Part II

Divergences and their cure

With all this, it seems we have a complete **recipe** to do particle physics:

* Identify the **weakly coupled** degrees of freedom.

* **Choose** an appropriate **interpolating field**.

* Write an **interacting** field theory compatible with the **symmetries** of the system.

* **Compute** the **correlation functions** in **perturbation theory**.

* Use the **LSZ** reduction formula to evaluate **perturbatively** the ***S*-matrix elements** and **cross sections.**

With all this, it seems we have a complete **recipe** to do particle physics:

* Identify the **weakly coupled** degrees of freedom.

* **Choose** an appropriate **interpolating field**.

* Write a ... em.

* **Comp**...

* Use th... **atrix elemen**...

Warning!

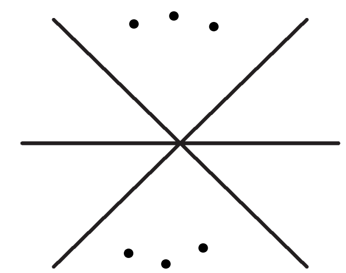
A particle can **exist** in the theory even if there is **no field** associated with it. Particles can appear as **poles** (i.e., **bound states**) in the Green functions of **other fields**.

However, when computing ***S*-matrix elements**, it is **convenient** to introduce them through their **interpolating fields**.

The problem comes when computing **quantum corrections**…

Restoring the powers of $\hbar$, the Feynman rules of a ϕ^n are

$$\Longrightarrow \quad \frac{i\hbar}{p^2 - m^2 + i\varepsilon}$$



$$\Longrightarrow \quad -i\frac{\lambda}{\hbar}$$

The power of $\hbar$ of a diagram with E external lines, I internal propagators, and V vertices is

$$\#(\hbar) = I - V$$

while the **number of loops** in the diagram is

$$L = I - (V - 1) = I - V + 1$$

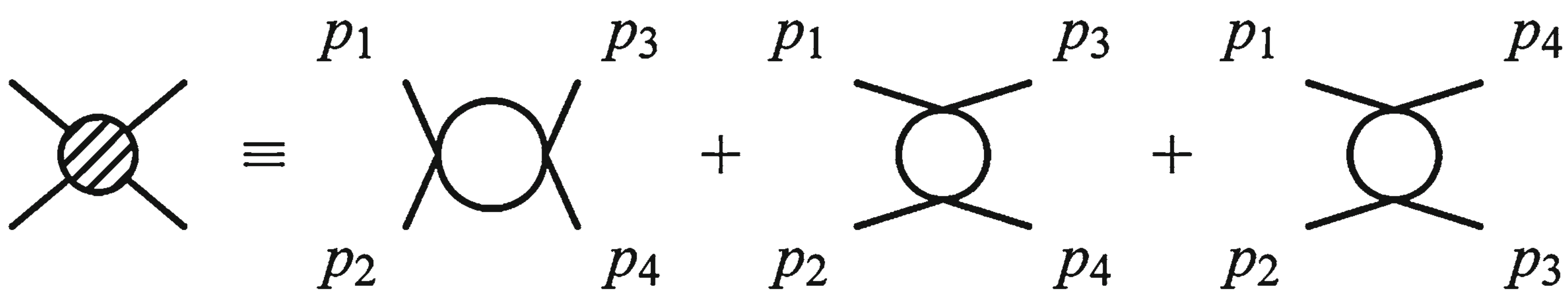
global conservation delta function

\# of integrations

\# of independent delta functions

Thus, $\#(\hbar) = I - V = L - 1$ and an ***L*-loop diagram** scales as $\hbar^{L-1}$

However, **loop diagrams** frequently give **divergent** results.



$$= \frac{\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} \left[\frac{1}{(k+p_1+p_2)^2 - m^2 + i\epsilon} + \frac{1}{(k+p_1+p_3)^2 - m^2 + i\epsilon} + \frac{1}{(k+p_1+p_4)^2 - m^2 + i\epsilon} \right]$$

These integrals are **logarithmically divergent**

$\sim k^3 dk$

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} \frac{1}{(k+p_1+p_2)^2 - m^2 + i\epsilon} \sim \int^{\infty} \frac{dk}{k} \longrightarrow \infty$$

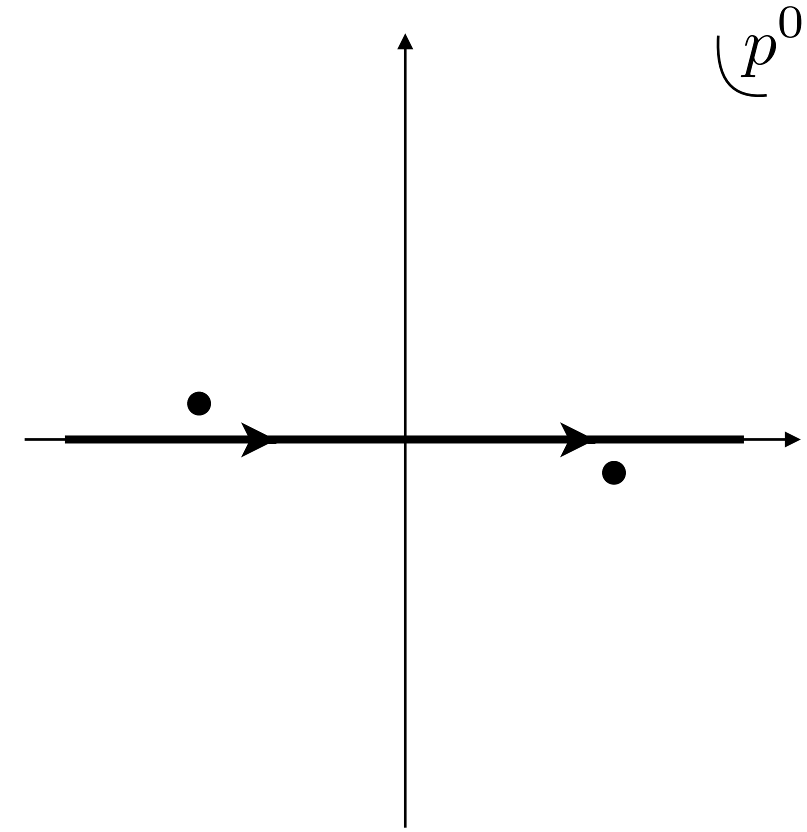
$\sim k^4$

To avoid meaningless results, we need to **regularize** our theory

Let us look at a **typical Feynman integral**:

$$I = \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2 + i\epsilon} \qquad \left(\sim \int^\infty pdp\right)$$

$$= -i \int \frac{d^4\ell_E}{(2\pi)^4} \frac{1}{\ell_E^2 + m^2}$$



There are many **ways** to **make sense** of this. For example:

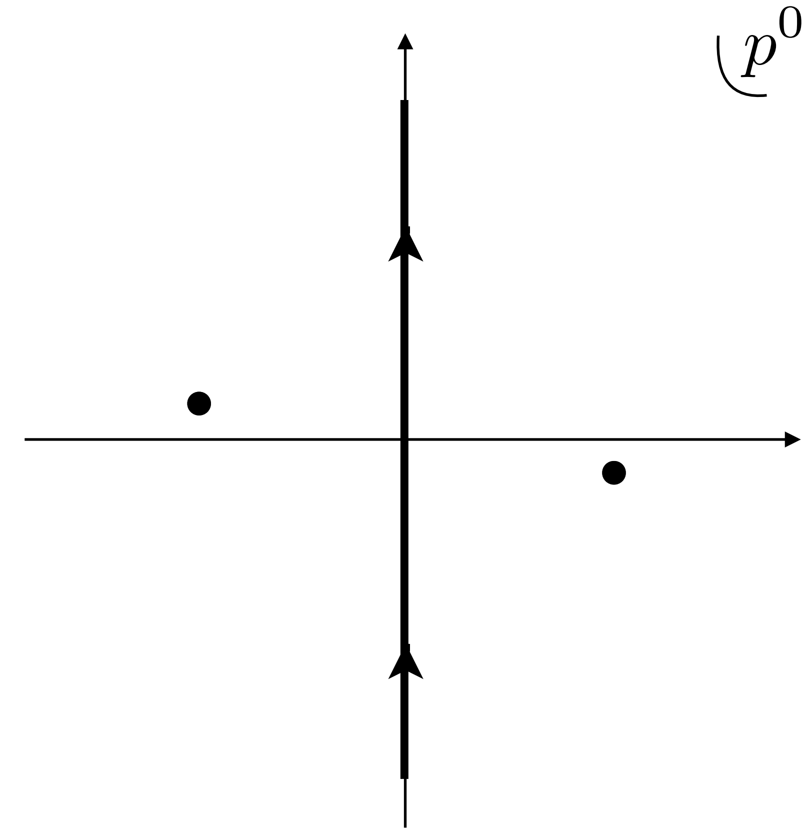
- **Sharp momentum cutoff** Λ

$$I(\Lambda) = -i \int_{|\ell_E|<\Lambda} \frac{d^4\ell_E}{(2\pi)^4} \frac{1}{\ell_E^2 + m^2} \sim \Lambda^2$$

This method, however, **breaks Lorentz** and **gauge invariance**.

To avoid meaningless results, we need to **regularize** our theory

Let us look at a **typical Feynman integral**:



$$I = \int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2 - m^2 + i\epsilon} \qquad \left(\sim \int^{\infty} pdp\right)$$

$$= -i\int \frac{d^4\ell_E}{(2\pi)^4}\frac{1}{\ell_E^2 + m^2}$$

There are many **ways** to **make sense** of this. For example:

- **Sharp momentum cutoff** Λ

$$I(\Lambda) = -i \int\limits_{|\ell_E|<\Lambda} \frac{d^4\ell_E}{(2\pi)^4}\frac{1}{\ell_E^2 + m^2} \sim \Lambda^2$$

This method, however, **breaks Lorentz** and **gauge invariance**.

- **Pauli-Villars method:** introduce a number of fictitious fields with large masses M_i and whose propagators have the **"wrong" sign**

$$I(M_i) = \int \frac{d^4p}{(2\pi)^4}\left(\frac{1}{p^2-m^2+i\epsilon} - \sum_{i=1}^{n}\frac{g_i}{p^2-M_i^2+i\epsilon}\right)$$

$$= -i\int \frac{d^4\ell_E}{(2\pi)^4}\left(\frac{1}{\ell_E^2+m^2} - \sum_{i=1}^{n}\frac{g_i}{\ell_E^2+M_i^2}\right)$$

Wolfgang Pauli (1900-1958)

Felix Villars (1921-2002)

Pauli-Villars regularization is **Lorentz and gauge invariant**, but rather cumbersome.

- **Dimensional regularization:** define the **Feynman integrals in d dimensions** and continue d to complex values.

$$I(d) = \int \frac{d^dp}{(2\pi)^d}\frac{1}{p^2-m^2+i\epsilon} = -i\int\frac{d^d\ell_E}{(2\pi)^d}\frac{1}{\ell_E^2+m^2}$$

This requires the introduction of an **energy scale μ** to preserve the **dimensions** of the coupling constant. E.g., for a scalar ϕ^4 theory $\lambda \longrightarrow \mu^{4-d}\lambda$

Dimensional regularization **preserves Lorentz and gauge invariance**, but one has to be **careful** when working with **chiral theories**!

Once the theory is **regularized**, we can compute **finite** scattering amplitudes

$$i\mathscr{M} = f(p_i; \lambda, m, \Lambda)$$

Once the theory is **regularized**, we can compute **finite** scattering amplitudes

external momenta

$$i\mathscr{M} = f(p_i; \lambda, m, \Lambda) \xrightarrow{\Lambda \to \infty} \infty$$

couplings

masses

cutoff

Once the theory is **regularized**, we can compute **finite** scattering amplitudes

external momenta

$$i\mathscr{M} = f(p_i; \lambda, m, \Lambda) \xrightarrow{\Lambda \to \infty} \infty$$

couplings, masses, cutoff

Hendrik A. Kramers
(1894-1952)

To handle the theory, we introduce the notion of **renormalization**:

* Only **measurable** quantities are **physical**.

* The **quantities** appearing in the **Lagrangian** (masses, couplings, fields, etc.) are **unphysical**.

$$\phi_0(x) = \sqrt{Z(\Lambda)}\phi(x)$$

* Divergences are **“absorbed”** in the unphysical parameters

$$i\mathscr{M} = f\Big(p_i; \lambda_0(\Lambda), m_0(\Lambda), \Lambda\Big) \xrightarrow{\Lambda \to \infty} f(p_i; \lambda, m)$$

renormalized quantities

* The cutoff dependence of the parameters is fixed by the **definition** of physical quantities (**renormalization conditions**).

Let us apply this program to a **scalar** ϕ^4 theory. The **renormalized Lagrangian** is

$$\mathscr{L}_{\rm ren} = \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{m_0(\Lambda)^2}{2}\phi_0^2 - \frac{\lambda(\Lambda)}{4!}\phi_0^4$$

$$= \frac{1}{2}Z(\Lambda)\partial_\mu\phi\partial^\mu\phi - \frac{m_0(\Lambda)^2 Z(\Lambda)}{2}\phi^2 - \frac{\lambda(\Lambda)Z(\Lambda)^2}{4!}\phi^4$$

$$\phi_0(x) = \sqrt{Z(\Lambda)}\phi(x)$$

It can be rewritten in terms of the **finite**, **renormalized**, **masses** and **couplings** as

$$\mathscr{L}_{\rm ren} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 + \frac{1}{2}\delta_Z\partial_\mu\phi\partial^\mu\phi - \frac{\delta_m}{2}\phi^2 + \frac{\delta_\lambda}{2}\phi^4$$

counterterms

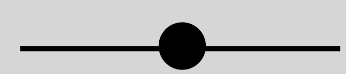
where

$$Z(\Lambda) = 1 + \delta_Z(\Lambda)$$

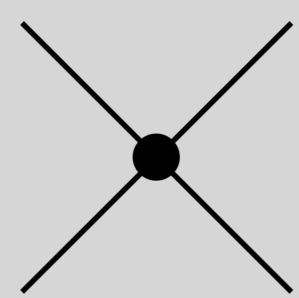
$$m_0(\Lambda)^2 = m^2 + \delta_m(\Lambda)$$

$$\lambda_0(\Lambda) = \lambda + \delta_\lambda(\Lambda)$$

Feynman rules for counterterms



$$= i(p^2\delta_Z - \delta_m)$$



$$= -i\delta_\lambda$$

$$\mathscr{L}_{\rm ren} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 + \frac{1}{2}\delta_Z\partial_\mu\phi\partial^\mu\phi - \frac{\delta_m}{2}\phi^2 + \frac{\delta_\lambda}{2}\phi^4$$

By construction, **quantities** computed from the **renormalized Lagrangian** are **finite**. Renormalization can now be **systematically** implemented:

- **Regularize** the theory.

- **Compute** loop diagrams using the Lagrangian

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

- Fix the **counterterms** to eliminate the **divergences** at each loop level.

- **Evaluate physical quantities** in terms of finite **renormalized parameters**.

- **Compute amplitudes**

Let us look at it **hands-on**: ϕ^4 at one loop.

At one loop there are two divergent diagrams by **power counting**:


$\sim \Lambda^2$


$\sim \log\Lambda$

Using a **hard cutoff**, we have

p 

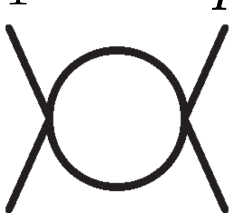
$$= -\frac{i\lambda}{2}\int^{\Lambda}\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon} = -\frac{i\lambda}{2}\int_{|\ell_E|<\Lambda}\frac{d^4\ell_E}{(2\pi)^4}\frac{1}{\ell_E^2+m^2}$$

$$= -\frac{im^2\lambda}{32\pi^2}\left[\frac{\Lambda^2}{m^2}-\log\left(\frac{\Lambda^2}{m^2}\right)\right] + \text{finite piece}$$

Mandelstam variables

$$s = (p_1+p_2)^2$$
$$t = (p_1-p_3)^2$$
$$u = (p_1-p_4)^2$$

p_1 p_3



p_2 p_4

+crossed

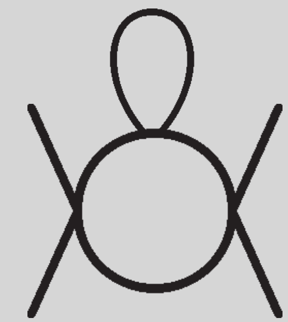
$$= \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left\{\log\left[\frac{\Lambda^2}{m^2-x(1-x)s}\right]+\log\left[\frac{\Lambda^2}{m^2-x(1-x)t}\right]\right.$$
$$\left.+\log\left[\frac{\Lambda^2}{m^2-x(1-x)u}\right]\right\} + \text{finite piece}$$

Let us look at it **hands-on**: ϕ^4 at one loop.

At one loop there are two divergent diagrams by **power**

$$\sim \Lambda^2$$

Diagrams with subdivergences



are dealt with by renormalizing the divergent subdiagram.

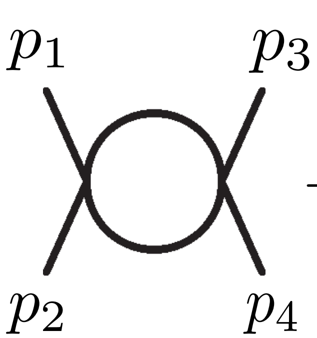
Using a **hard cutoff**, we have

$$= -\frac{i\lambda}{2}\int^{\Lambda}\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon} = -\frac{i\lambda}{2}\int_{|\ell_E|<\Lambda}\frac{d^4\ell_E}{(2\pi)^4}\frac{1}{\ell_E^2+m^2}$$

$$= -\frac{im^2\lambda}{32\pi^2}\left[\frac{\Lambda^2}{m^2}-\log\left(\frac{\Lambda^2}{m^2}\right)\right] + \text{finite piece}$$

Mandelstam variables

$$s=(p_1+p_2)^2$$
$$t=(p_1-p_3)^2$$
$$u=(p_1-p_4)^2$$



$$+\text{crossed} \;=\; \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left\{\log\left[\frac{\Lambda^2}{m^2-x(1-x)s}\right]+\log\left[\frac{\Lambda^2}{m^2-x(1-x)t}\right]\right.$$

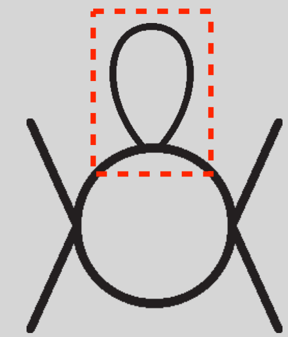
$$\left.+\log\left[\frac{\Lambda^2}{m^2-x(1-x)u}\right]\right\} + \text{finite piece}$$

Let us look at it **hands-on**: ϕ^4 at one loop.

At one loop there are two divergent diagrams by **power**

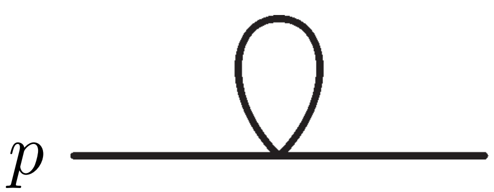
$$\sim \Lambda^2$$

Diagrams with subdivergences



are dealt with by renormalizing the divergent subdiagram.

Using a **hard cutoff**, we have

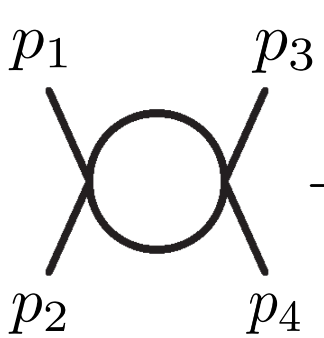


$$= -\frac{i\lambda}{2}\int^{\Lambda}\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon} = -\frac{i\lambda}{2}\int_{|\ell_E|<\Lambda}\frac{d^4\ell_E}{(2\pi)^4}\frac{1}{\ell_E^2+m^2}$$

$$= -\frac{im^2\lambda}{32\pi^2}\left[\frac{\Lambda^2}{m^2}-\log\left(\frac{\Lambda^2}{m^2}\right)\right] + \text{finite piece}$$

Mandelstam variables

$$s=(p_1+p_2)^2$$
$$t=(p_1-p_3)^2$$
$$u=(p_1-p_4)^2$$



$$+\text{crossed} \quad = \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left\{\log\left[\frac{\Lambda^2}{m^2-x(1-x)s}\right]+\log\left[\frac{\Lambda^2}{m^2-x(1-x)t}\right]\right.$$

$$\left.+\log\left[\frac{\Lambda^2}{m^2-x(1-x)u}\right]\right\} + \text{finite piece}$$

Let us look at it **hands-on**: ϕ^4 at one loop.

At one loop there are two divergent diagrams by **power counting**:

$\sim \Lambda^2$

$\sim \log\Lambda$

Using a **hard cutoff**, we have

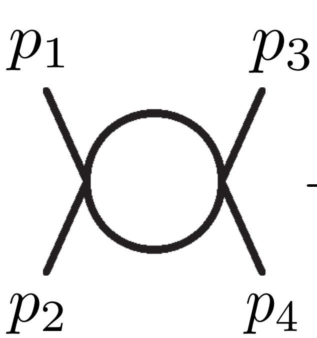
p 

$$= -\frac{i\lambda}{2}\int^{\Lambda}\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon} = -\frac{i\lambda}{2}\int_{|\ell_E|<\Lambda}\frac{d^4\ell_E}{(2\pi)^4}\frac{1}{\ell_E^2+m^2}$$

$$= -\frac{im^2\lambda}{32\pi^2}\left[\frac{\Lambda^2}{m^2}-\log\left(\frac{\Lambda^2}{m^2}\right)\right] + \text{finite piece}$$

Mandelstam variables

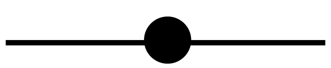
$$s = (p_1+p_2)^2$$
$$t = (p_1-p_3)^2$$
$$u = (p_1-p_4)^2$$



$$+\text{crossed} = \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left\{\log\left[\frac{\Lambda^2}{m^2-x(1-x)s}\right]+\log\left[\frac{\Lambda^2}{m^2-x(1-x)t}\right]\right.$$
$$\left.+\log\left[\frac{\Lambda^2}{m^2-x(1-x)u}\right]\right\} + \text{finite piece}$$

$$= -\frac{im^2\lambda}{32\pi^2}\left[\frac{\Lambda^2}{m^2} - \log\left(\frac{\Lambda^2}{m^2}\right)\right] + \text{finite piece}$$

From this result we can **identify** two of the **counterterms** at **one loop**:



$$= i(p^2\delta_Z - \delta_m) \quad \longrightarrow \quad \begin{cases} \delta_Z\Big|_{1-\text{loop}} = 0 \\ \delta_m\Big|_{1-\text{loop}} = -\dfrac{m^2\lambda}{32\pi^2}\left[\dfrac{\Lambda^2}{m^2} - \log\left(\dfrac{\Lambda^2}{\mu^2}\right)\right] \end{cases}$$

where we have introduced an **arbitrary energy scale** μ. The **"bare"**, cutoff-dependent **mass at one loop** to be

$$m_0(\Lambda)^2 = m^2 + \delta_m(\Lambda) \quad \longrightarrow \quad m_0(\Lambda)^2 = m^2\left\{1 - \frac{\lambda}{32\pi^2}\left[\frac{\Lambda^2}{m^2} - \log\left(\frac{\Lambda^2}{\mu^2}\right)\right]\right\}$$

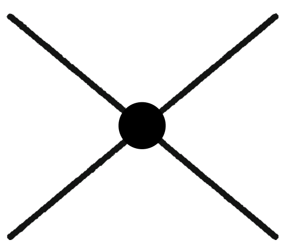
and

$$Z(\Lambda) = 1 + \delta_Z(\Lambda) \quad \longrightarrow \quad Z(\Lambda) = 1$$

no field renormalization at one loop!

$$+\text{crossed} = \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left\{\log\left[\frac{\Lambda^2}{m^2-x(1-x)s}\right]+\log\left[\frac{\Lambda^2}{m^2-x(1-x)t}\right] + \log\left[\frac{\Lambda^2}{m^2-x(1-x)u}\right]\right\} + \text{finite piece}$$

The logarithmic **divergence** is **cancelled** by choosing the **counterterm**



$$= -i\delta_\lambda \quad \longrightarrow \quad \delta_\lambda\Big|_{1-\text{loop}} = \frac{3\lambda^2}{32\pi^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)$$

where μ is an **arbitrary energy scale**.

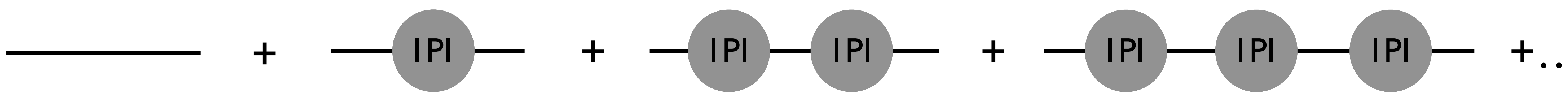
The "**bare**" coupling constant at **one-loop** is:

$$\lambda_0(\Lambda) = \lambda + \delta_\lambda(\Lambda) \quad \longrightarrow \quad \lambda_0(\Lambda) = \lambda + \frac{3\lambda^2}{32\pi^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)$$

Warning!!! Renormalized quantities are **not** necessarily **physical**!

Physical quantities are **defined operationally**. Let us look a the **mass**.

In general, the two-point function (**full propagator**) is given by



$$= \frac{i}{p^2 - m^2} + \frac{i}{p^2 - m^2}\Pi(p^2)\frac{i}{p^2 - m^2} + \frac{i}{p^2 - m^2}\Pi(p^2)\frac{i}{p^2 - m^2}\Pi(p^2)\frac{i}{p^2 - m^2} + \ldots$$

$$= \frac{i}{p^2 - m^2}\sum_{n=0}^{\infty}\left[\Pi(p^2)\frac{i}{p^2 - m^2}\right]^n = \frac{i}{p^2 - m^2}\frac{1}{1 - \Pi(p^2)\frac{i}{p^2 - m^2}}$$

$$= \frac{i}{p^2 - m^2 - i\Pi(p^2)}$$

We can define the **physical mass** as the **pole** of the **full propagator**

physical mass

$$m^2_{\text{phys}} - m^2 - i\Pi(m^2_{\text{phys}}) = 0$$

renormalized mass

mass renormalization condition

Warning!!! Renormalized quantities are **not** necessarily **physical**!

Physical quantities are **defined operationally**. Let us look a the **<u>mass</u>**.

In general, the two-point function (**full propagator**) is given by

In fact, we **also** have to require that the **residue** at the pole **equals** i

$$p^2 - m^2 - i\Pi(p^2) = \left(1 - i\frac{d\Pi}{dp^2}\bigg|_{p^2=m^2_{\text{phys}}}\right)(p^2 - m^2_{\text{phys}}) + \ldots$$

thus,

$$\frac{d\Pi}{dp^2}\bigg|_{p^2=m^2_{\text{phys}}} = 0$$

$\ldots$

$m^2 + \ldots$

$$= \frac{i}{p^2 - m^2 - i\Pi(p^2)}$$

We can define the **physical mass** as the **pole** of the **full propagator**

physical mass

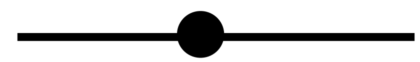
$$m^2_{\text{phys}} - m^2 - i\Pi(m^2_{\text{phys}}) = 0$$

renormalized mass

mass renormalization condition

$$m^2_{\text{phys}} - m^2 - i\Pi(m^2_{\text{phys}}) = 0$$

From our loop **calculation**,

$\Pi(p^2)_{1-\text{loop}} =$  + 

$$= -\frac{im^2\lambda}{32\pi^2}\left[\frac{\Lambda^2}{m^2} - \log\left(\frac{\Lambda^2}{m^2}\right)\right] + \frac{im^2\lambda}{32\pi^2}\left[\frac{\Lambda^2}{m^2} - \log\left(\frac{\Lambda^2}{\mu^2}\right)\right]$$

$$= -\frac{im^2\lambda}{32\pi^2}\log\left(\frac{m^2}{\mu^2}\right)$$

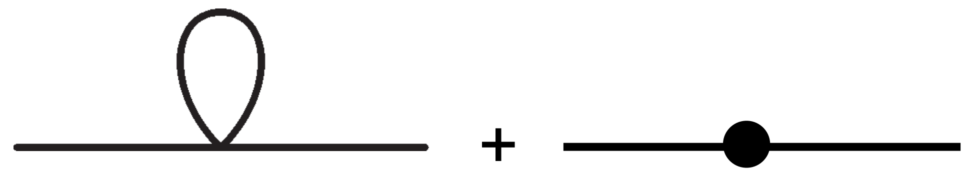
which **momentum independent**. Thus, the **physical mass** is given in terms of the **renormalized parameters** $\boldsymbol{m}$ and $\boldsymbol{\lambda}$ by

$$m^2_{\text{phys}} = m^2\left[1 + \frac{\lambda}{32\pi^2}\log\left(\frac{m^2}{\mu^2}\right)\right]$$

which is **independent** on the (unphysical) momentum **cutoff**.

$$m_{\text{phys}}^2 - m^2 - i\Pi(m_{\text{phys}}^2) = 0$$

From our loop **calculation**,

$\Pi(p^2)_{1-\text{loop}} =$ 

$$= -\frac{im^2\lambda}{32\pi^2}\left[\frac{\Lambda^2}{m^2} - \log\left(\frac{\Lambda^2}{m^2}\right)\right] + \frac{im^2\lambda}{32\pi^2}\left[\frac{\Lambda^2}{m^2} - \log\left(\frac{\Lambda^2}{\mu^2}\right)\right]$$

$$= -\frac{im^2\lambda}{32\pi^2}\log\left(\frac{m^2}{\mu^2}\right)$$

at one loop: $\frac{d\Pi}{dp^2} \equiv 0$

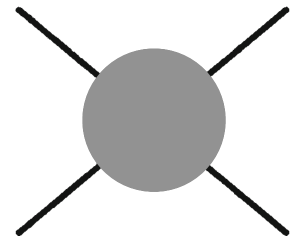
which **momentum independent**. Thus, the **physical mass** is given in terms of the **renormalized parameters** m and λ by

$$m_{\text{phys}}^2 = m^2\left[1 + \frac{\lambda}{32\pi^2}\log\left(\frac{m^2}{\mu^2}\right)\right]$$

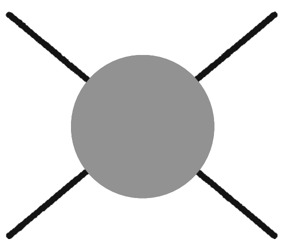
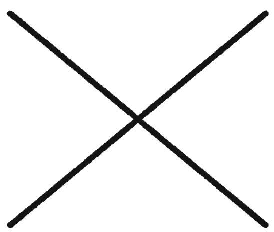
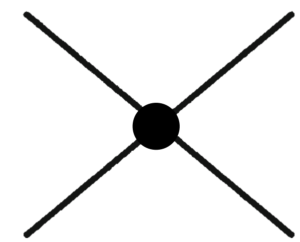
which is **independent** on the (unphysical) momentum **cutoff**.

Next we look at the **<u>coupling constant</u>**.

We can define the **physical coupling constant**, for example, as

$-i\lambda_{\text{phys}} \equiv$  $\Big|_{\substack{s=4m^2 \\ t=u=0}}$

From our calculation

 $=$  $+$  $+$ crossed $+$ 

$$= -i\lambda + \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left\{\log\left[\frac{\Lambda^2}{m^2 - x(1-x)s}\right] + \log\left[\frac{\Lambda^2}{m^2 - x(1-x)t}\right]\right.$$

$$\left. + \log\left[\frac{\Lambda^2}{m^2 - x(1-x)u}\right]\right\} - \frac{3i\lambda^2}{32\pi^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)$$

$$\begin{pmatrix} s = 4m^2 \\ t = u = 0 \end{pmatrix} = -i\lambda + \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left\{\log\left[\frac{\mu^2}{m^2(1-2x)^2}\right] + 2\log\left(\frac{\mu^2}{m^2}\right)\right\}$$

$$-i\lambda_{\rm phys} = -i\lambda + \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left\{\log\left[\frac{\mu^2}{m^2(1-2x)^2}\right] + 2\log\left(\frac{\mu^2}{m^2}\right)\right\}$$

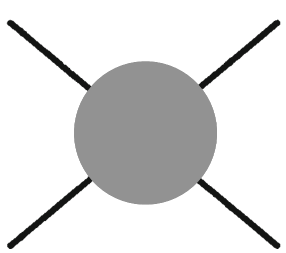
$$= -i\lambda + \frac{3i\lambda^2}{32\pi^2}\int_0^1 dx\left[\log\left(\frac{\mu^2}{m^2}\right) - \frac{1}{3}\log(1-2x)^2\right]$$

$$\int_0^1 dx\,\log(1-2x)^2 = -2$$

$$\lambda_{\rm phys} = \lambda - \frac{\lambda^2}{16\pi^2}\left[1 + \frac{3}{2}\log\left(\frac{\mu^2}{m^2}\right)\right]$$

Other definitions of the physical coupling lead to **different results**. For example:

$-i\lambda_{\rm phys} \equiv$



$\Big|_{s=t=u=\frac{4}{3}m^2}$

$$\lambda_{\rm phys} = \lambda - \frac{3\lambda^2}{32\pi^2}\left[2\left(1-\sqrt{2}\arctan\frac{1}{\sqrt{2}}\right) + \log\left(\frac{\mu^2}{m^2}\right)\right]$$

$$m^2_{\rm phys} = m^2\left[1+\frac{\lambda}{32\pi^2}\log\left(\frac{m^2}{\mu^2}\right)\right] \qquad \lambda_{\rm phys} = \lambda - \frac{\lambda^2}{16\pi^2}\left[1+\frac{3}{2}\log\left(\frac{\mu^2}{m^2}\right)\right]$$

Physical quantities **cannot depend** on the fiducial scale μ. The **explicit dependence** is **compensated** by the one of the **renormalized parameters**.

Let us begin with the **coupling**

$$\mu\frac{d\lambda_{\rm phys}}{d\mu} = 0$$

$$\Downarrow$$

$$\left(\mu\frac{d\lambda}{d\mu}\right) - \frac{\lambda}{8\pi^2}\left(\mu\frac{d\lambda}{d\mu}\right)\left[1+\frac{3}{2}\log\left(\frac{\mu^2}{m^2}\right)\right] - \frac{3\lambda^2}{16\pi^2} = 0$$

At **leading order** in **λ**

$$\mu\frac{d\lambda}{d\mu} - \frac{3\lambda^2}{16\pi^2} = 0 \quad \Longrightarrow \quad \beta(\lambda) \equiv \mu\frac{d\lambda}{d\mu} = \frac{3\lambda^2}{16\pi^2}$$

This defines the **beta function**.

$$m^2_{\text{phys}} = m^2\left[1+\frac{\lambda}{32\pi^2}\log\left(\frac{m^2}{\mu^2}\right)\right] \qquad \lambda_{\text{phys}} = \lambda - \frac{\lambda^2}{16\pi^2}\left[1+\frac{3}{2}\log\left(\frac{\mu^2}{m^2}\right)\right]$$

Physical quantities **cannot depend** on the fiducial scale μ. The **explicit dependence** is **compensated** by the one of the **renormalized parameters**.

Let us begin with the **coupling**

$$\mu\frac{d\lambda_{\text{phys}}}{d\mu} = 0$$

$$\Downarrow$$

$$\left(\mu\frac{d\lambda}{d\mu}\right) - \frac{\lambda}{8\pi^2}\left(\mu\frac{d\lambda}{d\mu}\right)\left[1+\frac{3}{2}\log\left(\frac{\mu^2}{m^2}\right)\right] - \frac{3\lambda^2}{16\pi^2} = 0$$

At **leading order** in **λ**

$$\mu\frac{d\lambda}{d\mu} - \frac{3\lambda^2}{16\pi^2} = 0 \quad \Longrightarrow \quad \beta(\lambda) \equiv \mu\frac{d\lambda}{d\mu} = \frac{3\lambda^2}{16\pi^2}$$

This defines the **beta function**.

$$m^2_{\rm phys} = m^2\left[1+\frac{\lambda}{32\pi^2}\log\left(\frac{m^2}{\mu^2}\right)\right] \qquad\qquad \lambda_{\rm phys} = \lambda - \frac{\lambda^2}{16\pi^2}\left[1+\frac{3}{2}\log\left(\frac{\mu^2}{m^2}\right)\right]$$

Next we deal with the **physical mass**

$$\mu\frac{dm^2_{\rm phys}}{d\mu} = 0$$

$$\left(\mu\frac{dm^2}{d\mu}\right)\left[1+\frac{\lambda}{32\pi^2}\log\left(\frac{m^2}{\mu^2}\right)\right] + m^2\left[\frac{1}{32\pi^2}\left(\mu\frac{d\lambda}{d\mu}\right)\log\left(\frac{m^2}{\mu^2}\right) + \frac{\lambda}{32\pi^2 m^2}\left(\mu\frac{dm^2}{d\mu}\right) - \frac{\lambda}{16\pi^2}\right] = 0$$

Dropping **subleading** terms in **λ**

$$\mu\frac{dm^2}{d\mu} - \frac{\lambda m^2}{16\pi^2} = 0 \quad \longrightarrow \quad \gamma_{m^2}(\lambda) \equiv \frac{\mu}{m^2}\frac{dm^2}{d\mu} = \frac{\lambda}{16\pi^2}$$

with is the **Callan-Symanzik gamma function**.

$$m^2_{\text{phys}} = m^2 \left[1 + \frac{\lambda}{32\pi^2}\log\left(\frac{m^2}{\mu^2}\right)\right] \qquad \lambda_{\text{phys}} = \lambda - \frac{\lambda^2}{16\pi^2}\left[1 + \frac{3}{2}\log\left(\frac{\mu^2}{m^2}\right)\right]$$

Next we deal with the **physical mass**

$$\mu\frac{dm^2_{\text{phys}}}{d\mu} = 0$$

$$\mu\frac{d\lambda}{d\mu} = \frac{3\lambda^2}{16\pi^2}$$

$$\left(\mu\frac{dm^2}{d\mu}\right)\left[1 + \frac{\lambda}{32\pi^2}\log\left(\frac{m^2}{\mu^2}\right)\right] + m^2\left[\frac{1}{32\pi^2}\left(\mu\frac{d\lambda}{d\mu}\right)\log\left(\frac{m^2}{\mu^2}\right) + \frac{\lambda}{32\pi^2 m^2}\left(\mu\frac{dm^2}{d\mu}\right) - \frac{\lambda}{16\pi^2}\right] = 0$$

Dropping **subleading** terms in **λ**

$$\mu\frac{dm^2}{d\mu} - \frac{\lambda m^2}{16\pi^2} = 0 \qquad \longrightarrow \qquad \gamma_{m^2}(\lambda) \equiv \frac{\mu}{m^2}\frac{dm^2}{d\mu} = \frac{\lambda}{16\pi^2}$$

with is the **Callan-Symanzik gamma function**.

$$m^2_{\text{phys}} = m^2\left[1+\frac{\lambda}{32\pi^2}\log\left(\frac{m^2}{\mu^2}\right)\right] \qquad \lambda_{\text{phys}} = \lambda - \frac{\lambda^2}{16\pi^2}\left[1+\frac{3}{2}\log\left(\frac{\mu^2}{m^2}\right)\right]$$

N

There is a **further relevant function** to be defined

$$\gamma(\lambda) \equiv \frac{1}{2}\mu\frac{d}{d\mu}\log Z$$

but at **one loop** for the ϕ^4 theory

$$\gamma(\lambda) = 0$$

(no field renormalization)

$$0$$

$$\mu\frac{d\lambda}{d\mu} = \frac{3\lambda^2}{16\pi^2}$$

$$\ldots\text{og}\left(\frac{m^2}{\mu^2}\right) + \frac{\lambda}{32\pi^2 m^2}\left(\mu\frac{dm^2}{d\mu}\right) - \frac{\lambda}{16\pi^2}\right] = 0$$

Dropping **subleading** terms in **λ**

$$\mu\frac{dm^2}{d\mu} - \frac{\lambda m^2}{16\pi^2} = 0 \quad \longrightarrow \quad \gamma_{m^2}(\lambda) \equiv \frac{\mu}{m^2}\frac{dm^2}{d\mu} = \frac{\lambda}{16\pi^2}$$

with is the **Callan-Symanzik gamma function**.

We can now compute the **four-point amplitude** in terms of our **physical** quantities:

$$m_{\text{phys}}^2 = m^2\left[1+\frac{\lambda}{32\pi^2}\log\left(\frac{m^2}{\mu^2}\right)\right] \qquad \lambda_{\text{phys}} = \lambda - \frac{\lambda^2}{16\pi^2}\left[1+\frac{3}{2}\log\left(\frac{\mu^2}{m^2}\right)\right]$$

Inverting them at **this order**, we have

$$m^2 = m_{\text{phys}}^2\left[1-\frac{\lambda_{\text{phys}}}{32\pi^2}\log\left(\frac{m_{\text{phys}}^2}{\mu^2}\right)\right] \qquad \lambda = \lambda_{\text{phys}} + \frac{\lambda_{\text{phys}}^2}{16\pi^2}\left[1+\frac{3}{2}\log\left(\frac{\mu^2}{m_{\text{phys}}^2}\right)\right]$$

while for the **amplitude** we have found

$$= -i\lambda + \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left\{\log\left[\frac{\mu^2}{m^2-x(1-x)s}\right]+\log\left[\frac{\mu^2}{m^2-x(1-x)t}\right]\right.$$

$$\left.+\log\left[\frac{\mu^2}{m^2-x(1-x)u}\right]\right\}$$

$$i\mathcal{M} = -i\lambda + \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left\{\log\left[\frac{\mu^2}{m^2 - x(1-x)s}\right] + \log\left[\frac{\mu^2}{m^2 - x(1-x)t}\right] + \log\left[\frac{\mu^2}{m^2 - x(1-x)u}\right]\right\}$$

$$m^2 = m^2_{\text{phys}}\left[1 - \frac{\lambda_{\text{phys}}}{32\pi^2}\log\left(\frac{m^2_{\text{phys}}}{\mu^2}\right)\right]$$

$$\lambda = \lambda_{\text{phys}} + \frac{\lambda^2_{\text{phys}}}{16\pi^2}\left[1 + \frac{3}{2}\log\left(\frac{\mu^2}{m^2_{\text{phys}}}\right)\right]$$

At order $\boldsymbol{\lambda^2}$ the corrections to the **mass** are **irrelevant**, thus

$$i\mathcal{M}(s,t,u) = -i\lambda_{\text{phys}} + \frac{i\lambda^2_{\text{phys}}}{32\pi^2}\int_0^1 dx\left\{\log\left[\frac{m^2_{\text{phys}}}{m^2_{\text{phys}} - x(1-x)s}\right] + \log\left[\frac{m^2_{\text{phys}}}{m^2_{\text{phys}} - x(1-x)t}\right]\right.$$

$$\left. + \log\left[\frac{m^2_{\text{phys}}}{m^2_{\text{phys}} - x(1-x)u}\right] - 2\right\}$$

The result is **independent of $\boldsymbol{\mu}$** and satisfies the **renormalization condition**

$$i\mathcal{M}(4m^2_{\text{phys}}, 0, 0) = -i\lambda_{\text{phys}}$$

$$i\mathcal{M} = -i\lambda + \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left\{\log\left[\frac{\mu^2}{m^2 - x(1-x)s}\right] + \log\left[\frac{\mu^2}{m^2 - x(1-x)t}\right] + \log\left[\frac{\mu^2}{m^2 - x(1-x)u}\right]\right\}$$

$$m^2 = m^2_{\rm phys}\left[1 - \frac{\lambda_{\rm phys}}{32\pi^2}\log\left(\frac{m^2_{\rm phys}}{\mu^2}\right)\right]$$

$$\lambda = \lambda_{\rm phys} + \frac{\lambda^2_{\rm phys}}{16\pi^2}\left[1 + \frac{3}{2}\log\left(\frac{\mu^2}{m^2_{\rm phys}}\right)\right]$$

At order $\boldsymbol{\lambda^2}$ the corrections to the **mass** are **irrelevant**, thus

$$i\mathcal{M}(s,t,u) = -i\lambda_{\rm phys} + \frac{i\lambda^2_{\rm phys}}{32\pi^2}\int_0^1 dx\left\{\log\left[\frac{m^2_{\rm phys}}{m^2_{\rm phys} - x(1-x)s}\right] + \log\left[\frac{m^2_{\rm phys}}{m^2_{\rm phys} - x(1-x)t}\right] + \log\left[\frac{m^2_{\rm phys}}{m^2_{\rm phys} - x(1-x)u}\right] - 2\right\}$$

The result is **independent of $\boldsymbol{\mu}$** and satisfies the **renormalization condition**

$$i\mathcal{M}(4m^2_{\rm phys}, 0, 0) = -i\lambda_{\rm phys}$$

$$i\mathcal{M} = -i\lambda + \frac{i\lambda^2}{32\pi^2}\int_0^1 dx \left\{\log\left[\frac{\mu^2}{m^2 - x(1-x)s}\right] + \log\left[\frac{\mu^2}{m^2 - x(1-x)t}\right] + \log\left[\frac{\mu^2}{m^2 - x(1-x)u}\right]\right\}$$

$$m^2 = m^2_{\rm phys}\left[1 - \frac{\lambda_{\rm phys}}{32\pi^2}\log\left(\frac{m^2_{\rm phys}}{\mu^2}\right)\right]$$

$$\lambda = \lambda_{\rm phys} + \frac{\lambda^2_{\rm phys}}{16\pi^2}\left[1 + \frac{3}{2}\log\left(\frac{\mu^2}{m^2_{\rm phys}}\right)\right]$$

At order $\boldsymbol{\lambda^2}$ the corrections to the **mass** are **irrelevant**, thus

$$= -\log(1-2x)^2 \qquad\qquad = 0$$

$$i\mathcal{M}(s,t,u) = -i\lambda_{\rm phys} + \frac{i\lambda^2_{\rm phys}}{32\pi^2}\int_0^1 dx \left\{\log\left[\frac{m^2_{\rm phys}}{m^2_{\rm phys} - x(1-x)s}\right] + \log\left[\frac{m^2_{\rm phys}}{m^2_{\rm phys} - x(1-x)t}\right]\right.$$

$$= 0$$

$$\left. + \log\left[\frac{m^2_{\rm phys}}{m^2_{\rm phys} - x(1-x)u}\right] - 2\right\}$$

$s = 4m^2_{\rm phys}$

$t = u = 0$

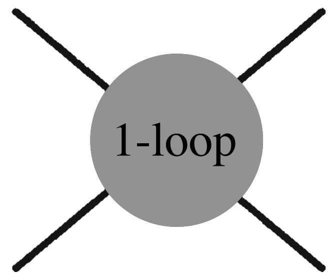
The result is **independent of μ** and satisfies the **renormalization condition**

$$i\mathcal{M}(4m^2_{\rm phys}, 0, 0) = -i\lambda_{\rm phys}$$

$$\int_0^1 dx\, \log(1-2x)^2 = -2$$

Effectively, once the **one-loop correction** has been included, the **effective coupling constant** is given by

$$-i\lambda_{\text{eff}}(q^2) =$$



$$\Bigg|_{s\sim t\sim u\sim q^2} = -i\lambda + \frac{3i\lambda^2}{32\pi^2}\int_0^1 dx\,\log\left[\frac{\mu^2}{m^2 - x(1-x)q^2}\right]$$

$$= -i\lambda + \frac{3i\lambda^2}{32\pi^2}\left[\log\left(\frac{\mu^2}{m^2}\right) + 2 - \sqrt{1-\frac{4m^2}{q^2}}\,\log\left(\frac{\sqrt{1-\frac{4m^2}{q^2}}+1}{\sqrt{1-\frac{4m^2}{q^2}}-1}\right)\right]$$

For **large momenta** $q^2 \gg m^2$, this is given by

$$\lambda_{\text{eff}}(q^2) = \lambda\left[1 + \frac{3\lambda}{32\pi^2}\log\left(\frac{q^2}{\mu^2}\right)\right]$$

Noticing that $\lambda_{\text{eff}}(\mu^2) = \lambda$, this can be written as

$$\lambda_{\text{eff}}(q^2) = \lambda_{\text{eff}}(\mu^2)\left[1 + \frac{3\lambda_{\text{eff}}(\mu^2)}{32\pi^2}\log\left(\frac{q^2}{\mu^2}\right)\right]$$

→ $\mu \equiv$ reference scale

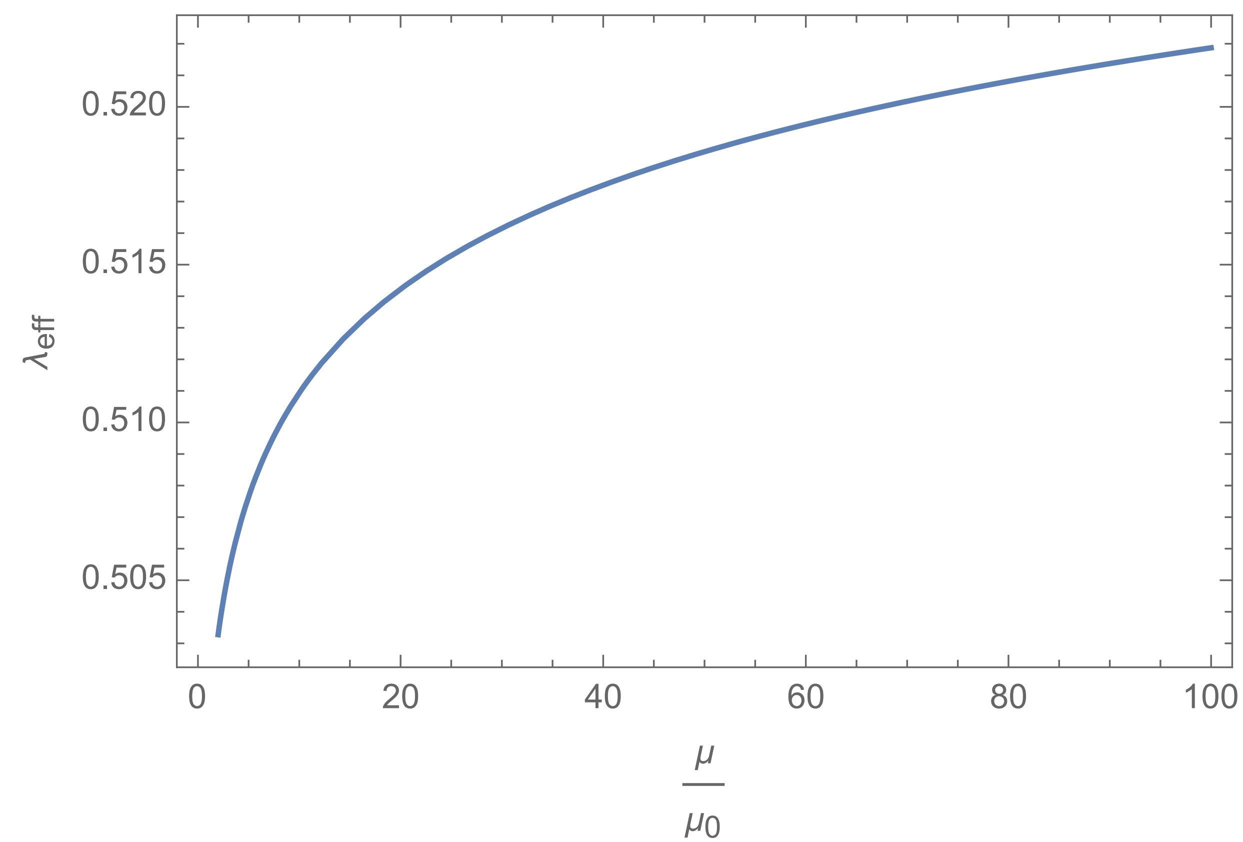
$$\lambda_{\rm eff}(\mu) = \lambda_{\rm eff}(\mu_0)\left[1 + \frac{3\lambda_{\rm eff}(\mu_0)}{32\pi^2}\log\left(\frac{\mu^2}{\mu_0^2}\right)\right]$$

Beware!
small change
in notation

Quantum corrections make **couplings run with energy**.

This **running** is also governed by the **one loop beta function**

$$\mu\frac{d\lambda_{\rm eff}}{d\mu} = \frac{3\lambda_{\rm eff}^2}{16\pi^2}$$



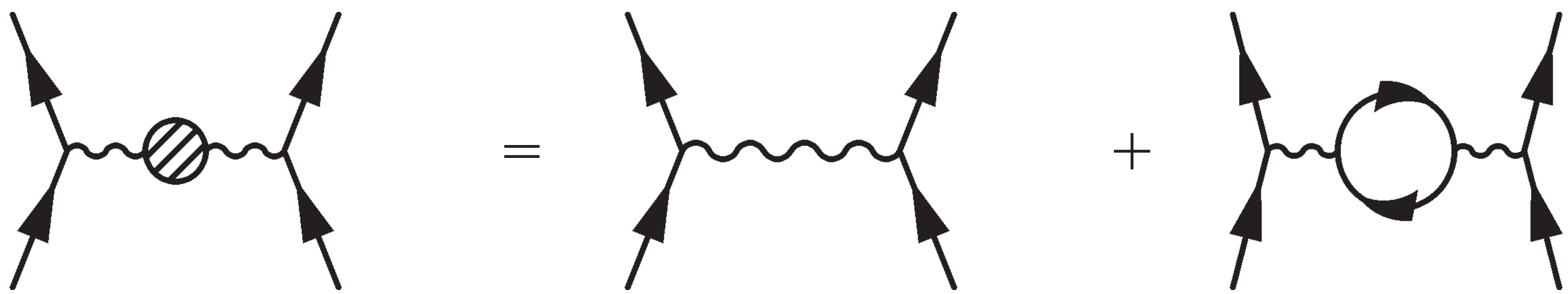
For the ϕ^4 theory, the effective coupling **grows** with energy → $\beta(\lambda) > 0$

Integrating the beta function equation we have

$$\lambda_{\rm eff}(\mu) = \frac{\lambda_{\rm eff}(\mu_0)}{1 - \frac{3\lambda_{\rm eff}(\mu_0)}{16\pi^2}\log\left(\frac{\mu}{\mu_0}\right)}$$

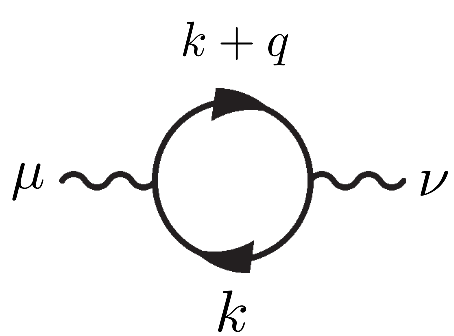
blows up at → $\mu = \mu_0 e^{\frac{16\pi^2}{3\lambda_{\rm eff}(\mu_0)}}$ **Landau pole**

A similar calculation of the **effective coupling** can be carried out in **QED**:



$$= \eta_{\alpha\beta}(\overline{v}_e\gamma^\alpha u_e)\frac{e^2}{4\pi q^2}(\overline{v}_\mu\gamma^\beta u_\mu) + \eta_{\alpha\beta}(\overline{v}_e\gamma^\alpha u_e)\frac{e^2}{4\pi q^2}\Pi(q^2)(\overline{v}_\mu\gamma^\beta u_\mu)$$

where



$$\equiv \Pi^{\mu\nu}(q) = i^2(-ie)^2(-1)\int\frac{d^4k}{(2\pi)^4}\frac{\mathrm{Tr}\left[(\not{k}+m_f)\gamma^\mu(\not{k}+\not{q}+m_f)\gamma^\nu\right]}{(k^2-m_f^2+i\epsilon)[(k+q)^2-m_f^2+i\epsilon]}$$

Regulating the divergence using a **sharp cutoff** Λ, we have

$$\Pi_{\mu\nu}(q) = c\Lambda^2\eta_{\mu\nu} + \Pi(q^2)(q^2\eta_{\mu\nu} - q_\mu q_\nu)$$

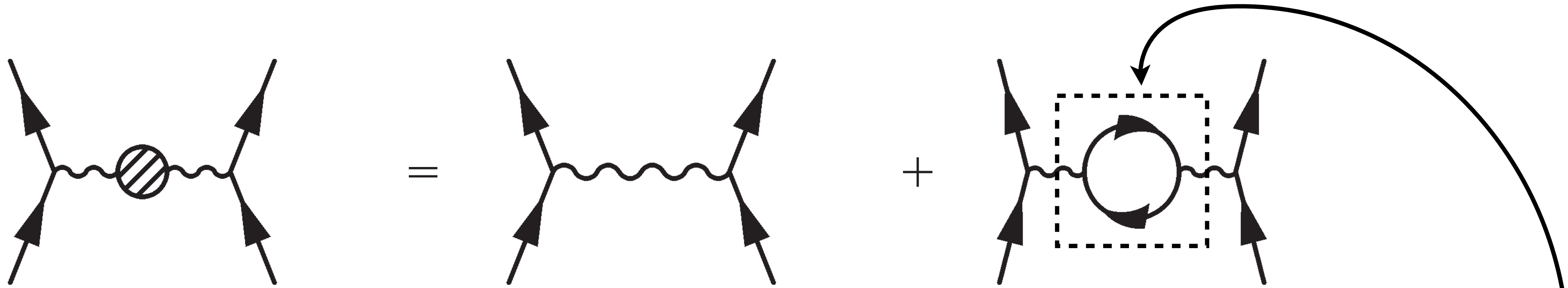
regularization artefact

Breaks gauge invariance, cured by adding a **local counterterm**

$$\Delta\mathscr{L} \sim \Lambda^2 A_\mu A^\mu$$

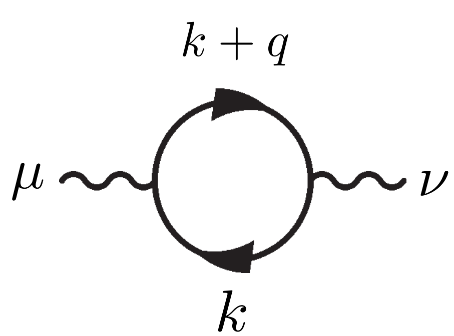
Gauge invariant and logarithmically divergent

A similar calculation of the **effective coupling** can be carried out in **QED**:



$$= \eta_{\alpha\beta}(\overline{v}_e\gamma^\alpha u_e)\frac{e^2}{4\pi q^2}(\overline{v}_\mu\gamma^\beta u_\mu) + \eta_{\alpha\beta}(\overline{v}_e\gamma^\alpha u_e)\frac{e^2}{4\pi q^2}\Pi(q^2)(\overline{v}_\mu\gamma^\beta u_\mu)$$

where



$$\equiv \Pi^{\mu\nu}(q) = i^2(-ie)^2(-1)\int\frac{d^4k}{(2\pi)^4}\frac{\mathrm{Tr}\left[(\not{k}+m_f)\gamma^\mu(\not{k}+\not{q}+m_f)\gamma^\nu\right]}{(k^2-m_f^2+i\epsilon)[(k+q)^2-m_f^2+i\epsilon]}$$

Regulating the divergence using a **sharp cutoff** Λ, we have

$$\Pi_{\mu\nu}(q) = c\Lambda^2\eta_{\mu\nu} + \Pi(q^2)(q^2\eta_{\mu\nu} - q_\mu q_\nu)$$

regularization artefact

Breaks gauge invariance, cured by adding a **local counterterm**

$$\Delta\mathscr{L} \sim \Lambda^2 A_\mu A^\mu$$

Gauge invariant and logarithmically divergent

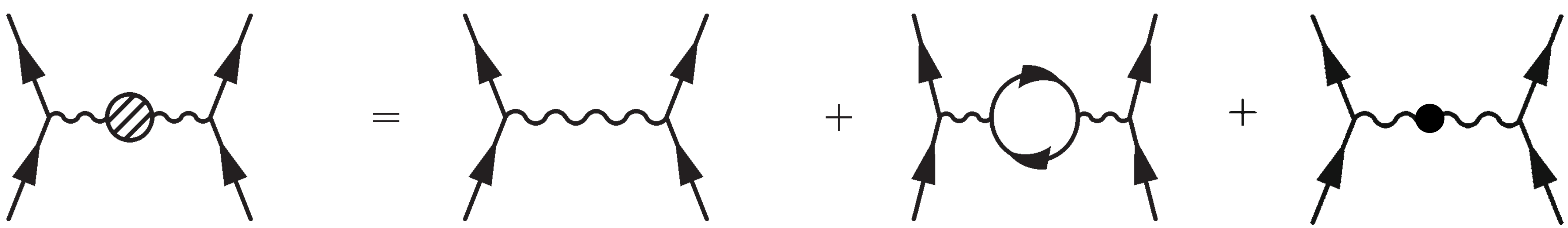
Forgetting about the **spurious quadratic divergence**, we have

$$\Pi_{\mu\nu}(q) = \left[\frac{e^2}{12\pi^2}\log\left(\frac{q^2}{\Lambda^2}\right) + \text{finite}\right](q^2\eta_{\mu\nu} - q_\mu q_\nu)$$

The logarithmic divergence can be cancelled by a **counterterm**

μ  ν $= -\dfrac{e^2}{12\pi^2}\log\left(\dfrac{\mu^2}{\Lambda^2}\right)(q^2\eta_{\mu\nu} - q_\mu q_\nu)$

The **total** contribution to the $e^-e^+ \to \mu^-\mu^+$ scattering is then



$$= \eta_{\alpha\beta}(\overline{v}_e\gamma^\alpha u_e)\left\{\frac{e^2}{4\pi q^2}\left[1 + \frac{e^2}{12\pi^2}\log\left(\frac{q^2}{\mu^2}\right)\right]\right\}(\overline{v}_\mu\gamma^\beta u_\mu)$$

$$\equiv \eta_{\alpha\beta}(\overline{v}_e\gamma^\alpha u_e)\left[\frac{e_{\text{eff}}(q^2)^2}{4\pi q^2}\right](\overline{v}_\mu\gamma^\beta u_\mu)$$

The **QED running effective charge** is then defined by

$$e_{\text{eff}}(q^2)^2 = e^2\left[1 + \frac{e^2}{12\pi^2}\log\left(\frac{q^2}{\mu^2}\right)\right]$$

$$e_{\text{eff}}(\mu)^2 = e_{\text{eff}}(\mu_0)^2\left[1 + \frac{e_{\text{eff}}(\mu_0)^2}{12\pi^2}\log\left(\frac{\mu^2}{\mu_0^2}\right)\right]$$

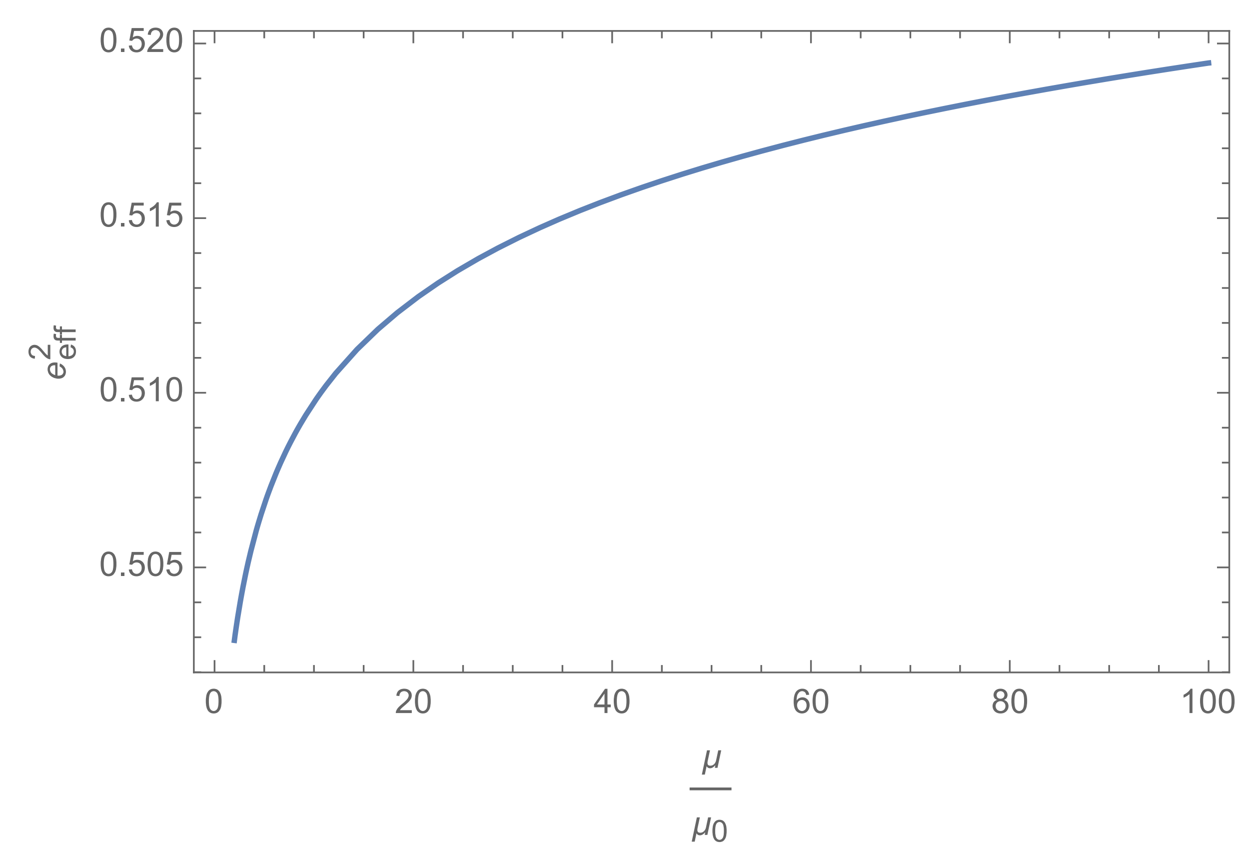
As in the ϕ^4 case, the **QED beta function** is **positive** and the coupling grows with energy

$$\beta(e)_{\text{QED}} = \frac{e^3}{12\pi^2} > 0$$

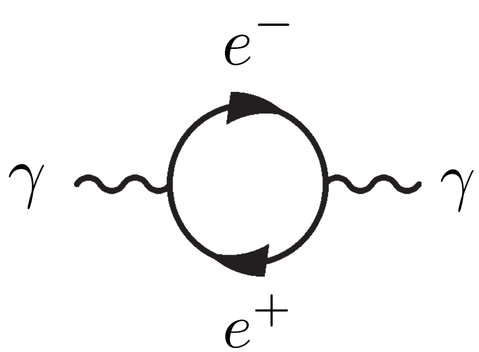
Again, there is a **Landau pole**, which for the **Standard Model** is located at

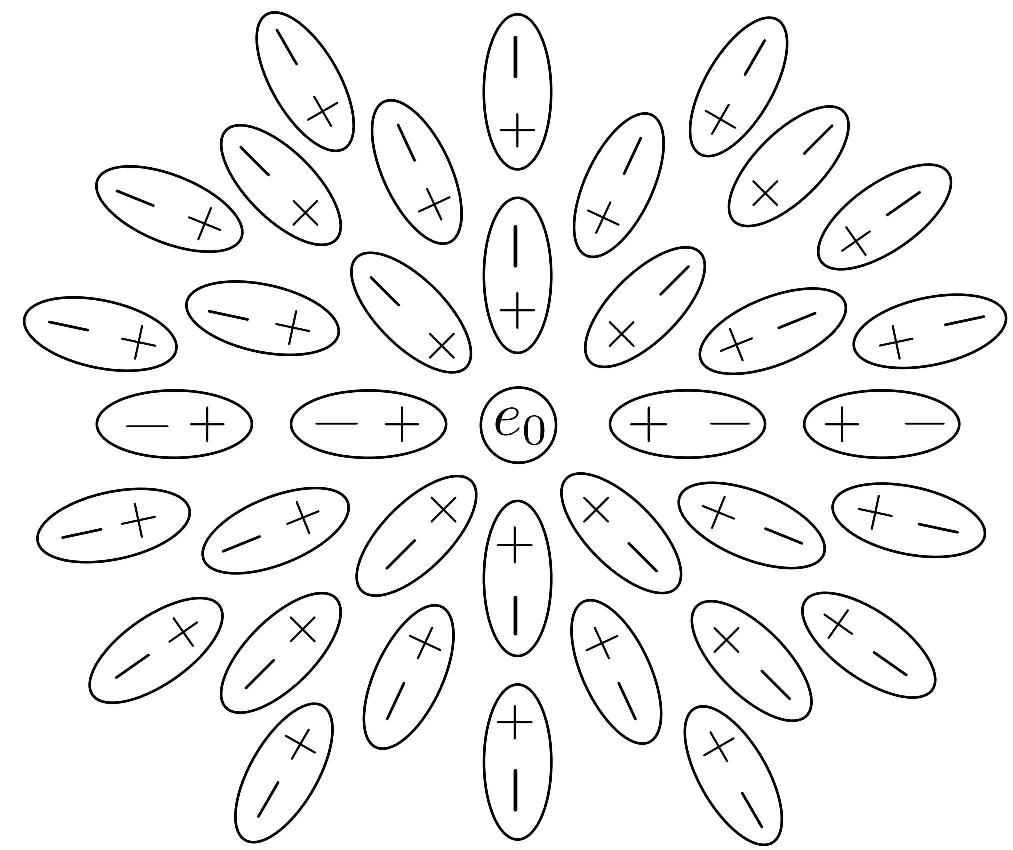
$$\mu_{\text{Landau}} \sim 10^{34}\ \text{GeV}$$

well **beyond** any other relevant energy scale.



Heuristically, the **running coupling** can be understood in terms of **screening**

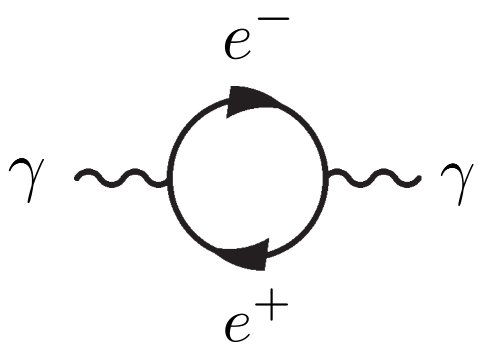


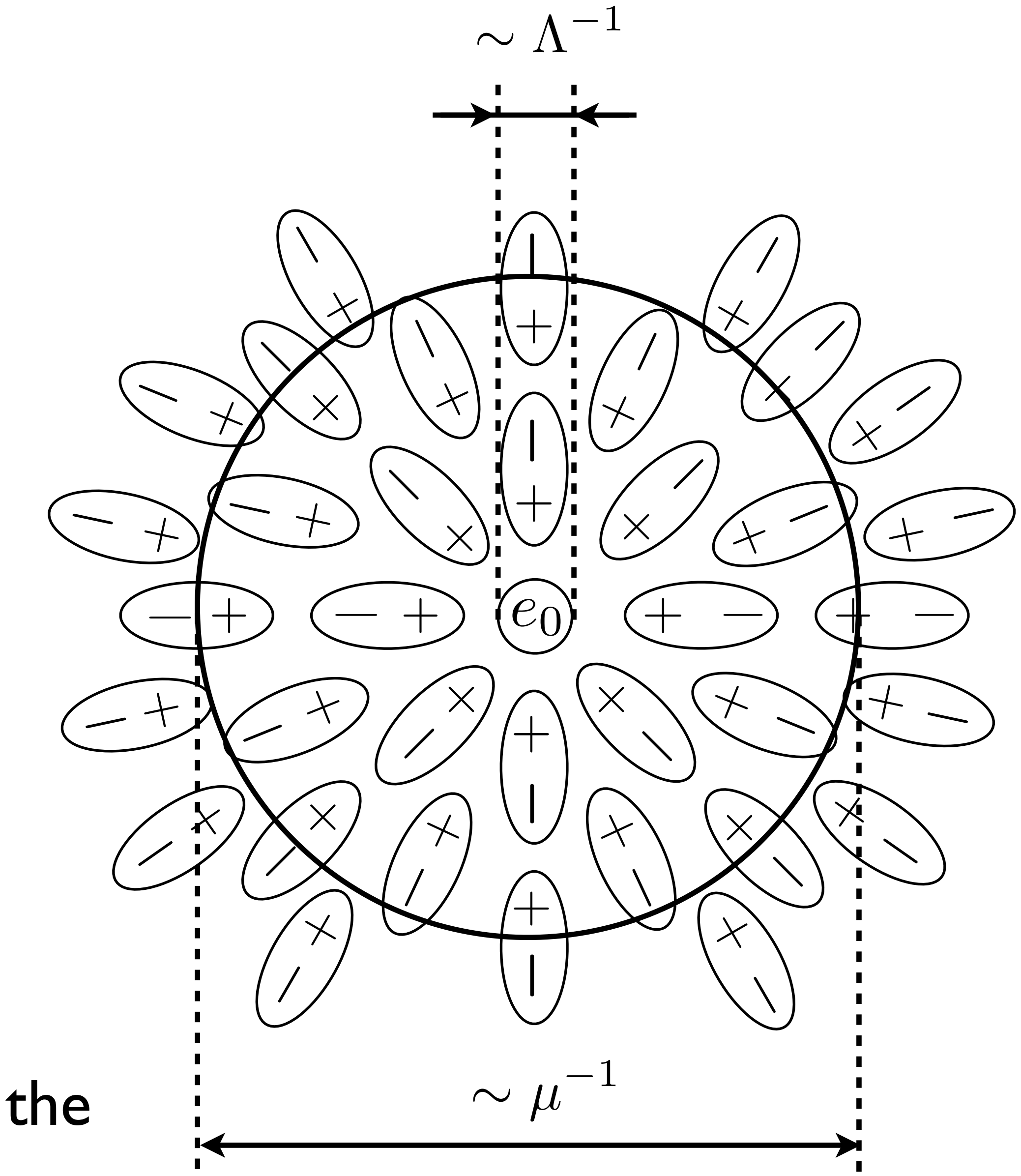


As in a dielectric medium, the **polarization** of the vacuum screens the **bare** charge

$$e(\mu)^2 = e_0(\Lambda)^2 \left[1 + \frac{e_0(\Lambda)^2}{12\pi^2}\log\left(\frac{\mu^2}{\Lambda^2}\right)\right]$$

Heuristically, the **running coupling** can be understood in terms of **screening**





As in a dielectric medium, the **polarization** of the vacuum screens the **bare** charge

$$e(\mu)^2 = e_0(\Lambda)^2 \left[1 + \frac{e_0(\Lambda)^2}{12\pi^2} \log\left(\frac{\mu^2}{\Lambda^2}\right)\right]$$