EFT Bootstrap and the pion S-matrix

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Based on arXiv: 2011.02802 + work in progress
with João Penedones and Pedro Vieira

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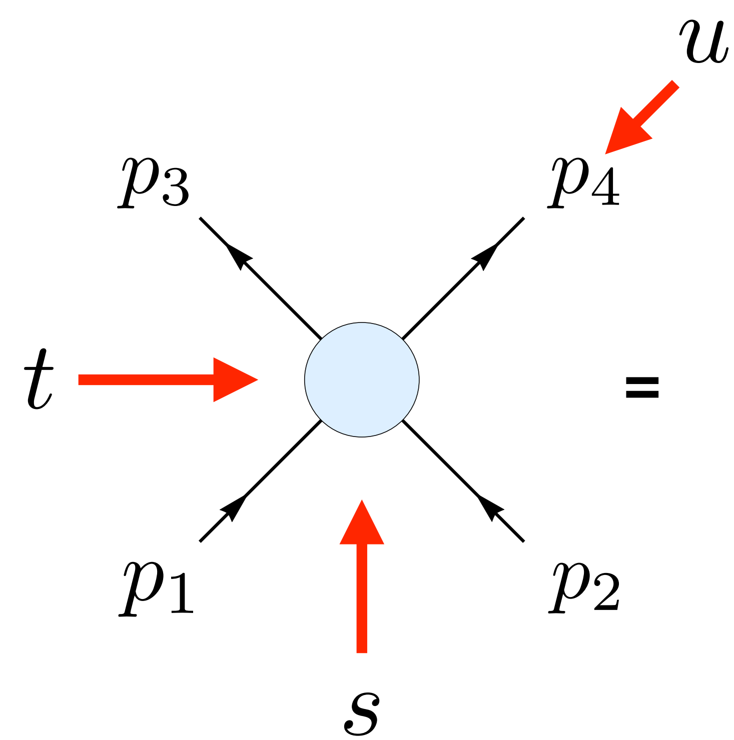
The space of scattering amplitudes

Identical particles in d+1 dim, and no other stable particles

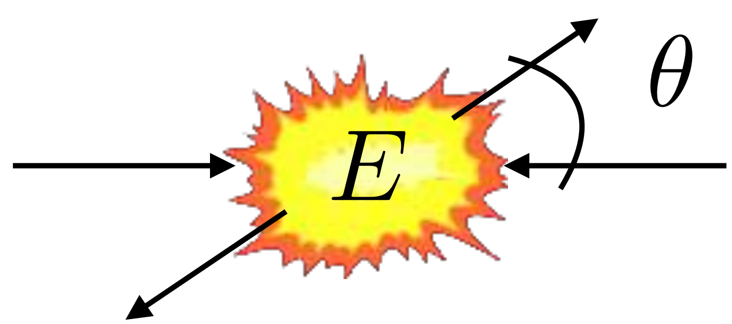
3 possible identical processes



$$= \mathcal{M}(s, t, u)$$

Crossing Symmetry

2 independent variables



$$s = 4E^2$$

$$t = -4E^2 \sin^2 \frac{\theta}{2}$$

$$u = 4m^2 - s - t$$

Unitarity

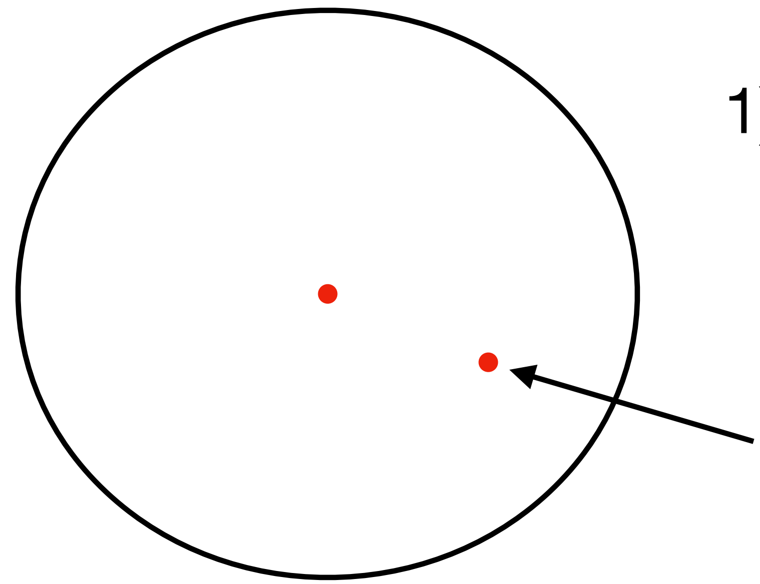
$$\sum_n |\mathrm{Prob}_{2\to n}(s, \ell)|^2 = 1$$

Crossing and Unitarity are not enough to constraint the space of amplitudes

Example:

$$f(z), \quad \text{st} \quad |f(e^{i\phi})|^2 \leq 1$$

$$\max f(0) = ?$$



1) if f holomorphic max f(0) = 1 (Max mod principle)

2) if f has singularities inside, f(0) is unbounded

Analyticity

We are going to assume Maximal Analyticity:
Amplitude analytic in the s, t planes except for singularities determined by on-shell processes

Bound state -> pole

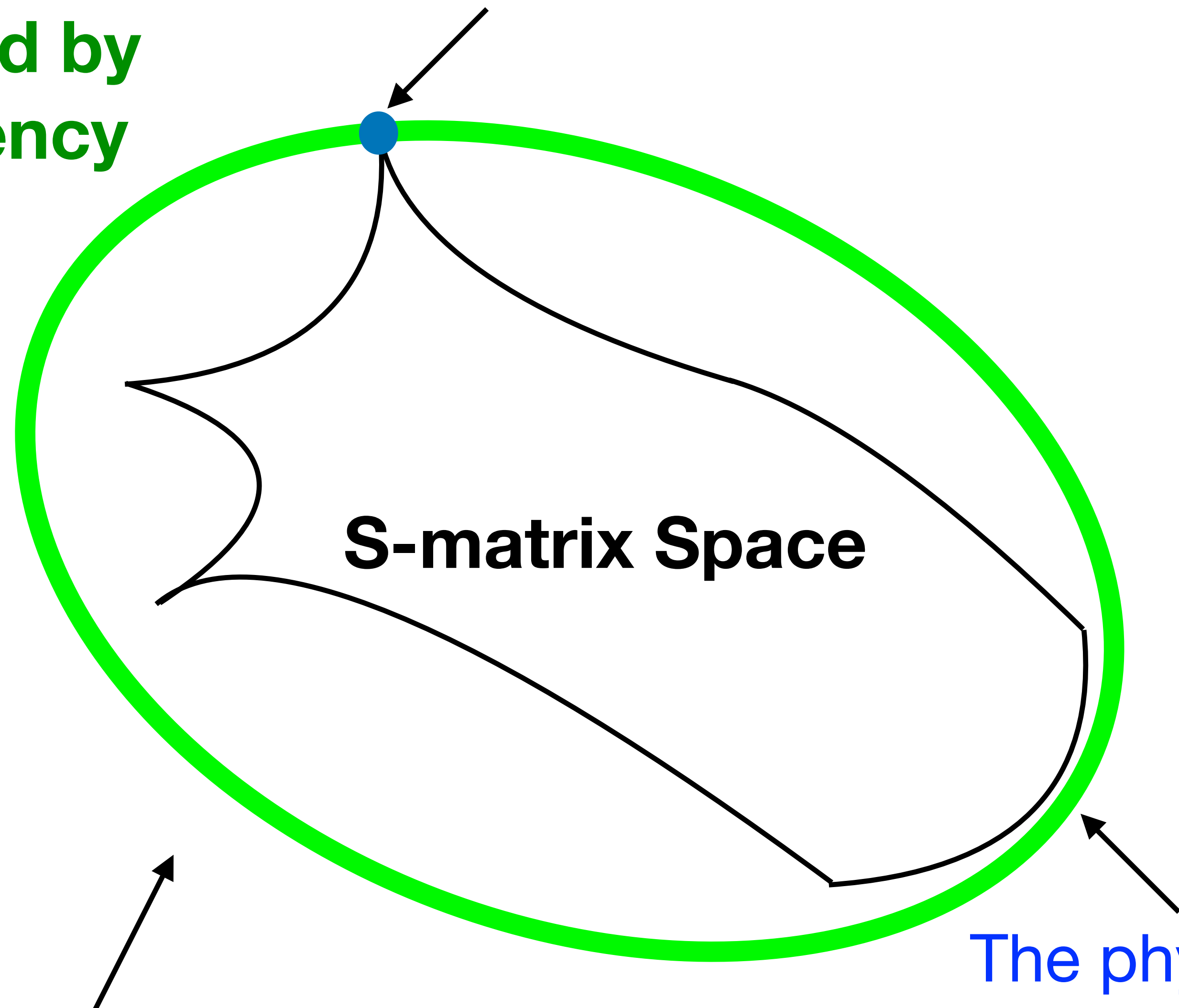
Multi-particle states -> branch point singularities

Full Unitarity: $|\mathrm{Prob}_{2\to 2}(s,\ell)|^2 + \cancel{\mathrm{positive}} = 1$

$$|\mathrm{Prob}_{2\to 2}(s,\ell)|^2 \leq 1$$

Integrable Theory (in 1+1 dim only)

Space Allowed by 2->2 consistency



The physical theory can be very close to the boundary

Space of S-matrix consistent with 2->2 scattering

Available tools

Primal == physicist approach

Challenging rigorous bounds,
but explicit amplitudes

More direct and generalizable to any dimension

Paulos, Penedones, Toledo, van Rees, Vieira ’17

Dual == mathematician approach

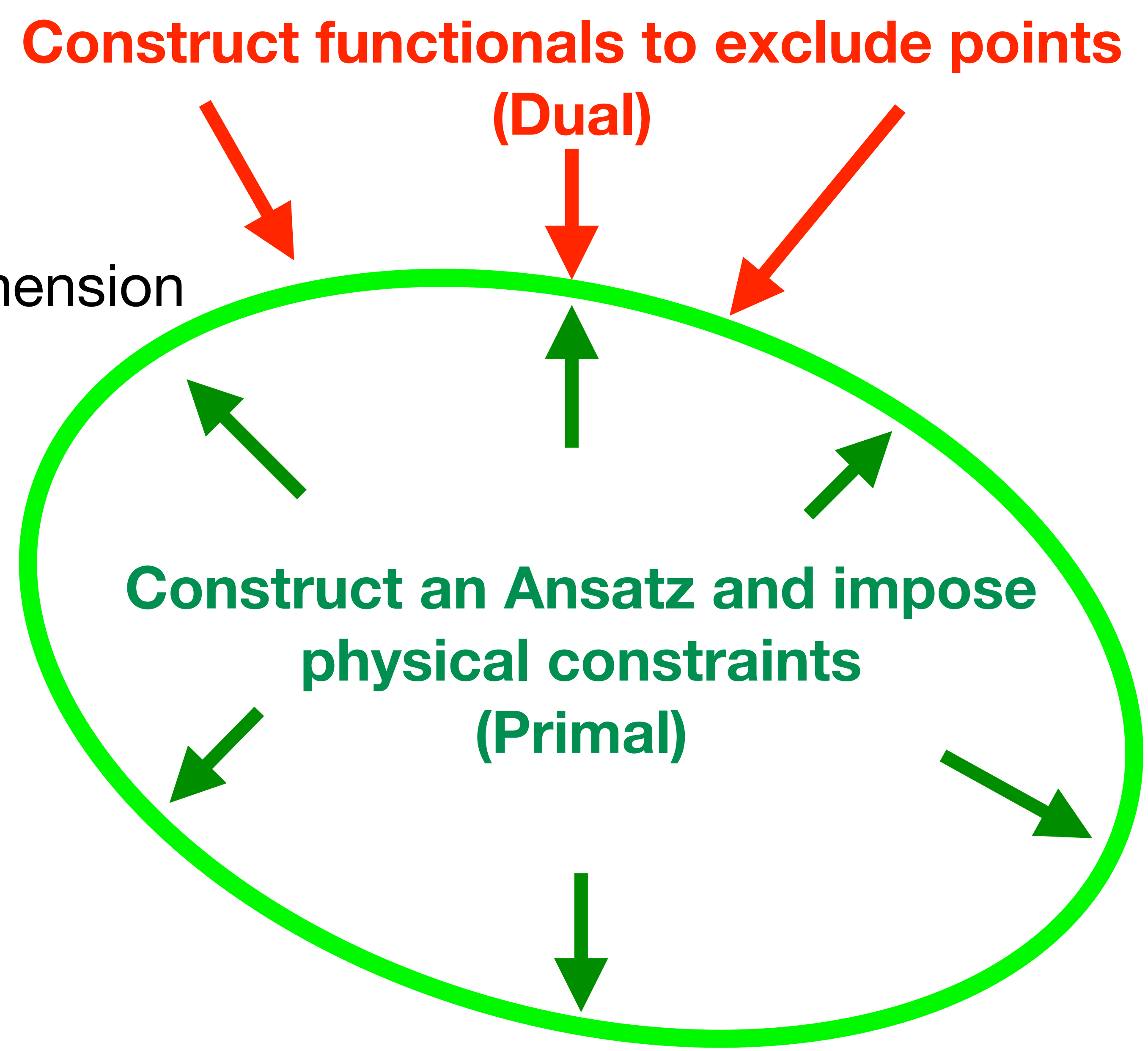
Rigorous bounds always

Well established in 1+1 D

Martin, Lukasczuk, Lopez,
Mennessier, Bonnier,…60s, 70s

Cordova, He, Kruczensky, Vieira ‘19

ALG, Hömrich, Vieira ‘20



Plan of the Talk

1) Low energy EFT for pions: what to bootstrap?

2) Primal S-matrix Bootstrap in d=4 for massless particles

3) Numerical bounds and phase shifts

4) A work in progress and future challenges

Our Target: QCD-like theories

$SU(N_c)$ **Gauge Theory**

$SU(2)_L \times SU(2)_R$ **Fermionic matter**

If quarks are massless

Spontaneous symmetry breaking

$$\langle \bar{\psi}\psi \rangle \neq 0 \qquad \text{Pions} \in \frac{SU(2)_L \times SU(2)_R}{SU(2)_V}$$

Theory of Goldstones with SO(3) symmetry

Exact soft theorems (on-shell)

If we add a a small mass term

$$m\bar{\psi}\psi$$

$$\frac{m}{\Lambda_{QCD}} \sim \frac{p}{\Lambda_{QCD}} << 1$$

Theory of quasi-Goldstones SO(3) symmetric

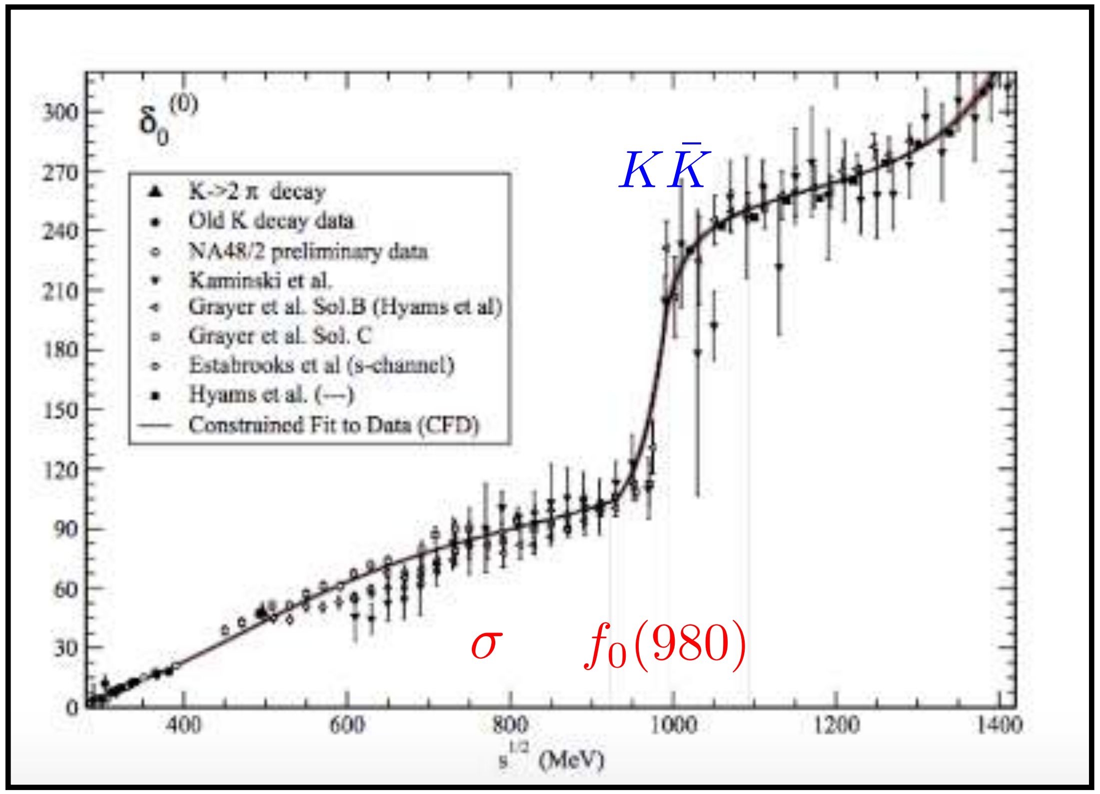
Adler zero condition (off-shell)

$$(m_u = m_d)$$

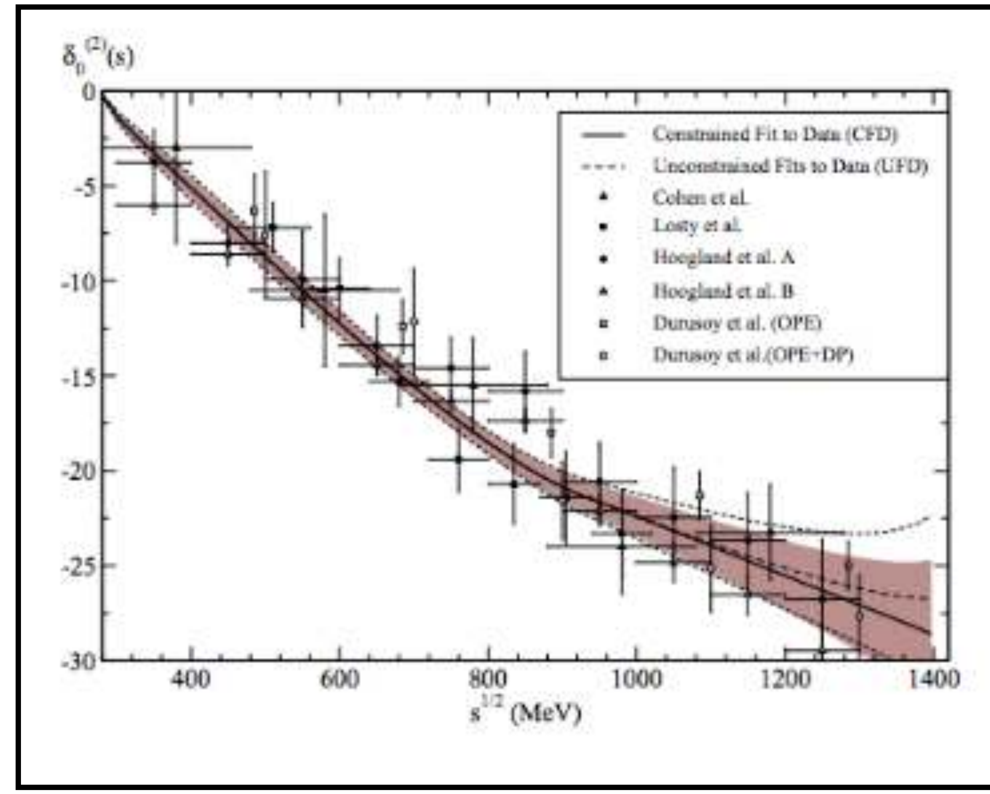
ALG, Penedones, Vieira '18

Some data

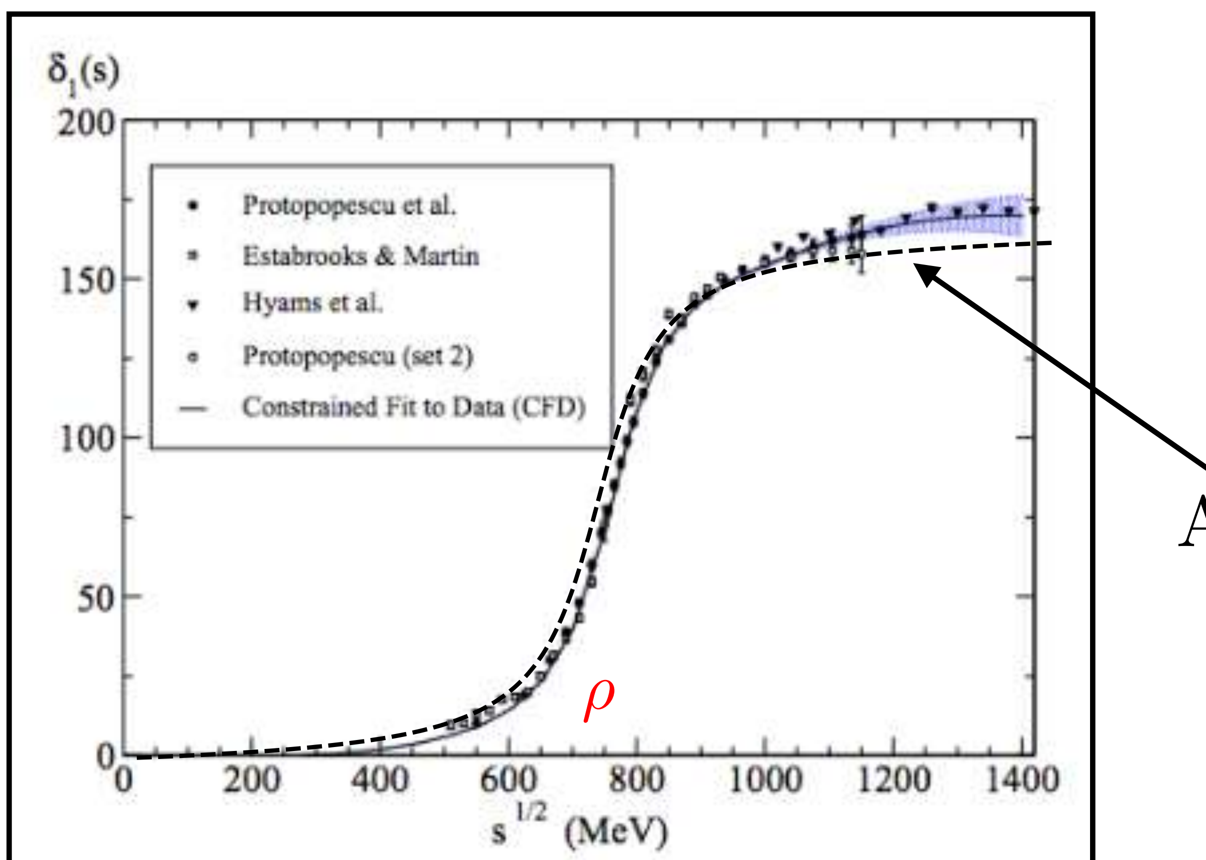
$\ell = 0,\ I = 0$



$\ell = 0,\ I = 2$



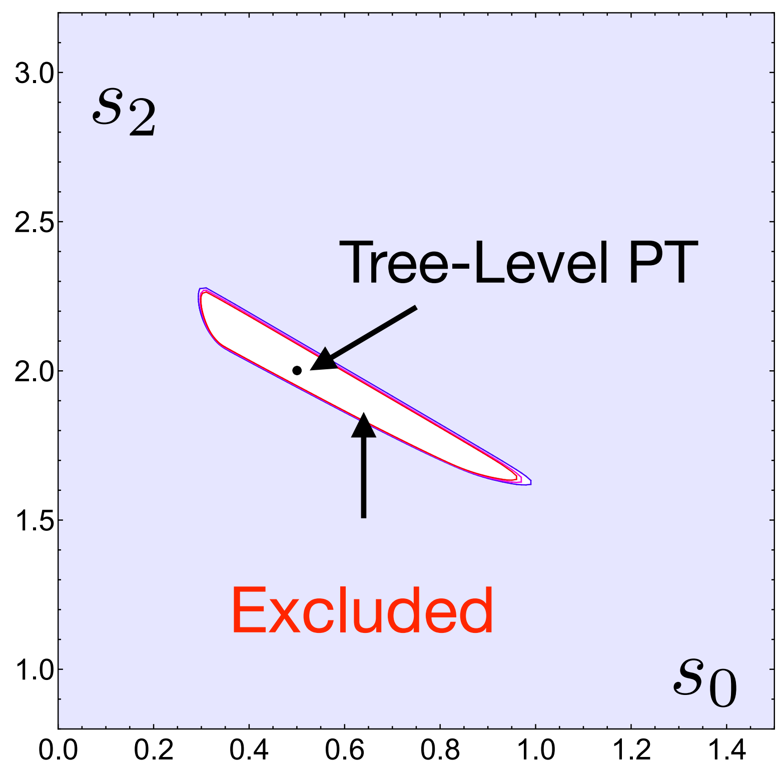
$\ell = 1,\ I = 1$



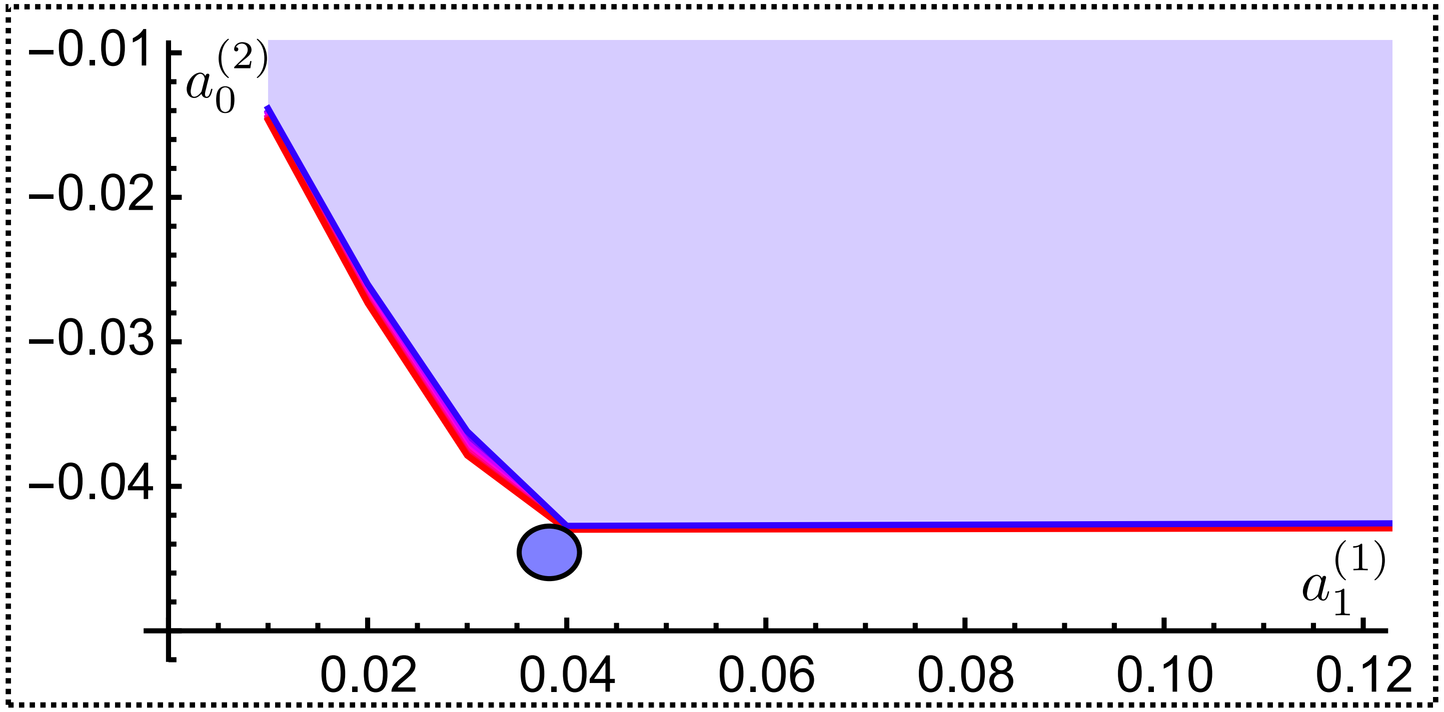
ArcTan

QCD low energy parameter space shows surprising structures

Pion Lake



Pion kink



ALG, Penedones, Vieira '18

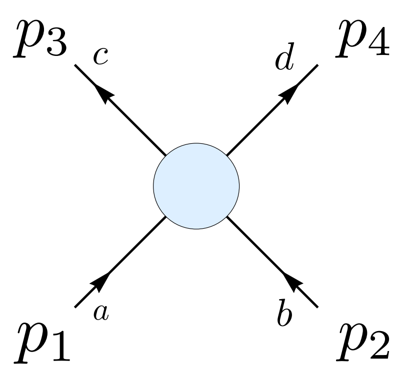
For any couple of zeros we fix we get a "kink"

The massive theory has a high dimensional space of parameters!

In the massless case we can focus only on the low energy constants!

$$m = 0 \implies s_0 = s_2 = 0$$

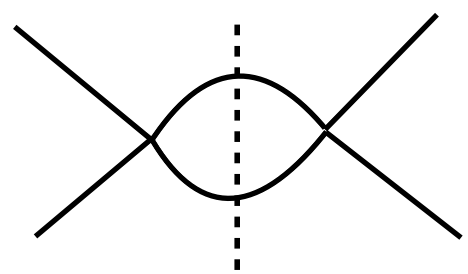
EFT expectations



$$= \quad \frac{s}{f^2}\delta_{ab}\delta^{cd} + \frac{t}{f^2}\delta_a^c\delta_b^d + \frac{u}{f^2}\delta_a^d\delta_b^c$$

With O(N) symmetry we have a linear tree-level interaction

$$A(s|t,u) = \frac{s}{f^2} + \mathcal{O}(s^2)$$



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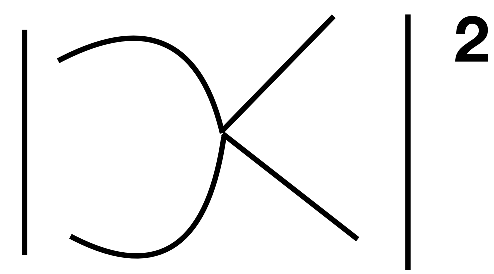


$$A(s|t,u)^{1-\mathrm{loop}} = \alpha\frac{s^2}{f^4} + \beta\frac{t^2+u^2}{f^4} - \frac{N_f-2}{32\pi^2}\frac{s^2}{f^4}\log\frac{-s}{f^2} - \frac{t-u}{96\pi^2 f^4}\left(t\log\frac{-t}{f^2} - u\log\frac{-u}{f^2}\right)$$

Theory dependent

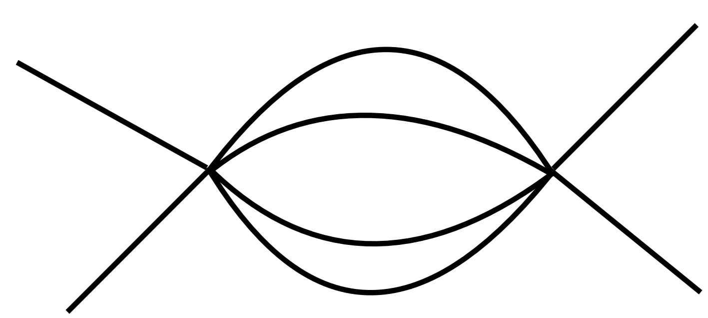
Completely fixed by elastic unitarity saturation

Theory dependent

$$A(s|t,u)^{\text{2-loops}} = \gamma s^3 + \delta(t^3+u^3) + \frac{N(9N-20)+19}{9216\,\pi^4} s^3 \log^2(-s)$$
$$-\frac{1}{18432\pi^4}((3N+11)(t^3\log^2(-t)+u^3\log^2(-u)) - 6(N-3)tu(t\log^2(-t)+u\log^2(-u)))$$
$$-\left((3N-1)\frac{\alpha}{48\pi^2} + (N+3)\frac{\beta}{24\pi^2} + \frac{11N-10}{27648\,\pi^4}\right) s^3\log(-s)$$
$$+\frac{\alpha}{96\pi^2}(t^2(t-2u)\log(-t) + u^2(u-2t)\log(-u))$$
$$+\frac{\beta}{96\pi^2}(t^2(2u+9t)\log(-t) + u^2(2t+9u)\log(-u))$$
$$+\frac{1}{110592\pi^4}((21N-17)(t^3\log(-t)+u^3\log(-u)) - 2(3N-5)tu(t\log(-t)+u\log(-u)))$$

Almost all fixed!!

How far can we go?



= O(s^3)

$$A(s|t,u)^{1-\text{loop}} = \alpha\frac{s^2}{f^4} + \beta\frac{t^2+u^2}{f^4} - \frac{N_f-2}{32\pi^2}\frac{s^2}{f^4}\log\frac{-s}{f^2} - \frac{t-u}{96\pi^2 f^4}\left(t\log\frac{-t}{f^2} - u\log\frac{-u}{f^2}\right)$$

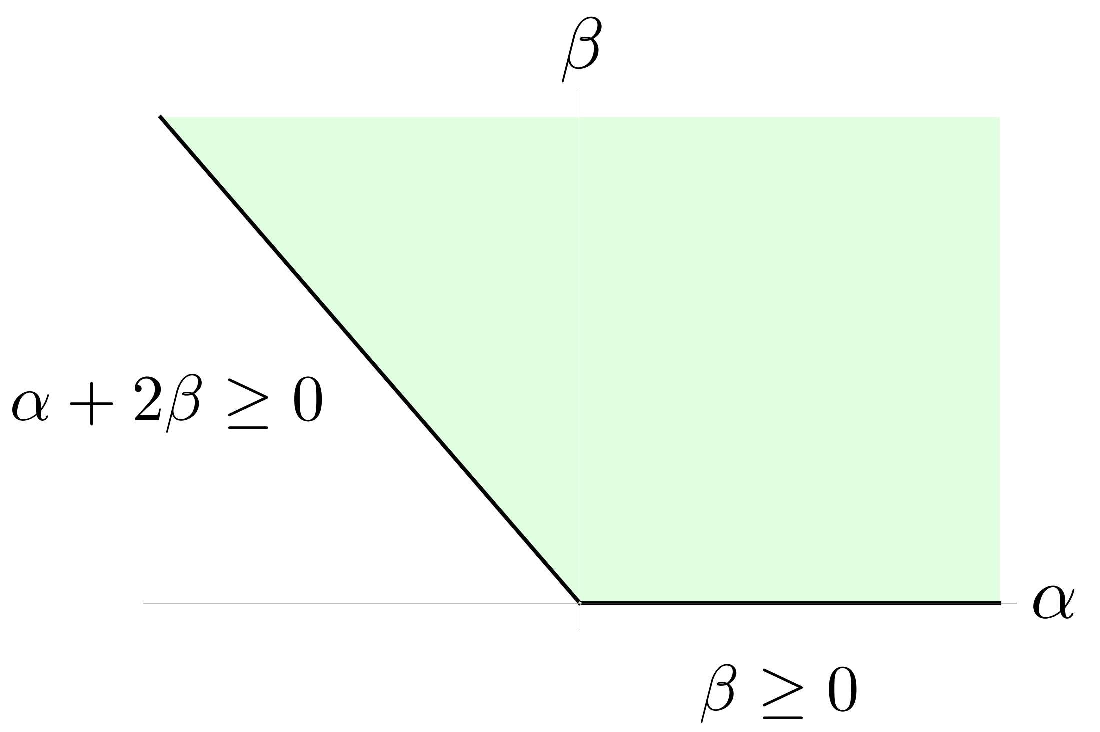
Once we have fixed the scale these are the first free parameters

What do we know about them?

Positivity Bounds

$$\text{Im}M(s,\theta=0) > 0$$

Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi ‘06



Non-perturbative crossing, analyticity, unitarity and UV completion?

Primal S-matrix Bootstrap formulation

We want to write an **Ansatz** analytic and crossing symmetric

Crossing symmetry

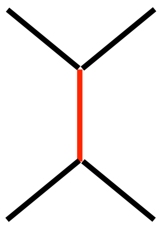
$$\mathcal{M}_{ab}^{cd}(s,t,u) = A(s|t,u)\delta_{ab}\delta^{cd} + A(t|s,u)\delta_a^c\delta_b^d + A(u|s,t)\delta_a^d\delta_b^c$$

$$A(s|t,u) = A(s|u,t)$$

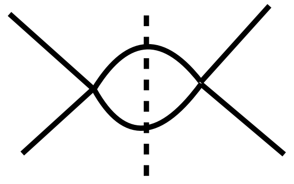
Analyticity

Holomorphic up to singularities determined by spectrum and unitarity

Only one stable particle -> no bound state poles!



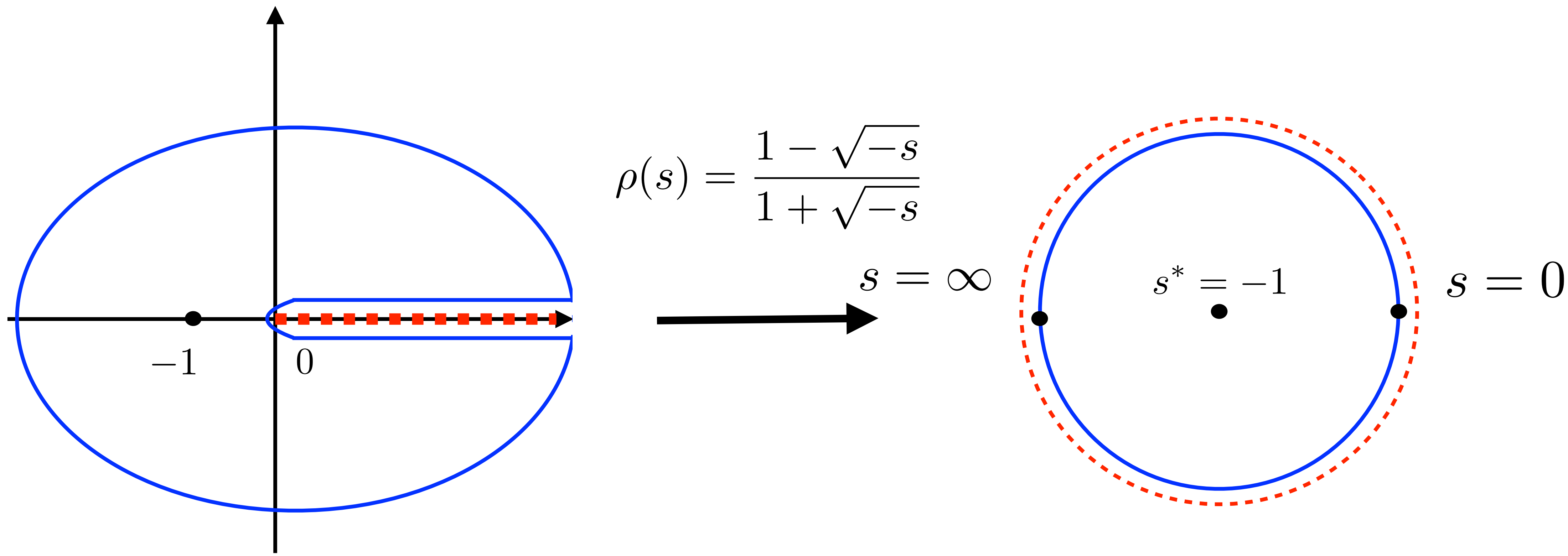
Elastic unitarity -> discontinuity for s, t, u > 0



Analytic Extension

We treat the 3 variables as independent and Taylor expand in a bigger space

$$\{s, t, u\} \in \mathbb{C}^3/\text{cuts}$$



We cannot expand on s+t+u=0, we expand in the enlarged space

$$A(s|t,u) = \texttt{low energy} + \sum{}' c_{ab}\rho(s)^a(\rho(t)^b + \rho(u)^b) + \sum{}' d_{ab}(\rho(t)^a\rho(u)^b + \rho(t)^b\rho(u)^a)$$

Analytic and crossing symmetric ansatz

Low Energy Behavior

$$A(s|t,u) = \texttt{low energy} + \sum' c_{ab}\rho(s)^a(\rho(t)^b + \rho(u)^b) + \sum' d_{ab}(\rho(t)^a\rho(u)^b + \rho(t)^b\rho(u)^a)$$

$$\texttt{low energy} \equiv -\frac{\chi(s)}{f_\pi^2} + \frac{1}{f_\pi^4}\left[\left(\alpha - 3 + \frac{\log f_\pi^2}{48\pi^2}\right)\chi(s)^2 + \left(\beta + \frac{\log f_\pi^2}{48\pi^2}\right)(\chi(t)^2 + \chi(u)^2) - \frac{3\chi(s)^2\log\chi(s) + (\chi(t)-\chi(u))(\chi(t)\log\chi(t) - \chi(u)\log\chi(u))}{96\pi^2}\right]$$

Impose the low energy EFT behavior at s=0

$$\chi(s) \equiv \frac{1}{4}(\rho(s)-1)^2 + \frac{1}{4}(\rho(s)-1)^3 = -s - 3s^2 + \mathcal{O}(s^{5/2})$$

Need to cancel spurious square roots

$$\rho(s) = 1 - 2\sqrt{-s} + 2(-s) - 2(-s)^{3/2} + 2(-s)^2 + \ldots$$

$$A(s|t,u) = \texttt{low energy} + {\sum}' c_{ab}\rho(s)^a(\rho(t)^b + \rho(u)^b) + {\sum}' d_{ab}(\rho(t)^a\rho(u)^b + \rho(t)^b\rho(u)^a)$$

The naive counting $a + b \leq N_{\max}$

$N_{\max}$	s^0	$s^{1/2}$	s^1	$s^{3/2}$	s^2	$s^{5/2}$
10	89	88	85	82	79	74

Number of free parameters once we impose low energy up to O(s^2)

Unitarity will be imposed numerically as a set of quadratic constraints

1) Project on the irreps of O(N)

2) Project on the various spin components

Let's look at s-channel unitarity

Singlet $$A^{(0)}(s,t,u) = N_f A(s|t,u) + A(t|s,u) + A(u|s,t)$$

Antisymmetric $$A^{(1)}(s,t,u) = A(t|s,u) - A(u|s,t)$$

Symmetric traceless $$A^{(2)}(s,t,u) = A(t|s,u) + A(u|s,t)$$

Partial wave projections

$$S_\ell^{(I)} = 1 + i s^{d/2-2} \int_{-1}^{1} (1-x^2)^{d/2-2} P_\ell^{(d)}(x) A^{(I)}(s,x)$$

$$x = \cos\theta$$

Linear in the ansatz parameters

d=4 $$\boxed{S_\ell^{(I)} = 1 + \frac{i}{64\pi} \int_{-1}^{1} P_\ell(x) A^{(I)}(s,x)}$$

d=10 $$S_\ell^{(I)} = 1 + \frac{i s^3}{2^{18} 3 \pi^4} \int_{-1}^{1} (1-x^2)^3 C_\ell^{(7/2)}(x) A^{(I)}(s,x)$$

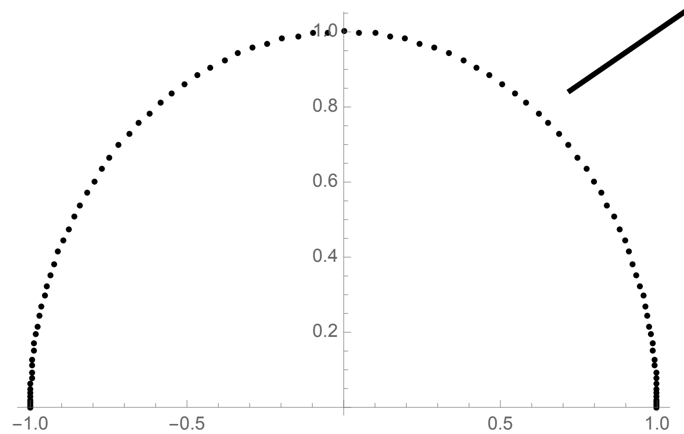
Harder as you go higher in dimensions!!

Primal optimization problem

$$\min \beta, \qquad \text{with} \quad \alpha = \alpha^*$$

Over the space of crossing symmetric and analytic functions of s, t, u

$$|S_\ell^{(I)}(s)|^2 \leq 1, \qquad s > 0, \qquad \ell = 0, \ldots, \infty$$



$$\ell = 0, \ldots, L_{\text{max}}$$

Since particles are massless,
the higher spin partial waves are not exponentially suppressed!

Discretize unitarity on a grid of points M_{max}

For each N_max we want to have M_max and L_max very large!!

$$M_{\text{max}} = 200 \qquad L_{\text{max}} = 90 \qquad N_{\text{max}} = 12, \ldots, 23$$

400 variables, 18x10^3 quadratic constraints, ~ 7 h per point on 40 cores for N_max=23

Bootstrap Summary

Crossing Symmetric Ansatz

$$\mathcal{M}_{ab}^{cd}(s,t,u) = A(s|t,u)\delta_{ab}\delta^{cd} + A(t|s,u)\delta_a^c\delta_b^d + A(u|s,t)\delta_a^d\delta_b^c$$

Mandelstam Analyticity + Real Analyticity

$$A(s|t,u) = \sum_{n\leq m}^{\infty} a_{nm}\left(\rho_t^n\rho_u^m + \rho_t^m\rho_u^n\right) + \sum_{n,m}^{\infty} b_{nm}\left(\rho_t^n + \rho_u^n\right)\rho_s^m \qquad \rho(s) = \frac{1-\sqrt{-s}}{1+\sqrt{-s}}$$

Impose Low Energy Behavior

$$A(s|t,u)^{1-\text{loop}} = \alpha\frac{s^2}{f^4} + \beta\frac{t^2+u^2}{f^4} - \frac{N_f-2}{32\pi^2}\frac{s^2}{f^4}\log\frac{-s}{f^2} - \frac{t-u}{96\pi^2 f^4}\left(t\log\frac{-t}{f^2} - u\log\frac{-u}{f^2}\right)$$

Check Unitarity Numerically

$$S_\ell^{(I)}(s) = 1 + i\int_{-1}^{1} P_\ell(x)A^{(I)}(s,x)dx \qquad \boxed{|S_\ell^{(I)}(s)|^2 \leq 1}$$

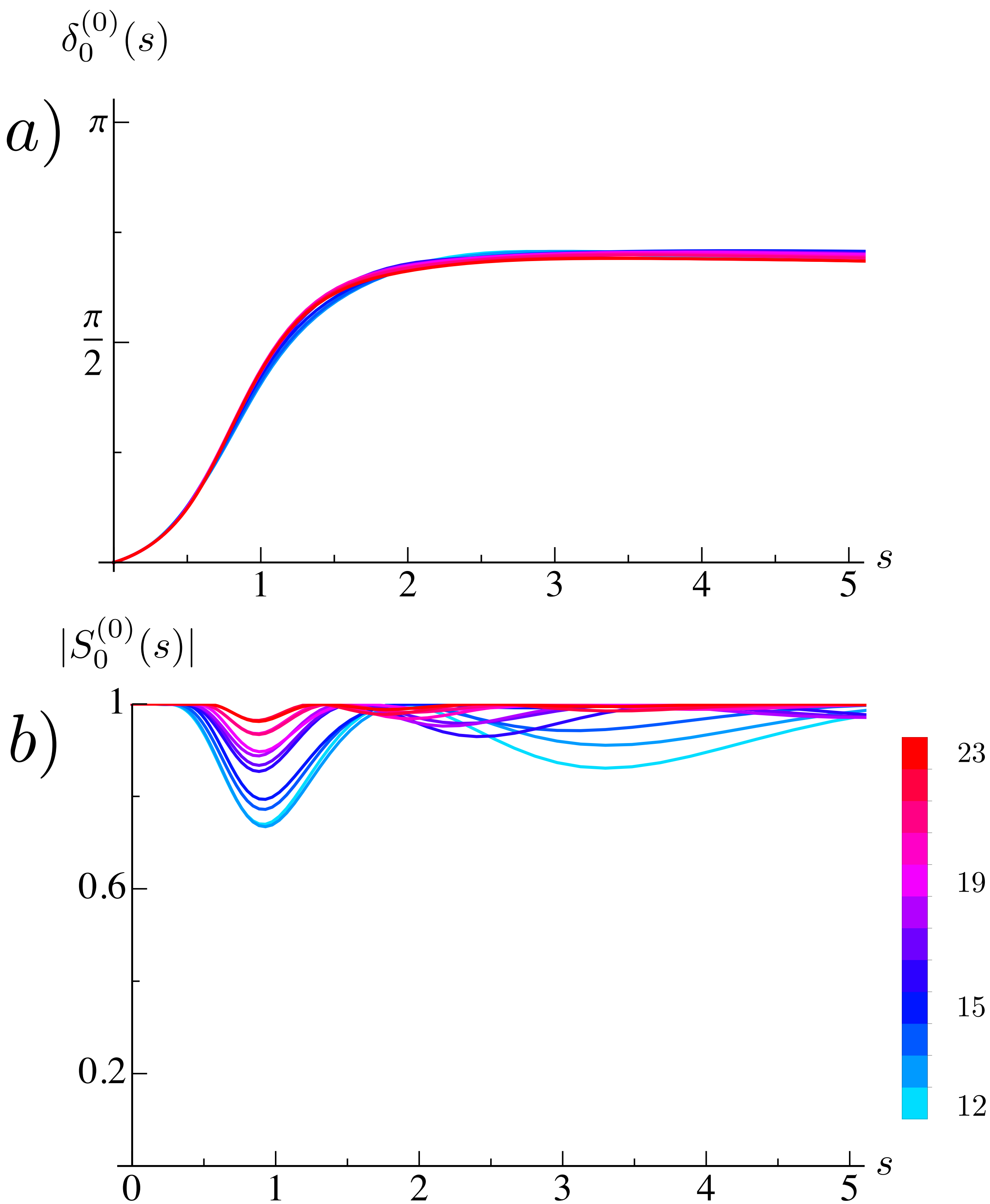
Our Target

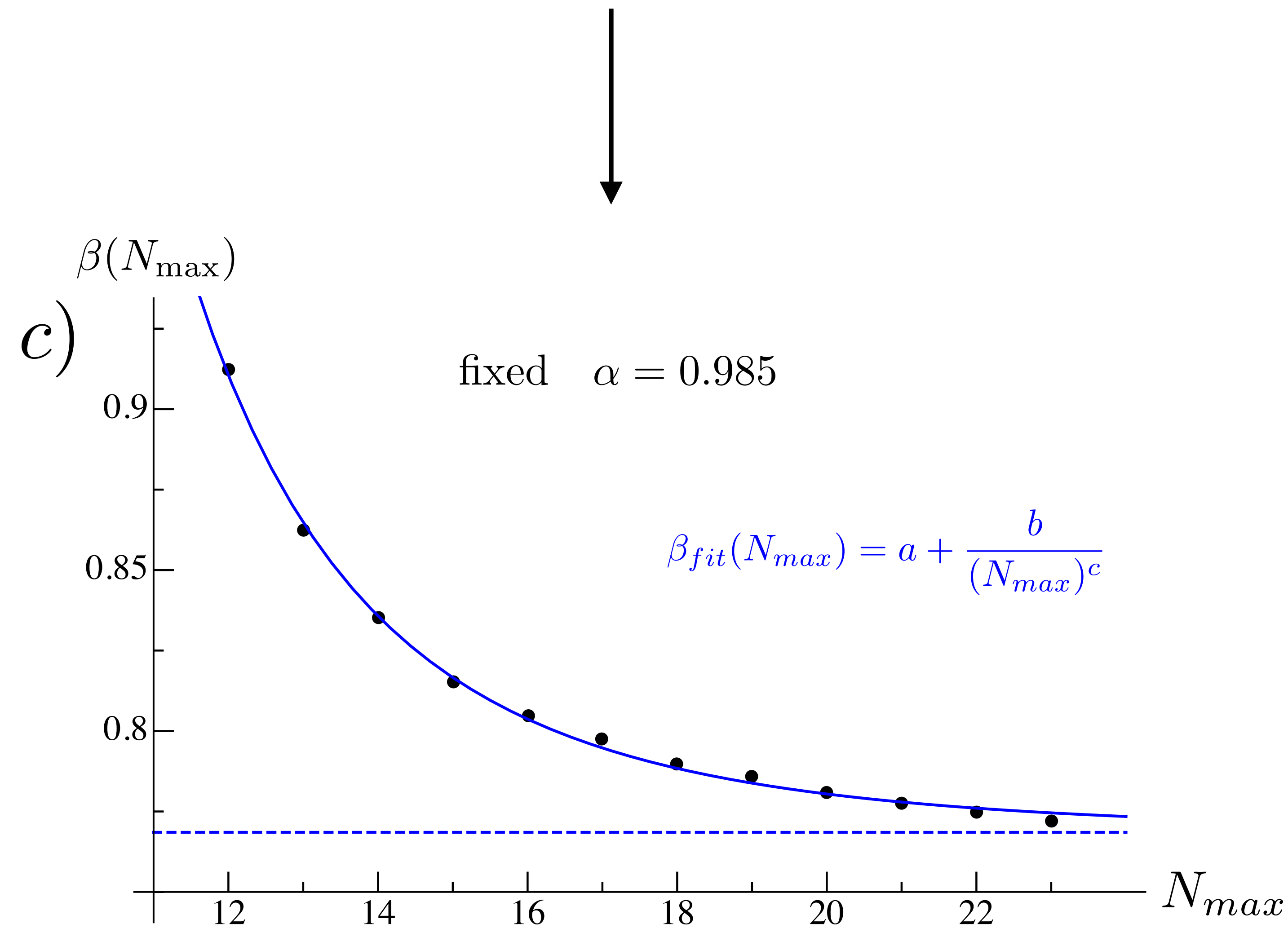
$$\{\alpha, \beta\}$$

We fix alpha and minimize beta

1) Scan in alpha

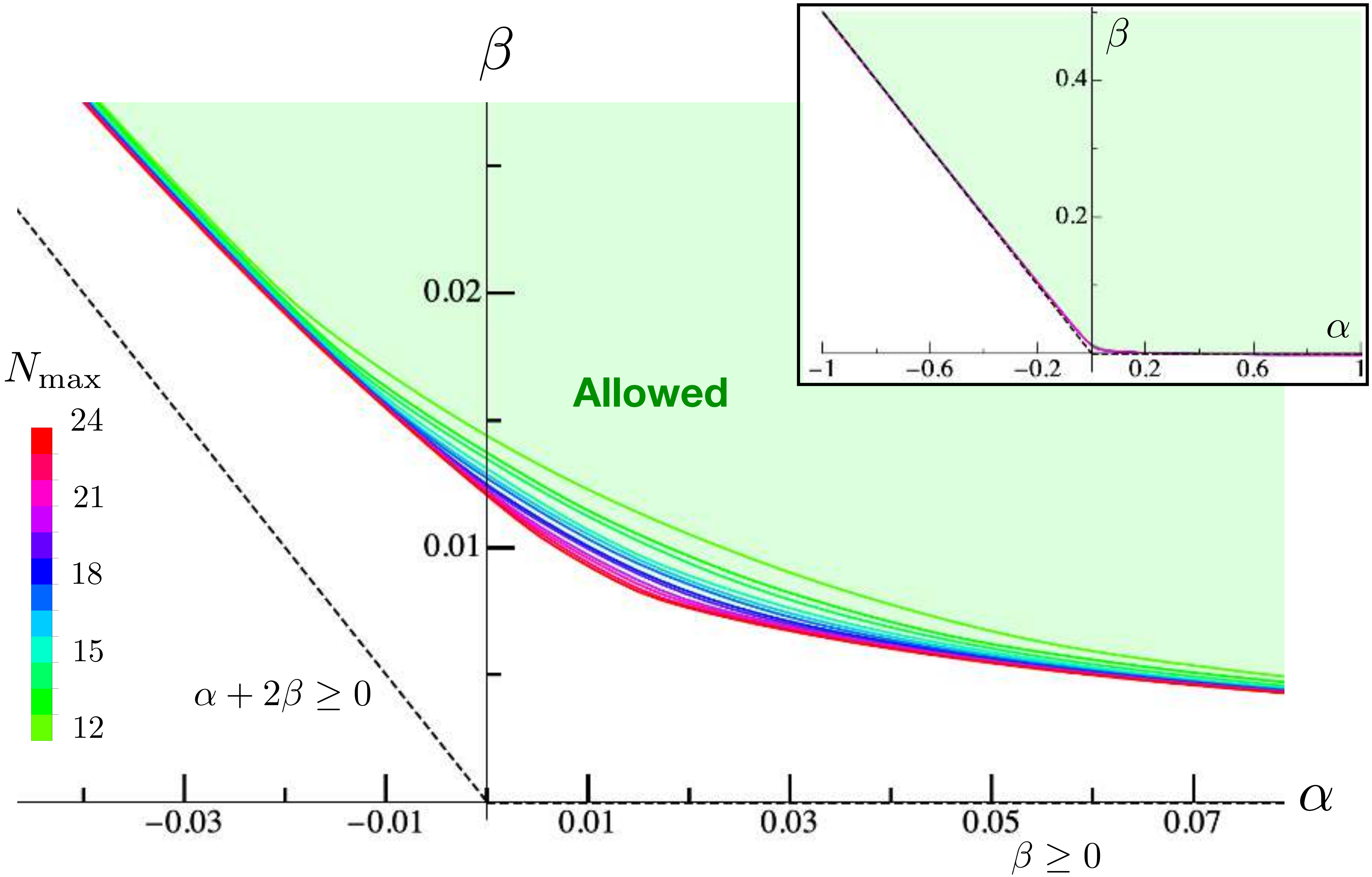
2) Make sure Nmax is large enough





We observe unitarity tries to saturate in all partial waves!

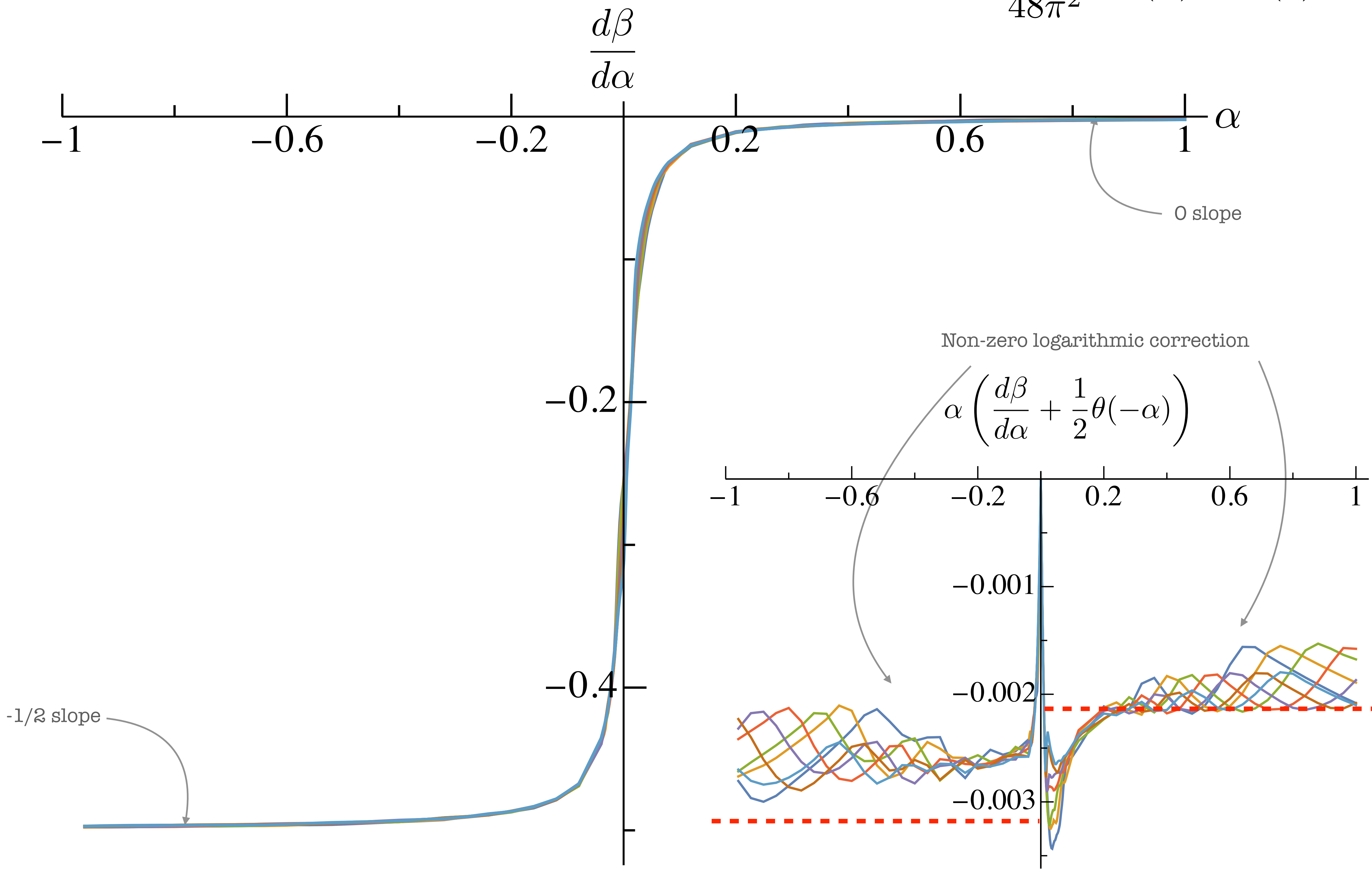
$$A(s|t,u) = \frac{s}{f^2} + \frac{\alpha}{f^4}s^2 + \frac{\beta}{f^4}(t^2+u^2) + \text{logarithms} + \dots$$



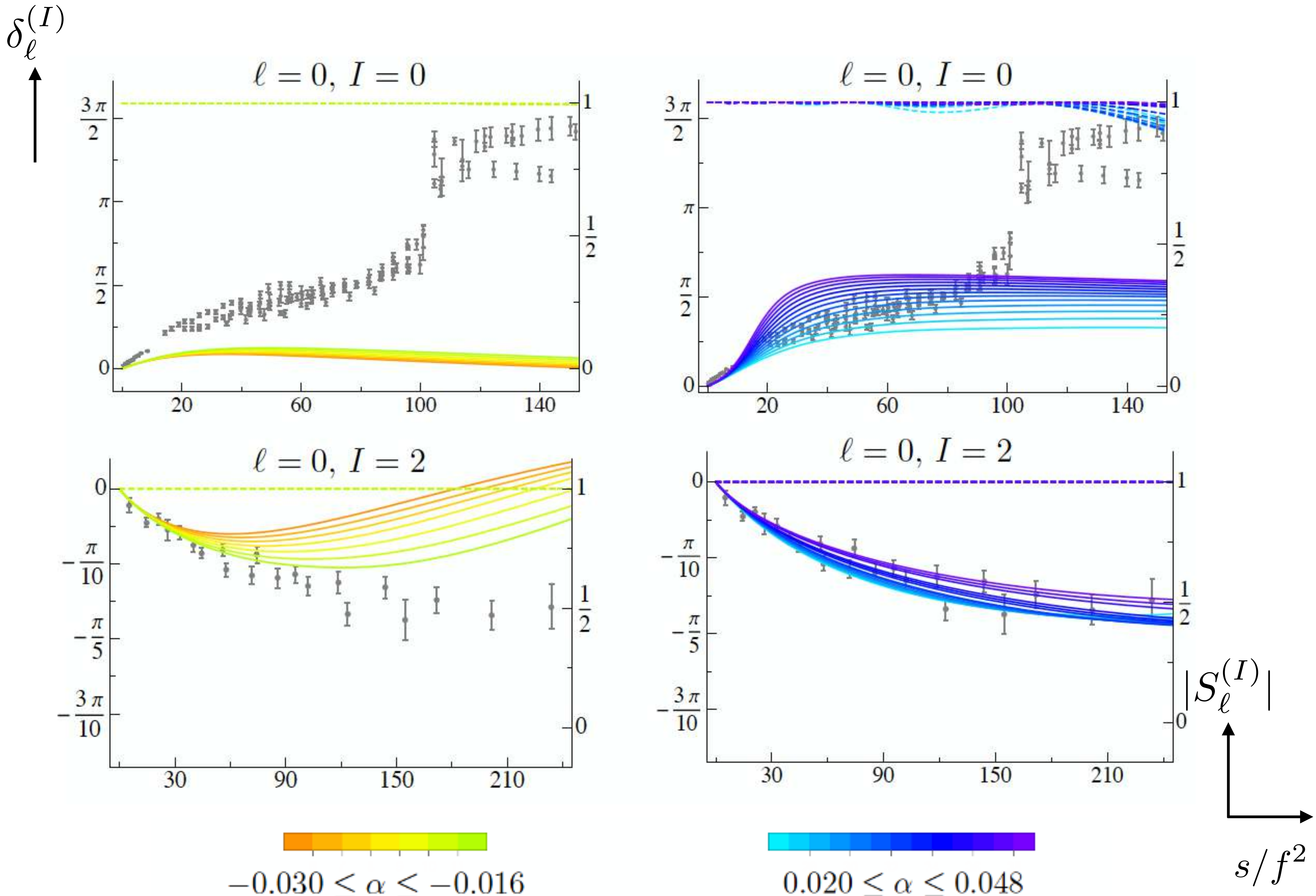
There are asymptotic logarithmic corrections!

$$\alpha + 2\beta \geq -\frac{1}{16\pi^2}\log(-\alpha) + \mathcal{O}(1)$$

$$\beta \geq -\frac{1}{48\pi^2}\log(\alpha) + \mathcal{O}(1)$$

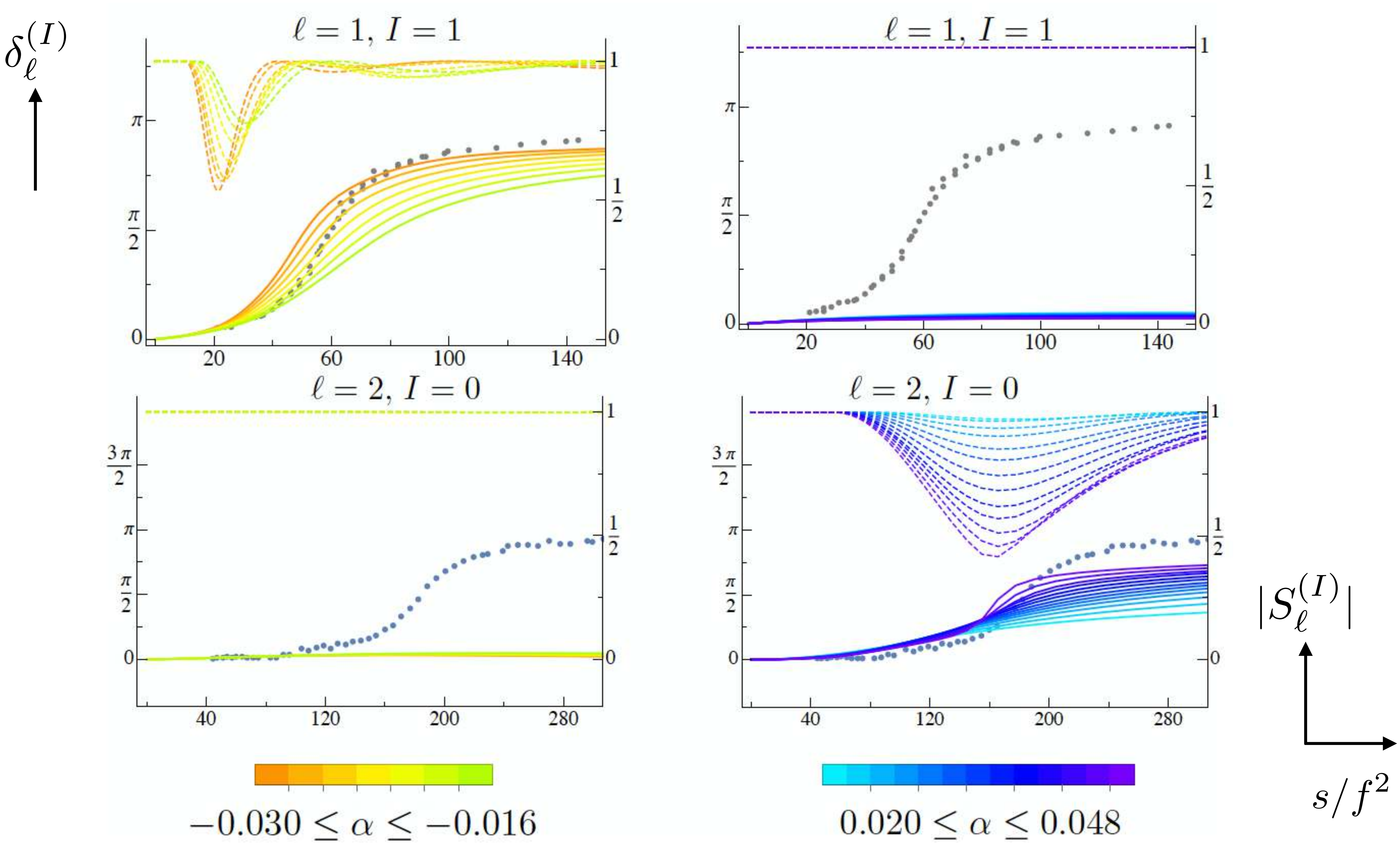


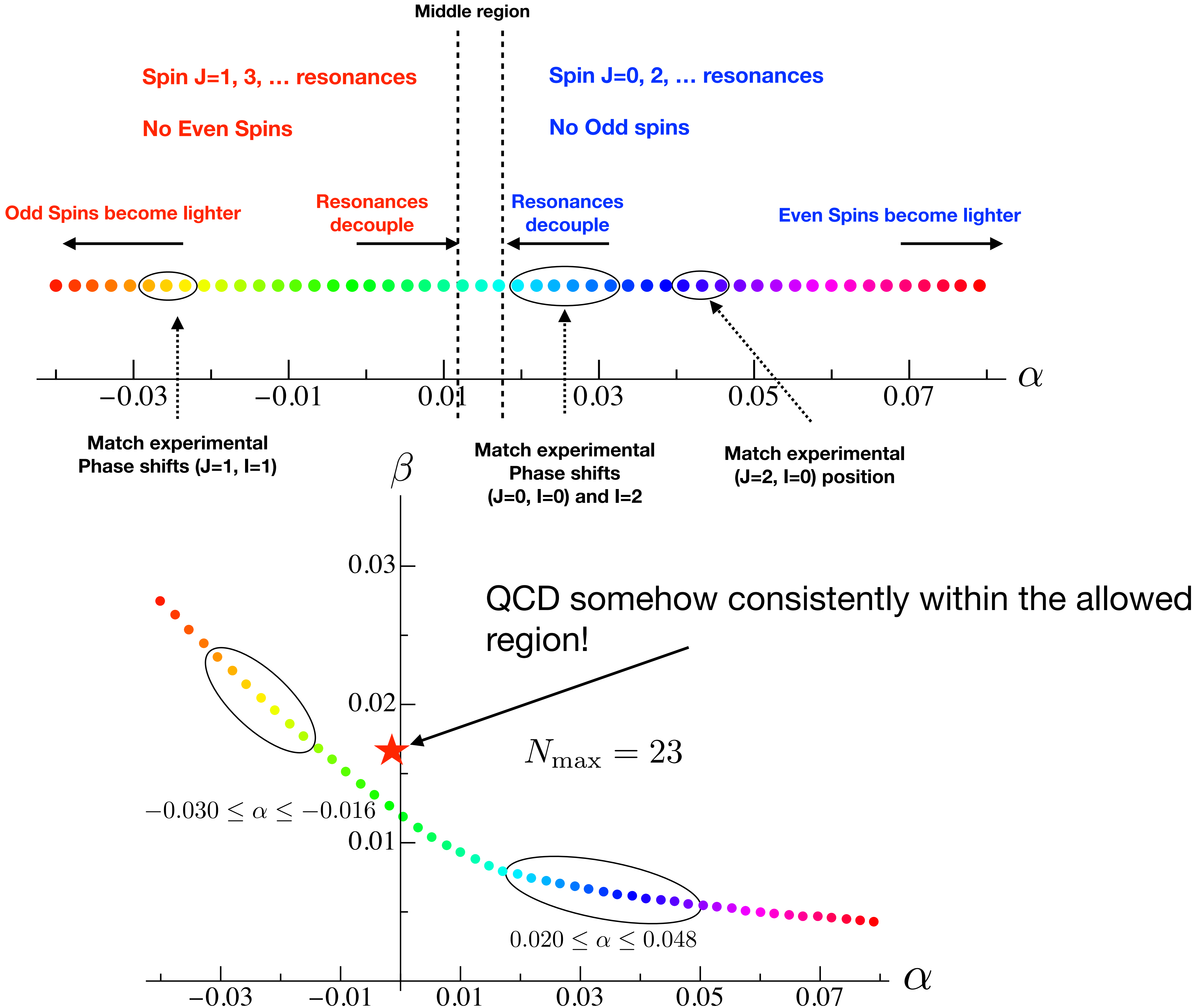
Phase shifts along the boundary: Spin=0



Higher spins

Clear Spin 1, 2! Evidence for Spin 3, 4.
For higher spins we would need an insane amount of time to converge




Middle region
Spin J=1, 3, ... resonances
No Even Spins
Spin J=0, 2, ... resonances
No Odd spins
Odd Spins become lighter
Resonances decouple
Resonances decouple
Even Spins become lighter
−0.03
−0.01
0.01
0.03
0.05
0.07
α
Match experimental Phase shifts (J=1, I=1)
Match experimental Phase shifts (J=0, I=0) and I=2
Match experimental (J=2, I=0) position
β
0.03
0.02
0.01
QCD somehow consistently within the allowed region!
$N_{\max} = 23$
$-0.030 \leq \alpha \leq -0.016$
$0.020 \leq \alpha \leq 0.048$
−0.03
−0.01
0.01
0.03
0.05
0.07
α

Sugra bootstrap and String theory

Why study maximal Supergravity in 10 D?

1) Gravity in D > 4 is IR safe

2) Maximal Susy: scalars instead of gravitons

(Work in progress with J. Penedones, P. Vieira)

$$\mathbb{A} = \frac{\delta^{16}(Q)}{stu} M(s,t,u)$$

Axidilaton scattering $$A(s|t,u) = \frac{s^4}{stu} M(s,t,u)$$

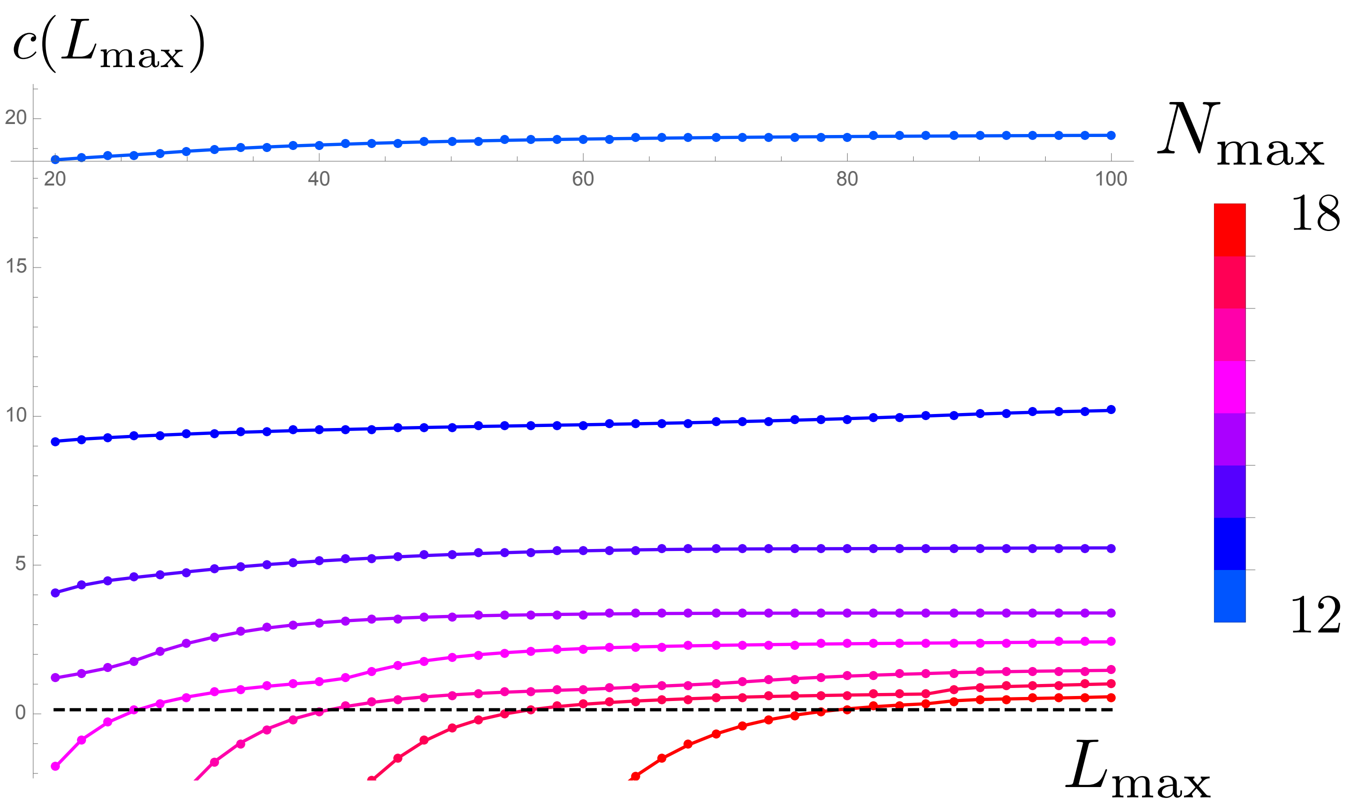
$$M(s,t,u) = 1 + c(stu) + \mathcal{O}(s^5)$$

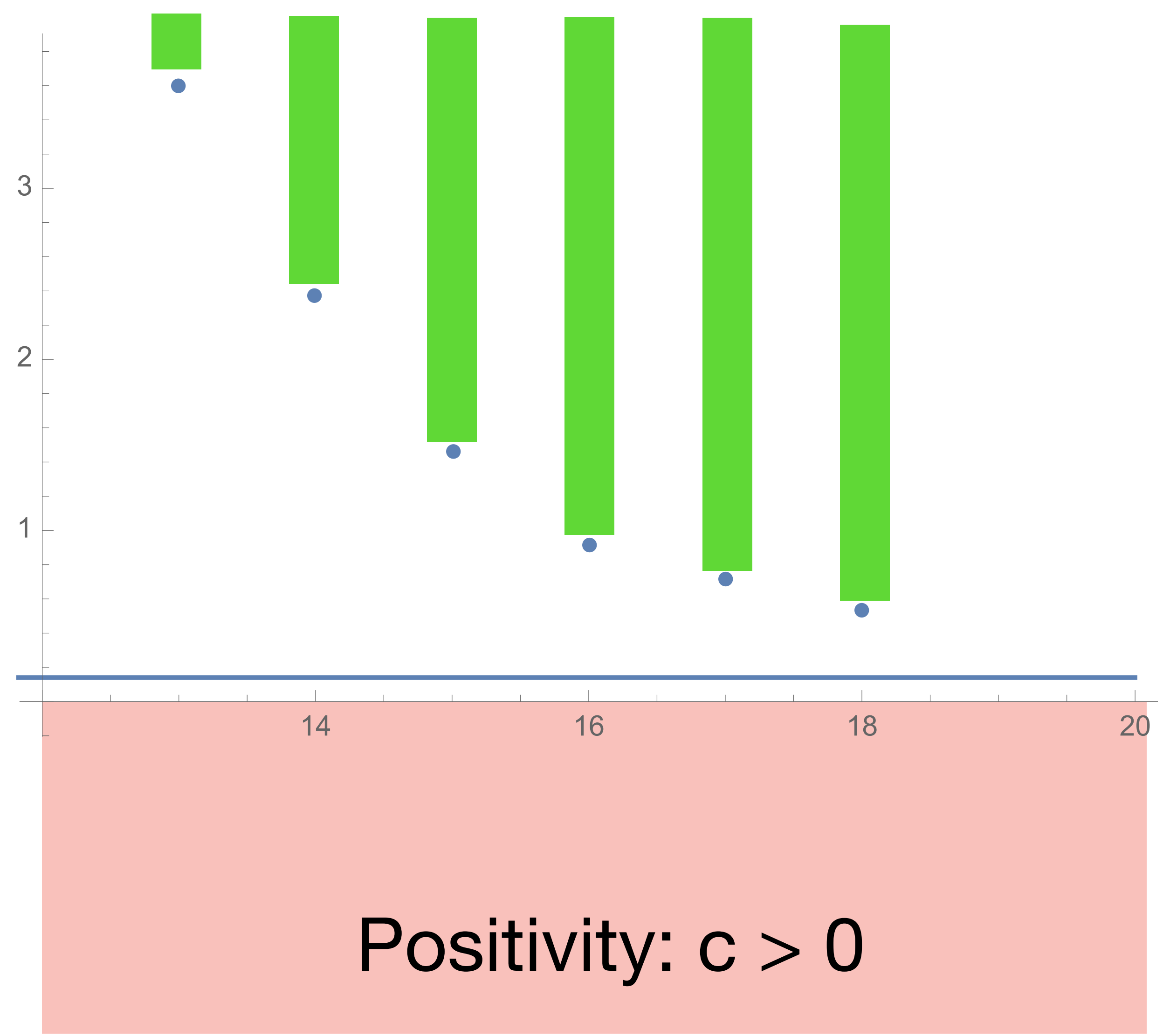
Not special from an EFT point of view, but bounded in String Theory!

Type IIA, type IIB predicts a similar value

$$c \geq 0.14$$

Chester, Green, Pufu, Wang, Wen ‘19





Works in progress and future challenges

Maximal Supergravity in 10 D shows that even in the 2->2 complexity grows pretty fast

Higher D dual problems? A dual formulation might provide analytical insights and speed up numerics

We will ever be able to go beyond 2->2?

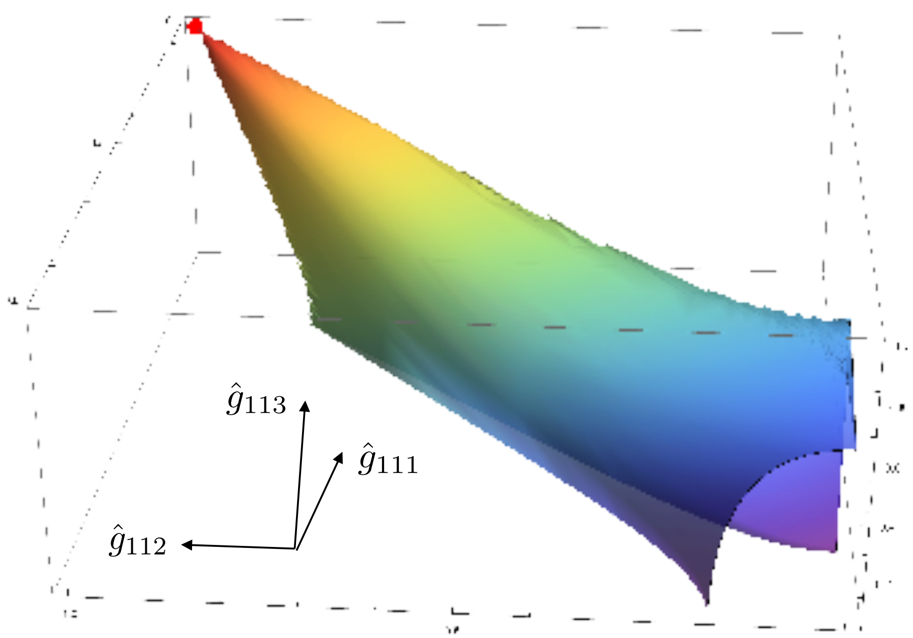
Including 2->3 processes in massless EFTs

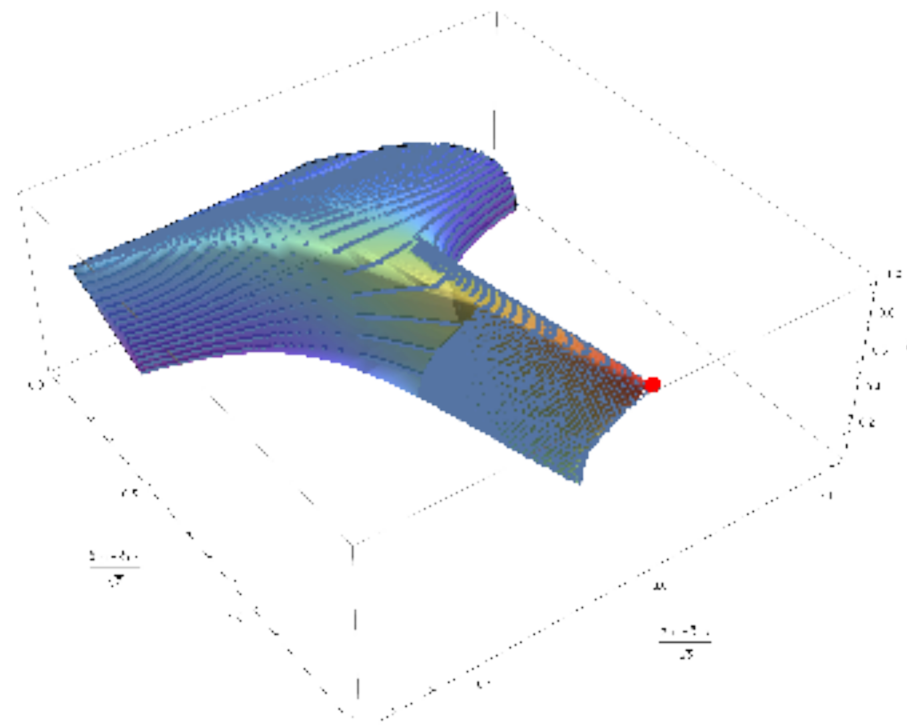
(Work in progress with A. Homrich, J. Penedones, P. Vieira)

Explore more the dual formulation!!!

Works in progress in 1+1 D

1) Ising Field Theory with thermal and magnetic deformations including form factors





2) Z2 preserving deformation: Tri-critical Ising

3) Dual flux-tube bootstrap for massless particles in 1+1 D with EFT description

(Work in progress with J. Elias Miro)