Shedding some light on the light-by-light

Oscar Catà

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(in collaboration with L. Cappiello, G. D'Ambrosio, D. Greynat and A. Iyer)

Outline

- Motivation
- A toy model
- MV limit and the anomaly
- Numerical results
- Conclusions

Status of the muon $(g-2)_\mu$

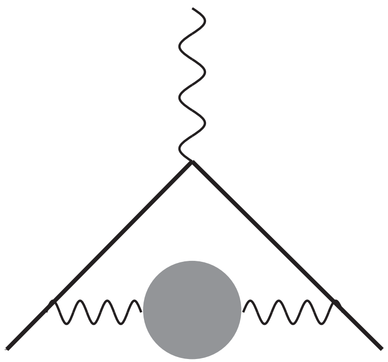
- While awaiting for the FNL (and J-PARC) numbers, currently:

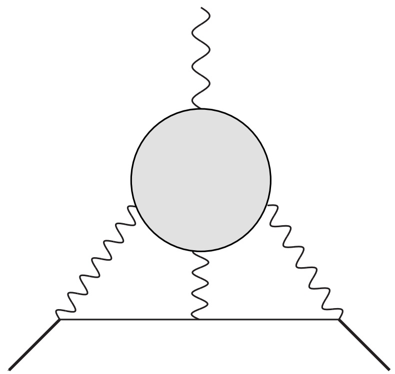
$$a_\mu^{\exp} = 116592091(54)(33) \times 10^{-11}$$

- Long-standing discrepancy[a] with the SM estimate (3 to 4σ):

$$a_\mu^{\rm SM} = 116591823(1)(34)(26) \times 10^{-11}$$

- Excellent control over the dominant EW and EM corrections. Hadronic contributions (HVP and HLbL) small but dominate the uncertainty. Difficult to evaluate.





[a] Only one experiment, not yet challenged...

Hadronic contributions

- HVP leading effect $(\sim 700 \times 10^{-10})$. Uncertainties can be reduced with e^+e^--based and/or τ-based analyses. [Davier et al, Teubner et al]

  Lattice QCD at a really advanced stage. [Mainz, BMWc, RBC/UKQCD]

- HLbL much harder to estimate. Connection to experiment more convoluted, albeit dispersion analyses promising. [Bern, Bonn, Mainz]

  Lattice QCD catching up fast. [Mainz, RBC/UKQCD]

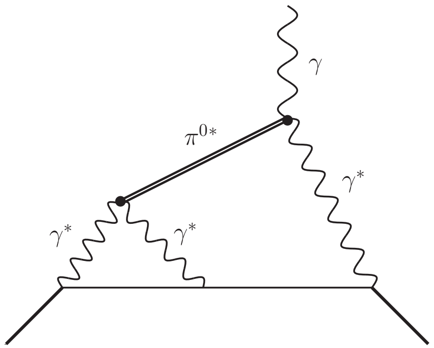
- Experimentally, FNL and J-PARC have a much improved projected uncertainty, 16×10^{-11}.

- Hadronic contributions cannot account for the present discrepancy, but we need better control of theoretical uncertainties to claim NP interpretations, when/if the time comes.

HLbL estimates

- Contributions ranked using large-N_c and χPT arguments. [de Rafael'93]

- Dominance from π^0 exchange:



- However, single resonance exchange (axials) have a sizeable effect (kinematical kernels peaked at $1-2$ GeV).

- Scalar exchange, Goldstone loops and quark loops also important.

- The final outcome is complicated by large cancellations of the different contributions.

- Three main routes for HLbL: form factor ansatz, lattice QCD, dispersion relations.

Form factor analysis

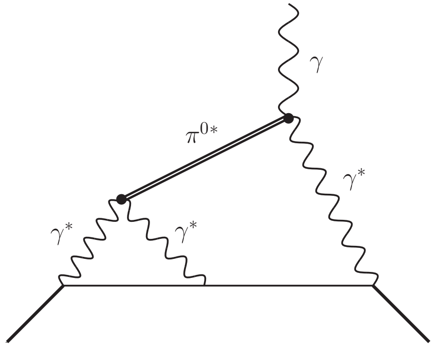
- Main contributions from form factor analyses (in units of 10^{-11}):

Contribution	BPP	HKS,HK	KN	MV	PdRV	N,JN
π^0, η, η'	$85(13)$	$82.7(6.4)$	$83(12)$	$114(10)$	$114(13)$	$99(16)$
axial vectors	$2.5(1.0)$	$1.7(1.7)$	–	$22(5)$	$15(10)$	$22(5)$
scalars	$-6.8(2.0)$	–	–	–	$-7(7)$	$-7(2)$
π, K loops	$-19(13)$	$-4.5(8.1)$	–	–	$-19(19)$	$-19(13)$
π,K loops +subl. N_C	–	–	–	$0(10)$	–	–
quark loops	$21(3)$	$9.7(11.1)$	–	–	2.3	$21(3)$
Total	$83(32)$	$89.6(15.4)$	$80(40)$	$136(25)$	$105(26)$	$116(39)$

- Overall agreement with the 'pion-pole' contribution, main discrepancies in other contributions.
- Caveat: not all the entries above come from independent calculations.
- Axial- and Goldstone-exchange contributions not settled.
- A number of theoretical issues still open, which affect not just the uncertainty.

Form factor analysis

Pion-pole contribution:



- Vertices given by the $\pi^0\gamma^*\gamma^*$ form factor,

$$\int d^4x\; e^{iq_1\cdot x}\langle 0|T\left\{J^{\mu}_{\rm EM}(x)\, J^{\nu}_{\rm EM}(0)\right\}|\pi^0(p)\rangle = \epsilon^{\mu\nu\alpha\beta} q_{1\,\alpha} q_{2\,\beta}\, F_{\pi\gamma\gamma}\left(Q_1^2, Q_2^2\right)$$

- $F_{\pi\gamma\gamma}$ not known from first principles. Information only on certain kinematical limits:

(a) $F_{\pi\gamma\gamma}\left(0,0\right) = -\dfrac{N_c}{12\pi^2 f_\pi} \equiv \mathcal{A}$ (Anomaly)

(b) $\lim\limits_{Q^2\to\infty} F_{\pi\gamma\gamma}(Q^2,Q^2) = -\dfrac{2f_\pi}{3Q^2} + \ldots$ (OPE)

(c) $\lim\limits_{Q^2\to\infty} F_{\pi\gamma\gamma}(0,Q^2) = -\dfrac{2f_\pi}{Q^2} + \ldots$ (Brodsky-Lepage)

Form factor analysis

- Ansätze with different short and long-distance constraints: [see e.g. Knecht et al'01]

$$F^{(1)}_{\gamma^*\gamma^*\pi^0}(q_1,q_2)=\mathcal{A};\qquad F^{(2)}_{\gamma^*\gamma^*\pi^0}(q_1,q_2)=\mathcal{A}\frac{m_V^4}{(q_1^2-m_V^2)(q_2^2-m_V^2)}$$

$$F^{(3)}_{\gamma^*\gamma^*\pi^0}(q_1,q_2)=\mathcal{A}\frac{m_V^2}{m_V^2-q_1^2-q_2^2};\qquad F^{(4)}_{\gamma^*\gamma^*\pi^0}(q_1,q_2)=\mathcal{A}\frac{m_V^4-\frac{4\pi^2 f_\pi^2}{N_c}(q_1^2+q_2^2)}{(q_1^2-m_V^2)(q_2^2-m_V^2)}$$

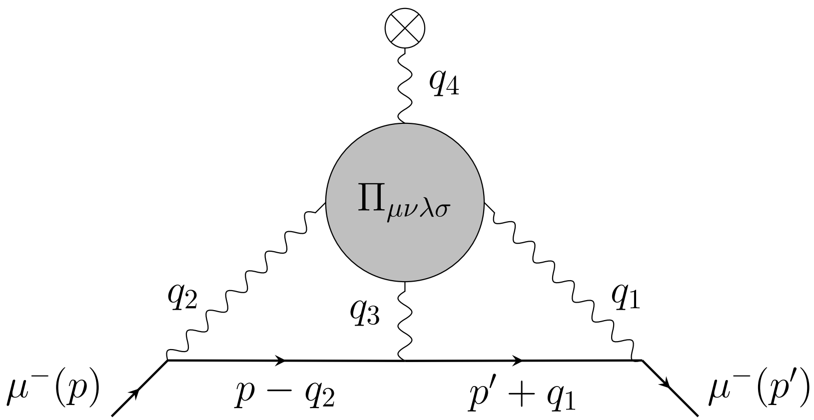
- In principle, the more constraints the better (closer to QCD). However, interesting to play with them to test which constraints are numerically important.
- The same strategy can be repeated for the other contributions.
- Important: the previous models are interpolators, i.e., the parameters are not the physical masses. They encode OPE information (inclusive)...

MAIN HURDLES:

- Hard to pin down the discrepancies: different interpolators for different channels, subject to different constraints.
- Not always clear how/if the short distances can be incorporated into form factors.

The Melnikov-Vainshtein limit

- OPE condition on the electromagnetic correlator (not on a form factor!)



- Main object:

$$W^{\mu\nu}(q_2, q_3) = \int d^4x \int d^4y\, e^{i(q_2\cdot x + q_3\cdot y)} T\{j^\mu_{\rm em}(x), j^\nu_{\rm em}(y)\}$$

- In the kinematical limit $Q_2^2 \simeq Q_3^2 \gg Q_1^2 \gtrsim \Lambda_{\rm QCD}$

$$\lim_{\xi\to\infty} W^{\mu\nu}\left(\xi Q - \frac{Q_3}{2}, -\xi Q - \frac{Q_3}{2}\right) = \frac{1}{\xi}\frac{2i}{Q^2}\epsilon^{\mu\nu\lambda\rho} Q_\lambda \sum_a \hat{d}^{a\gamma\gamma} \int d^4z e^{-iq_1\cdot z} j^{(a)}_{5\rho}(z)$$

- The OPE links VVVV to the (anomalous) VVA. [Melnikov-Vainshtein'03]

The Melnikov-Vainshtein limit

- The resulting short-distance constraint allegedly leads to a (sizable) increase

Contribution	BPP	HKS,HK	KN	MV	PdRV	N,JN
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- The increase is ascribed to the Goldstone and axial contributions, but this is misleading...

- Attempts to implement it with form factors not entirely successful. E.g., [Melnikov-Vainshtein'03]

$$\mathcal{A}_{\pi^0} = F_{\pi\gamma\gamma}(q_2, q_3)\frac{1}{q_1^2 - m_\pi^2}F_{\pi\gamma\gamma}(q_1, 0)$$

consistent only if $F_{\pi\gamma\gamma}(q_1, 0) = 1$. Hard to argue phenomenologically: symmetry arguments, OPE of the pion form factor...

Open issues:

- Does a form factor analysis capture this effect?

- Which resonances are responsible for it?

Correlators vs form factors

- Melnikov and Vainshtein redux:

  The OPE constraint is solid, but a model is needed to extrapolate it to all energies. With the model chosen, there is a substantial increase in the HLbL

- The main problem ([Melnikov'11]) is that, in general,

$$\textbf{Correlator} \neq \sum (\text{particle exchange})$$

- Sometimes a finite number of particles will fail to satisfy short distances, e.g.

$$\lim_{Q^2\to\infty} \langle VV \rangle = \lim_{Q^2\to\infty} \sum_n^{\infty} \frac{F_{Vn}}{Q^2 + M_{Vn}} \simeq \log\left(\frac{Q^2}{\mu^2}\right)$$

- Contact terms are important to fulfill general properties, e.g. gauge invariance:

$$\lim_{q^2\to 0} \Pi^{\mu\nu}_{AA}(q) = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) f_\pi^2$$

  Pion propagation is not enough. Contact terms are fundamental.

Correlators vs form factors

- Relevant issue for the HLbL: this mismatch between form factors and correlators is at the root of the so-called pion 'on-shell' vs 'off-shell' contributions.
- How to make sure that correlators contain (possible) contact terms? Lagrangian description with external sources, e.g. ChPT.
- Problem: Lagrangians for strong interactions only known at specific kinematical regimes (pQCD, ChPT).

Shopping list:

- Hadronic model with a (regulated) infinite number of resonances.
- Lagrangian formulation.
- Correct low and high energies at correlator level.
- Anomalies correctly implemented.
- Simplicity.

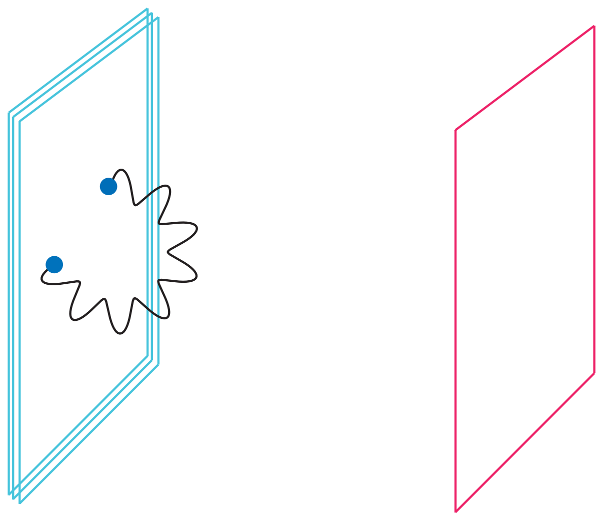
In other words, a consistent realization of large-N_c QCD.

A toy model

- 5-dimensional model: [Hirn-Sanz'05]

$$S_5 = \int d^4x \int_0^{z_0} dz \Big\{ -\lambda\sqrt{g}\,\mathrm{tr}\big[F_{(L)}^{MN}F_{(L)MN} + F_{(R)}^{MN}F_{(R)MN}\big] + c\,\mathrm{tr}\big[\omega_5(L_M) - \omega_5(R_M)\big]\Big\}$$

with $\omega_5(L) = \mathrm{tr}\left[LF_{(L)}^2 + \frac{i}{2}L^3F_{(L)} - \frac{1}{10}L^5\right]$



$z = \epsilon$ $z = z_0$

- AdS$_5$ space: $ds^2 = \frac{1}{z^2}(-dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu)$

A toy model

- Lagrangian-based theory of infinite massive vector and axial mesons (Kaluza-Klein modes):

$$V_\mu(x,z) = \sum_{n=1} V_\mu^{(n)}(x)\varphi_n^V(z), \qquad A_\mu(x,z) = \sum_{n=1} A_\mu^{(n)}(x)\varphi_n^A(z)$$

- (Spontaneous) chiral symmetry breaking via IR boundary conditions:

$$L_\mu(x,z_0) - R_\mu(x,z_0) = 0, \qquad F_L^{z\mu}(x,z_0) + F_R^{z\mu}(x,z_0) = 0$$

- Pion multiplet related to the axial zero mode $A_5^{(0)}(x,z)$. Via Wilson lines, e.g.

$$\xi_L(x,z) = P\exp\left\{-i\int_z^{z_0} dz'\, L_5(x,z')\right\}$$

the change of variables $L_M^\xi(x,z) = \xi_L^\dagger(x,z)\left[L_M(x,z) + i\partial_M\right]\xi_L(x,z)$ replaces A_5 by

$$U(x) \equiv \xi_L(x)\xi_R^\dagger(x) = \exp\left[\frac{2i\pi^a(x)t^a}{f_\pi}\right]$$

- Short-distance QCD: through (conformal) AdS$_5$ metric, one reproduces all the (leading) constraints tested so far.

- Simplicity: only 3 free parameters, λ, z_0, c.

A toy model

- Important: the change of variables does not leave the CS term invariant, but induces a shift

$$\omega_5(L^\xi) = \omega_5(L) + \omega_5(\Sigma_L) + d\alpha_4(L, \Sigma_L), \qquad \Sigma_L = d\xi_L \xi_L^\dagger$$

where $\alpha_4(L, \Sigma_L) = \frac{1}{2}\mathrm{tr}\left[\Sigma_L(LF_{(L)} + F_{(L)}L) + i\Sigma_L L^3 - \frac{1}{2}\Sigma_L L \Sigma_L L - i\Sigma_L^3 L\right]$.

- UV boundary conditions (AdS/CFT prescription): fields on the boundary are sources of the 4d theory, *i.e.*,

$$L_\mu^\xi(x, 0) = l_\mu^\xi(x) = u^\dagger(x)\left[l_\mu(x) + i\partial_\mu\right]u(x)$$

Holographic recipe: Given an action $S_5(A_M)$,

(1) Split the fields as $A_\mu(x,z) = a(x,z)\,\hat{a}_\mu^\perp(x) + \bar{a}(x,z)\,\hat{a}_\mu^\parallel(x) + \frac{\alpha(z)}{f_\pi}\partial_\mu\pi(x)$

(2) Solve the EoM for $a(x,z)$ et al. and plug them back into the action. This defines the four-dimensional generating functional $S_{\mathrm{eff}}(\hat{a}_\mu(x))$.

(3) Compute correlators, e.g.

$$\Pi_{VVVV}^{\mu\nu\lambda\rho} = \frac{\delta^4 S_{\mathrm{eff}}}{\delta\hat{v}_\mu \delta\hat{v}_\nu \delta\hat{v}_\lambda \delta\hat{v}_\rho}$$

How far can we go with the toy model?

Not QCD, clear limitations:

- From the pQCD quark loop in Π_{AA} and the chiral anomaly:

$$\lambda = \frac{N_c}{48\pi^2}; \qquad c = \frac{N_c}{24\pi^2}$$

  but confinement scale too rough:

$$m_{Vn} = \frac{\gamma_{0,n}}{z_0}, \qquad f_\pi^2 = \frac{N_c}{6\pi^2 z_0^2}, \qquad \frac{m_\rho}{f_\pi} \sim 10.7(!)$$

- No explicit chiral symmetry breaking: massless Goldstones (easily fixed).
- Mass scalings:

$$\frac{m_\rho}{m_{a_1}} = \frac{\gamma_{0,1}}{\gamma_{1,1}} \sim 0.63$$

  good, but the spectrum of V and A excitations is not accurately reproduced.

- No mass splitting within multiplets.

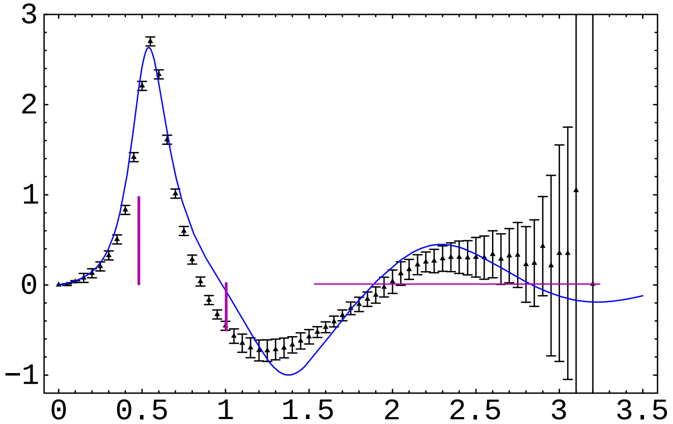
Minkowski vs Euclidean

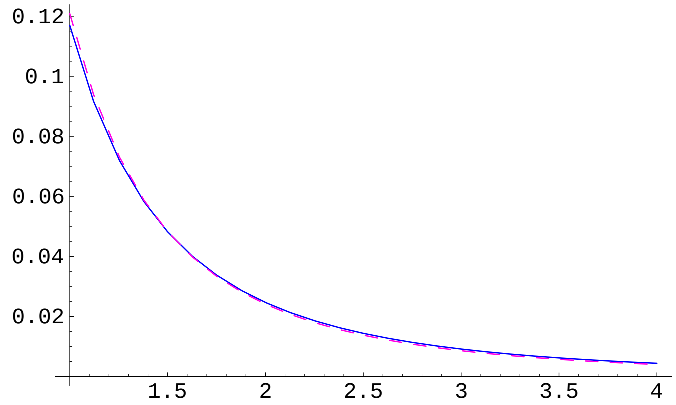
- However, the HLbL comes from an integral over Euclidean space.

$$a_\mu^{\mathrm{HLbL}} = -\frac{e^6}{48 m_\mu} \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} K_{\mu\nu\lambda\rho} \left(\frac{\partial}{\partial q_4^\rho} \Pi^{\mu\nu\lambda\sigma}(q_1, q_2, -q_4 - q_1 - q_2) \right) \Bigg|_{q_4=0}$$

- From experience: as long as short- and long-distances are fulfilled, the form of the interpolator in Minkowski space has a minor impact.[Knecht, Peris, Perrottet, de Rafael, ... ca. 2000]

- Consider e.g. Π_{LR},

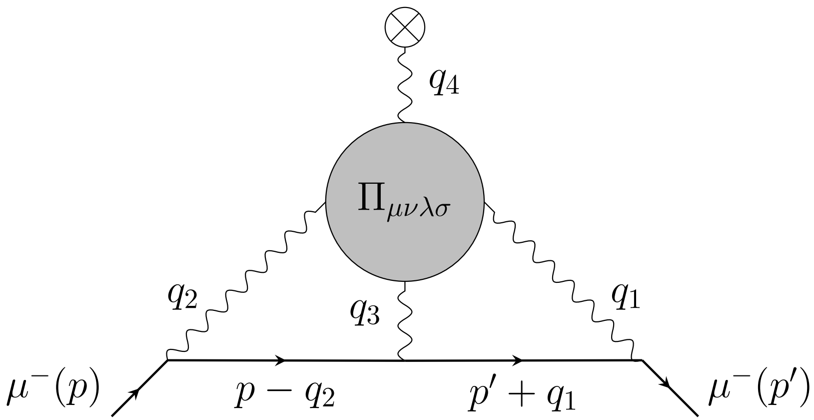




Very different pole distributions in Minkowski can give similar Euclidean continuations.

The toy model is expected to be an excellent laboratory to explore QFT issues in HLbL.

The HLbL tensor



Using the effective action, one finds a close expression for it:

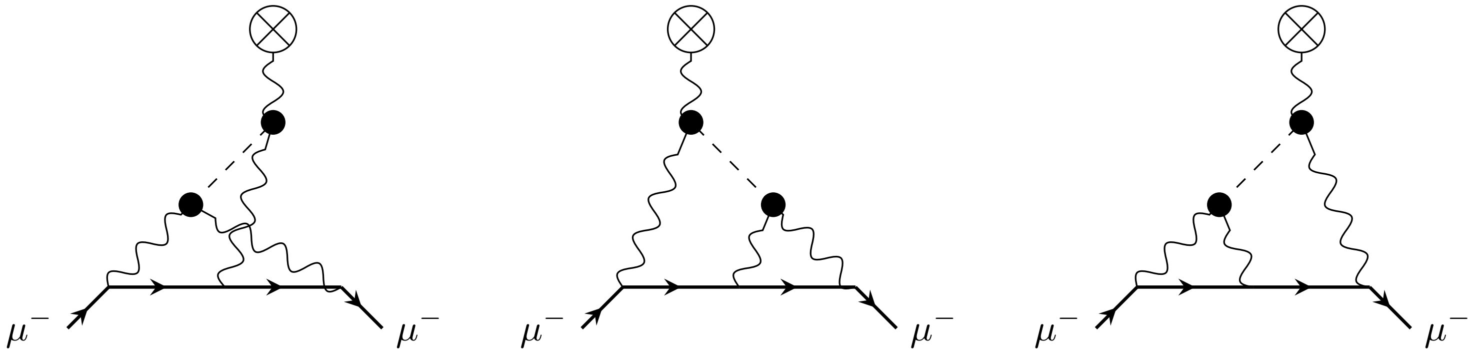
$$\Pi_{\mu\nu\lambda\rho} = \varepsilon_{\mu\nu\alpha\beta}\varepsilon_{\lambda\rho\alpha'\beta'}\left[\frac{2c^2}{\lambda}\int dz \int dz' T_{12}^{\beta}(z) G_A^{\alpha\alpha'}(z,z';p) T_{34}^{\beta'}(z') + F_{\pi\gamma\gamma}^{(12)} \frac{q_1^{\alpha} q_2^{\beta} q_3^{\alpha'} q_4^{\beta'}}{p^2 - m_\pi^2} F_{\pi\gamma\gamma}^{(34)}\right]$$

plus permutations, where $T_{ij}^{\mu}(z) = \Big[q_i^{\mu} v_i(z)\partial_z v_j(z) - q_j^{\mu} v_j(z)\partial_z v_i(z)\Big]$ and

$$F_{\pi\gamma\gamma}(Q_1^2, Q_2^2) = \frac{2c}{f_\pi}\int_0^{z_0} dz\, \alpha'(z)\, v(z, Q_1) v(z, Q_2)$$

- This represents the contribution of the Goldstone modes and axial excitations.
- The (inclusive) calculation of the HLbL in the large-N_c limit is straightforward.

The HLbL tensor



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plus permutations, where $\alpha(z) = 1 - \dfrac{z^2}{z_0^2}$,

$$v(z,Q) = Qz\left[K_1(Qz) + \frac{K_0(Qz_0)}{I_0(Qz_0)} I_1(Qz)\right], \quad a(z,Q) = Qz\left[K_1(Qz) - \frac{K_1(Qz_0)}{I_1(Qz_0)} I_1(Qz)\right]$$

$$G_A^{\perp}(z,z';Q) = -\frac{1}{Q} z' I_1(Qz') a(z,Q)\theta(z-z') + (z \leftrightarrow z')$$

$$G_A^{L}(z,z';Q) = -\frac{(z')^2}{2}\alpha(z)\theta(z-z') + (z \leftrightarrow z')$$

Pion transition form factor

- General expression:

$$F_{\pi\gamma\gamma}(Q_1^2, Q_2^2) = \frac{2c}{f_\pi} \int_0^{z_0} dz\, \alpha'(z)\, v(z, Q_1) v(z, Q_2)$$

- In the zero-momentum limit, $F_{\pi\gamma\gamma}$ is determined by the chiral anomaly. Using that $v(z, 0) = 1$,

$$F_{\pi\gamma\gamma}(0, 0) = -\frac{N_c}{12\pi^2 f_\pi}$$

- At very high energies,

$$\lim_{Q^2\to\infty} F_{\pi\gamma\gamma}(Q^2, Q^2) = -\frac{2f_\pi}{3Q^2} + \mathcal{O}\left(e^{-Qz_0}\right)$$
$$\lim_{Q^2\to\infty} F_{\pi\gamma\gamma}(0, Q^2) = -\frac{2f_\pi}{Q^2} + \mathcal{O}\left(e^{-Qz_0}\right)$$

- Only the leading term in the OPE is correctly reproduced by the model (enough).

- Parametrically there is agreement, but numerically not necessarily (more on this later on).

Longitudinal piece and pion-exchange dominance

- The longitudinal piece of the HLbL tensor can be projected via

$$G^A_{\mu\nu}(z,z';q) = P^{\perp}_{\mu\nu} G^A_{\perp}(z,z';q) + P^{\parallel}_{\mu\nu} G^A_{\parallel}(z,z';q)$$

- The electromagnetic tensor reads

$$\Pi^{\parallel}_{\mu\nu\lambda\rho} = T_{\mu\nu\lambda\rho}\left[\frac{F^{(12)}_{\pi\gamma\gamma}F^{(34)}_{\pi\gamma\gamma}}{s-m_\pi^2} - \frac{2c^2}{\lambda}\int dz \int dz' v_1(z)v_2(z)\frac{\partial_z\partial_{z'}G^{\parallel}_A(z,z')}{s}v_3(z')v_4(z')\right]$$

- The axial contribution can be split into a factorizable and a nonfactorizable piece with

$$\mathcal{W}^{\parallel} = \frac{F^{(12)}_{\pi\gamma\gamma}F^{(34)}_{\pi\gamma\gamma}}{s-m_\pi^2} - \frac{F^{(12)}_{\pi\gamma\gamma}F^{(34)}_{\pi\gamma\gamma}}{s} - \left(\frac{2c}{f_\pi}\right)^2\frac{1}{s}\int dz\alpha'(z)v_1(z)v_2(z)v_3(z)v_4(z)$$

- Limits:

$$\lim_{s\to\infty}\mathcal{W}^{\parallel} = -\left(\frac{2c}{f_\pi}\right)^2\frac{1}{s}\int dz\alpha'(z)v_1(z)v_2(z)v_3(z)v_4(z) + \mathcal{O}(m_\pi^2)$$

$$\lim_{s\to 0}\mathcal{W}^{\parallel} = \frac{F^{(12)}_{\pi\gamma\gamma}F^{(34)}_{\pi\gamma\gamma}}{s-m_\pi^2}$$

- There is no pion dominance! Axials play a fundamental role at short distances

Anomaly matching in the VVA triangle

- Consider the correlator

$$\Gamma_{\mu\nu\lambda}(q_3) = i\int d^4x d^4y\; e^{iq_3\cdot(x-y)}\langle 0|T\left\{j_\mu^{\rm em}(x)j_\nu^{\rm em}(y)j_\lambda^5(0)\right\}|0\rangle$$
$$= \frac{1}{24\pi^2}\left[\omega_L(q_3^2)t^{\parallel}_{\mu\nu\lambda} + \omega_T(q_3^2)t^{\perp}_{\mu\nu\lambda}\right]$$

- It is known that the chiral anomaly imposes (at all energies) [Vainshtein; Knecht et al]

$$\omega_L(q^2) = -\frac{2N_c}{q^2}$$

to all orders in pQCD. Corrections are $\mathcal{O}(m_\pi^2)$.

- This relation could be a consequence of pion dominance. [Melnikov-Vainshtein'03]

- However, this would entail that $F_{\pi\gamma\gamma}(Q_3,0)=1$. Puzzle: how such a contribution could be structureless?

- Additionally, at short distances,

$$\lim_{Q_3\to\infty}\left[\omega_L(Q_3^2) - 2\omega_T(Q_3^2)\right] = 0$$

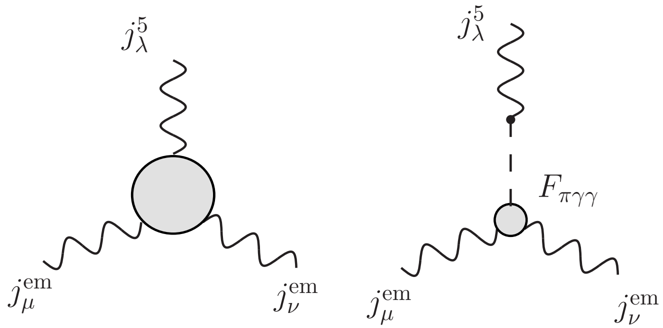
The VVA triangle

In the model, the triangle can be computed from the effective action:

$$(S_{\rm CS}^{(3)})^{\perp} = \frac{2c}{3}\varepsilon^{\mu\nu\lambda\rho}\int d^4x\, \hat{a}_\mu^{\perp}(x)\partial_\nu\hat{v}_\lambda(x)\hat{v}_\rho(x)\left[1+3\int_0^{z_0} dz\, a(x,z)v(x,z)v'(x,z)\right]$$

$$(S_{\rm CS}^{(3)})^{\parallel} = \frac{c}{3}\varepsilon^{\mu\nu\lambda\rho}\int d^4x\, \frac{\partial^\alpha\hat{a}_\alpha^{\parallel}(x)}{\Box}\partial_\nu\hat{v}_\lambda(x)\partial_\mu\hat{v}_\rho(x)\left[1+3\int_0^{z_0} dz\, \alpha'(z)v(x,z)v(x,z)\right]$$

plus the pion propagation. Notice the existence of a contact term.



Result:

$$\omega_L(Q_3) = \frac{2N_c}{Q_3^2} - \left(\frac{2N_c}{Q_3^2} - \frac{2N_c}{Q_3^2+m_\pi^2}\right)\frac{F_{\pi\gamma\gamma}(Q_3,0)}{F_{\pi\gamma\gamma}(0,0)},$$

$$\omega_T(Q_3) = \frac{N_c}{Q_3^2} - \frac{N_c}{2}z_0^2\left(\frac{K_0(Qz_0)}{I_0(Qz_0)} + \frac{K_1(Qz_0)}{I_1(Qz_0)}\right)$$

Longitudinal component: cancellation of the energy-dependent parts (exact in the chiral limit).
The anomaly is the contact term.

The VVA triangle

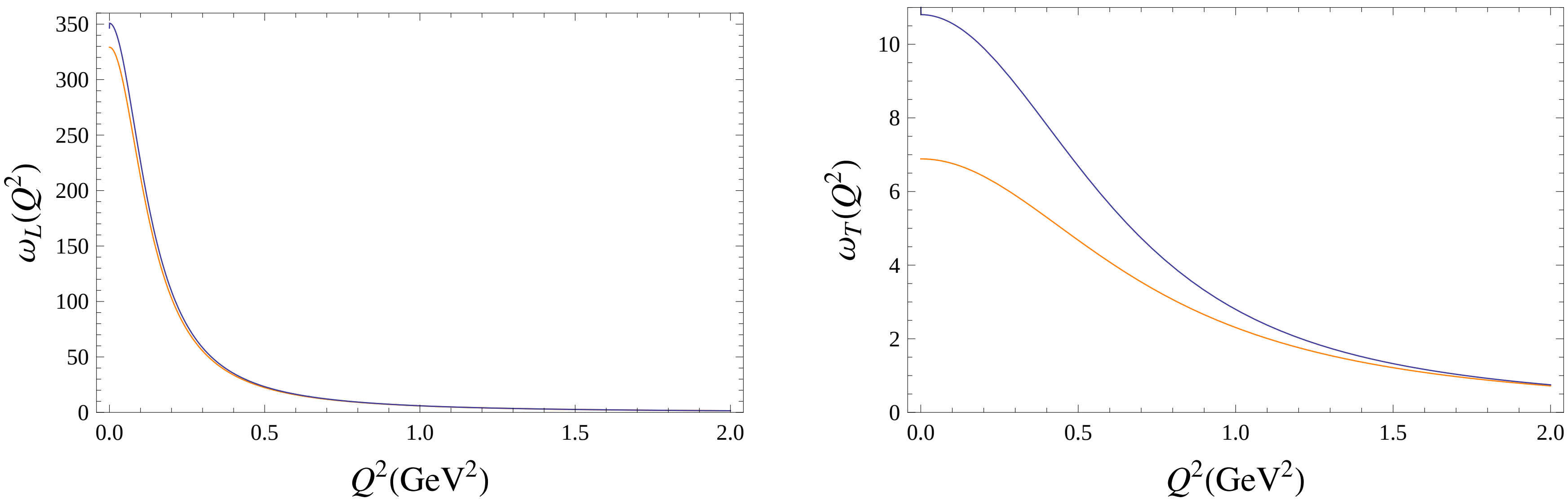
- Notice that one finds, as expected from QCD:

$$\omega_L(Q_3^2) = \frac{2N_c}{Q_3^2} + \mathcal{O}(m_\pi^2)$$

$$\lim_{Q_3\to\infty}\left[\omega_L(Q_3^2) - 2\,\omega_T(Q_3^2)\right] \sim \mathcal{O}\left(\frac{m_\pi^2}{Q_3^6}, Q_4^2\right)$$

- Pion dominance does not hold but it cannot be a bad numerical approximation. Compare with the continuation [Melnikov-Vainshtein'03]

$$\omega_L(Q_3^2) = \frac{2N_c}{Q_3^2 + m_\pi^2}; \qquad \omega_T(Q_3^2) = \frac{N_c}{m_{a_1}^2 - m_\rho^2}\left[\frac{m_{a_1}^2 - m_\pi^2}{Q_3^2 + m_\rho^2} - \frac{m_\rho^2 - m_\pi^2}{Q_3^2 + m_{a_1}^2}\right]$$



Numerical analysis

- Short distances convincingly implemented (parametrically), but numerical limitations of the model:

$$\frac{m_\rho}{f_\pi} \sim 10.7(!)$$

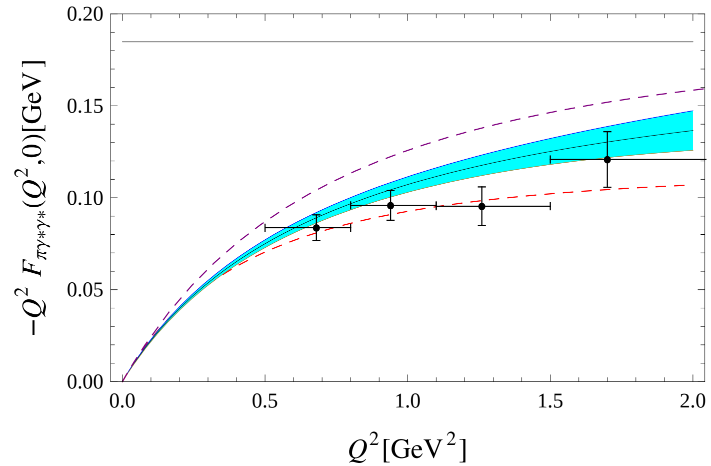
- Fixing m_ρ to the physical value important at low energies to match the slope of $F_{\pi\gamma\gamma}$:

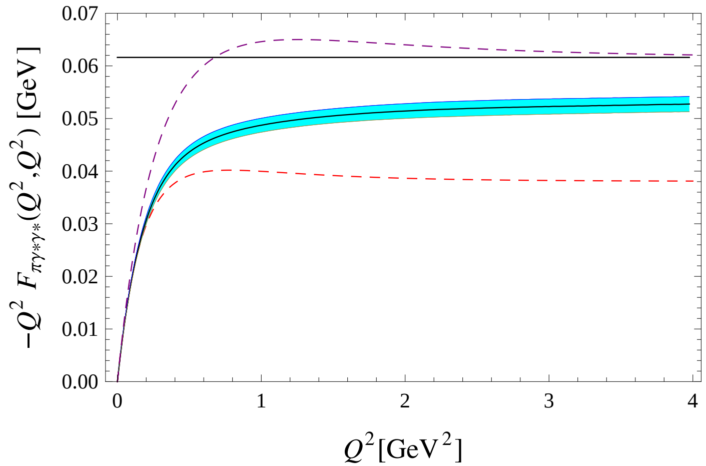
$$a_\pi = -m_\pi^2 \int_0^{z_0} dz\alpha'(z) \left[1 - 2\log\frac{z}{z_0}\right] \frac{z^2}{4} = 0.033$$

- No clear optimal choice of parameters:

$$\frac{f_\pi}{N_c} = 31\,\mathrm{MeV}; \qquad m_\rho = 776\,\mathrm{MeV}\,, \qquad (\textbf{Set 1})$$

$$f_\pi = 93\,\mathrm{MeV}; \qquad N_c = 3\,. \qquad (\textbf{Set 2})$$





Numerical analysis

	Set 1	Set 2
$a_\mu^{\rm PS}(\pi^0+\eta+\eta')$	8.1 (5.7+1.4+1.0)	11.2 (7.5+2.1+1.6)
$a_\mu^{A_L}(a_1+f_1+f_1^*)$	1.4 (0.4+0.4+0.6)	1.4 (0.4+0.4+0.6)
$a_\mu^{L}(a_\mu^{\rm PS}+a_\mu^{A_L})$	9.6	12.6
$a_\mu^{T}(a_1+f_1+f_1^*)$	1.4 (0.4+0.4+0.6)	1.4 (0.4+0.4+0.6)
a_μ	11.0	14.0

$a_\mu^{(\pi)} = 5.7(0.3)\cdot 10^{-10}$ [Hayakawa et al]

$a_\mu^{(\pi)} = 5.9(0.9)\cdot 10^{-10}$ [Bijnens et al]

$a_\mu^{(\pi)} = 5.8(1.0)\cdot 10^{-10}$ [Knecht et al]

$a_\mu^{(\pi)} = 6.8(0.3)\cdot 10^{-10}$ [Greynat et al]

$a_\mu^{(\pi)} = 6.3(0.3)\cdot 10^{-10}$ [Hoferichter et al]

Numerical analysis

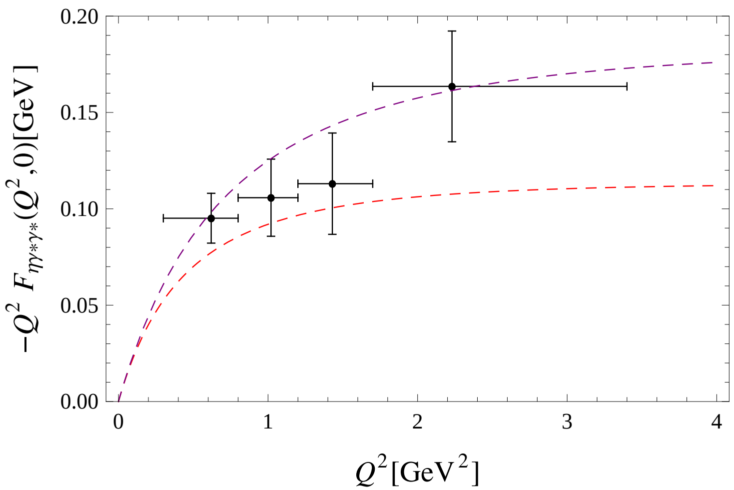
	Set 1	Set 2
$a_\mu^{\rm PS}(\pi^0+\eta+\eta')$	8.1 (5.7+1.4+1.0)	11.2 (7.5+2.1+1.6)

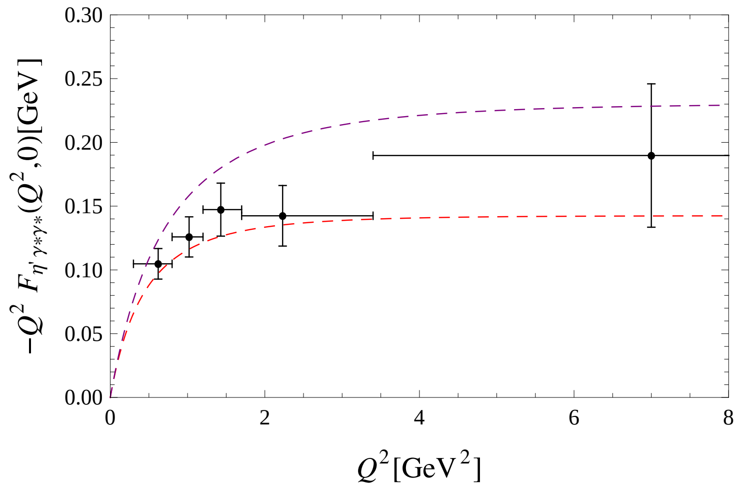
- Change of numerical input (flavour copies), while keeping the correlations between Goldstone and axials (anomaly!):

$$\frac{f_{\eta'}}{N_c} = 24.7\,\mathrm{MeV}; \qquad m_\rho = 776\,\mathrm{MeV}\,, \qquad (\textbf{Set 1})$$
$$f_{\eta'} = 74\,\mathrm{MeV}; \qquad N_c = 3\,, \qquad (\textbf{Set 2})$$

- Previous results: [Knecht et al]

$$a_\mu^{(\eta)} = 1.3(0.1)\cdot 10^{-10}; \qquad a_\mu^{(\eta')} = 1.2(0.1)\cdot 10^{-10}$$





Numerical analysis

	Set 1	**Set 2**
$a_\mu^L(a_\mu^{\rm PS}+a_\mu^{A_L})$	9.6	12.6
$a_\mu^T(a_1+f_1+f_1^*)$	1.4 (0.4+0.4+0.6)	1.4 (0.4+0.4+0.6)
a_μ	11.0	14.0

Final number:

$a_\mu^{(\rm AV+PS)} = 12.5(1.5)\cdot 10^{-10}$

$a_\mu^{(\rm AV+PS)} = 13.6(1.5)\cdot 10^{-10}$ [Melnikov et al]

$a_\mu^{(\rm AV+PS)} = 12.9(2.7)\cdot 10^{-10}$ [Prades et al]

$a_\mu^{(\rm AV+PS)} = 12.1(2.1)\cdot 10^{-10}$ [Jegerlehner et al]

$a_\mu^{(\rm AV+PS)} = 11.0(0.6)\cdot 10^{-10}$ [Leutgeb et al]

Numerical analysis

	Set 1	Set 2
$a_\mu^L(a_\mu^{\rm PS} + a_\mu^{A_L})$	9.6	12.6
$a_\mu^T(a_1 + f_1 + f_1^*)$	1.4 (0.4+0.4+0.6)	1.4 (0.4+0.4+0.6)
a_μ	11.0	14.0

Longitudinal and transverse breakout:

$$a_\mu^L = 11.1(1.5)\cdot 10^{-10} \qquad a_\mu^T = 1.4(0.2)\cdot 10^{-10}$$

$$a_\mu^{\rm (PS)} = 11.4(1.0)\cdot 10^{-10} \qquad a_\mu^{\rm (AV)} = 2.2(0.5)\cdot 10^{-10} \qquad \text{[Melnikov et al]}$$
$$a_\mu^{\rm (PS)} = 11.4(1.3)\cdot 10^{-10} \qquad a_\mu^{\rm (AV)} = 1.5(1.0)\cdot 10^{-10} \qquad \text{[Prades et al]}$$
$$a_\mu^{\rm (PS)} = 9.9(1.6)\cdot 10^{-10} \qquad a_\mu^{\rm (AV)} = 2.2(0.5)\cdot 10^{-10} \qquad \text{[Jegerlehner et al]}$$

However

$$a_\mu^{\rm (PS)} = 9.6(1.6)\cdot 10^{-10}\,; \qquad a_\mu^{\rm (AV)} = 2.8(0.2)\cdot 10^{-10}$$

The axial contribution is underestimated, in benefit of the Goldstone one.

Conclusions

- A (holographic) Lagrangian approach to HLbL provides an inclusive analysis of the leading large-N_c effects. One has access to full correlators and can clarify unresolved issues from form factor analyses. Nice perks: generating functional, simplicity, chiral anomaly and short-distance constraints correctly implemented.

- The chiral anomaly in VVA is saturated by a contact term (structureless). This is the reason for a significant axial increase to HLbL. We exclude a relevant contribution of massive pseudoscalars, as claimed elsewhere.

- The contact term is crucial to fulfill the MV short-distance constraint and is due to a collective effect, i.e. it is not saturated with a form factor. The puzzle of a structureless pion form factor is just a manifestation of this limitation.

- Pion dominance is not compatible with the correct implementation of the chiral anomaly at all energy scales, but numerically it gives an excellent estimate.

- The previous results are generic QFT consequences for the leading large-N_c contributions to HLbL. Specific numbers will of course depend on the model, but the bulk of the number should be model-independent. Our results seem to indicate so.

- Lattice simulations should be able to confirm some of (if not all) the previous points.