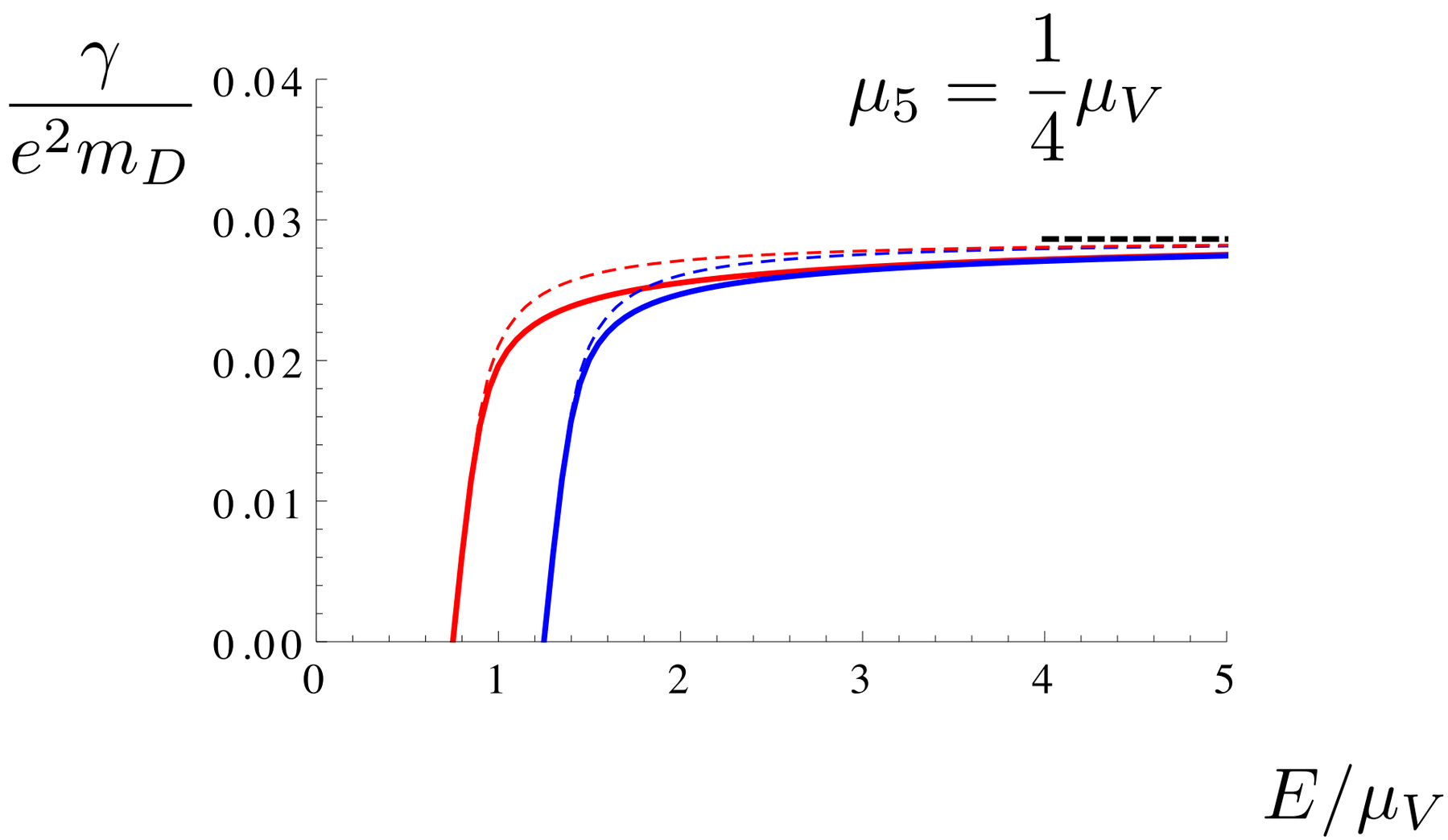
Close to the Fermi surface chirality-dependent effects turn out to be subleading

Same behavior as symmetric case

$$\gamma_\chi(E) = \frac{e^2}{24\pi}(E - \mu_\chi) + \dots$$

Away from the Fermi surface: full numerical result

Fermion damping



SC and C.Manuel, arXiv:1811.06394

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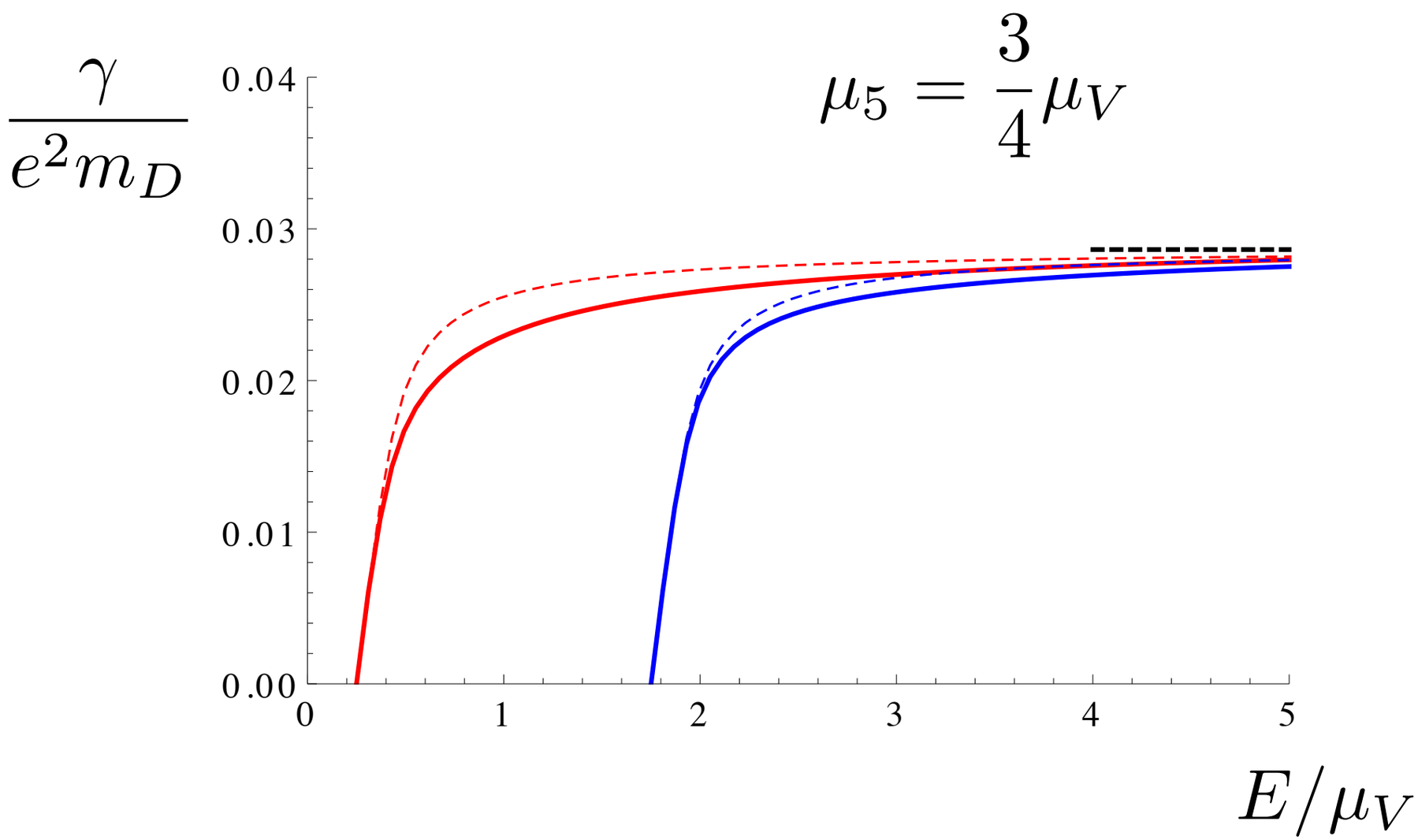

SC and C.Manuel, arXiv:1811.06394

Take-home message #3

- Chiral plasmas provide a nice playground for new physical effects with several real-world applications

- OSEFT is a nice tool to simplify our lives (or at least give us some guidance) in thermal field theory computations

- Still a lot of work to be done: maybe some (more or less accepted) ideas need rethinking

Inhomogeneous chiral condensates in NJL

- Allow for a spatially modulated chiral condensate

$$\langle\bar{\psi}\psi\rangle = S(\mathbf{x}) \qquad \langle\bar{\psi}i\gamma^5\tau_a\psi\rangle = P_a(\mathbf{x})$$

(we can also build $M(\mathbf{x}) = -2G(S(\mathbf{x}) + iP_3(\mathbf{x}))$

- Diagonalize the mean-field quark Hamiltonian in momentum space

$$\mathcal{H}_{\vec{p}_m,\vec{p}_n} = \begin{pmatrix} -\vec{\sigma}\cdot\vec{p}_m\,\delta_{\vec{p}_m,\vec{p}_n} & \sum_{\vec{q}_k} M_{\vec{q}_k}\delta_{\vec{p}_m,\vec{p}_n+\vec{q}_k} \\ \sum_{\vec{q}_k} M^*_{\vec{q}_k}\delta_{\vec{p}_m,\vec{p}_n-\vec{q}_k} & \vec{\sigma}\cdot\vec{p}_m\,\delta_{\vec{p}_m,\vec{p}_n} \end{pmatrix}$$

Inhomogeneous chiral condensates in NJL

- Then, minimize the thermodynamic potential

$$\Omega(T,\mu;M(\vec{x})) = -\frac{T}{V}\mathrm{Log}\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(\int_{x\in[0,\frac{1}{T}]\times V}(\mathcal{L}_{MF}+\mu\bar{\psi}\gamma^0\psi)\right)$$
$$= -\frac{TN_c}{V}\sum_n \mathrm{Tr}_{D,f,V}\,\mathrm{Log}\left(\frac{1}{T}\left(i\omega_n+\mathcal{H}_{MF}-\mu\right)\right)+\frac{1}{V}\int_V \frac{|M(\vec{x})-m|^2}{4G_s}$$

with respect to the mass function M(x)

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with respect to the mass function M(x)

- Not so easy for an arbitrary M(x) !

- To make the problem tractable, assume specific ansatz for the functional form of M, minimize thermodynamic potential, compare free energies for different modulations…

Ginzburg-Landau analysis

- Tackling arbitrary modulations is numerically intensive
- Can we avoid having to diagonalize the quark hamiltonian?

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➡ Ginzburg-Landau expansion of the free energy:

  - Systematic expansion in terms of order parameter and its gradients
  - For inhomogeneous phases, expected to be valid where both amplitudes and gradients are small

Ginzburg-Landau analysis

$$\Omega_{\rm GL} = \Omega[0] + \frac{1}{V}\int d\mathbf{x}\left[\alpha_2 M^2 + \alpha_4\left(M^4 + (\nabla M)^2\right) + \alpha_6\left(M^6 + 3(\nabla M)^2 M^2 + \frac{1}{2}(\nabla M^2)^2 + \frac{1}{2}(\nabla^2 M)^2\right)\right.$$
$$\left. + \alpha_8\left(M^8 + 14M^4(\nabla M)^2 - \frac{1}{5}(\nabla M)^4 + \frac{18}{5}M(\nabla^2 M)(\nabla M)^2 + \frac{14}{5}M^2(\nabla^2 M)^2 + \frac{1}{5}(\nabla^3 M)^2\right) + \ldots\right]$$

D.Nickel, Phys.Rev.Lett.103 (2009)
H.Abuki, D.Ishibashi, K.Suzuki, Phys.Rev.D85 (2012)

Ginzburg-Landau analysis

$$\Omega_{\mathrm{GL}} = \Omega[0] + \frac{1}{V}\int d\mathbf{x}\left[\alpha_2 M^2 + \alpha_4\left(M^4 + (\nabla M)^2\right) + \alpha_6\left(M^6 + 3(\nabla M)^2 M^2 + \frac{1}{2}(\nabla M^2)^2 + \frac{1}{2}(\nabla^2 M)^2\right)\right.$$
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Restored +

Ginzburg-Landau analysis

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Restored + "homogeneous" +

Ginzburg-Landau analysis

$$\Omega_{\mathrm{G}} = \Omega[0] + \frac{1}{V}\int d\mathbf{x}\Big[\alpha_2 M^2 + \alpha_4\left(M^4 + (\nabla M)^2\right) + \alpha_6\left(M^6 + 3(\nabla M)^2 M^2 + \frac{1}{2}(\nabla M^2)^2 + \frac{1}{2}(\nabla^2 M)^2\right)$$
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Restored + "homogeneous" + gradient terms

- In principle straightforward: for each order add all possible independent terms (considering gradients are of the same order as M)

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- GL coefficients $\alpha_n(T,\mu)$
  are independent from the shape of the modulation

  -> can be computed relatively easily
  in a chirally restored background!

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- GL coefficients $\alpha_n(T,\mu)$
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- But: calculating the relative prefactors between terms of the same order is an extremely tedious task..
  There are tricks, however.

SC, M. Mannarelli, F. Anzuini, O.Benhar, Phys.Rev.D97 (2018)