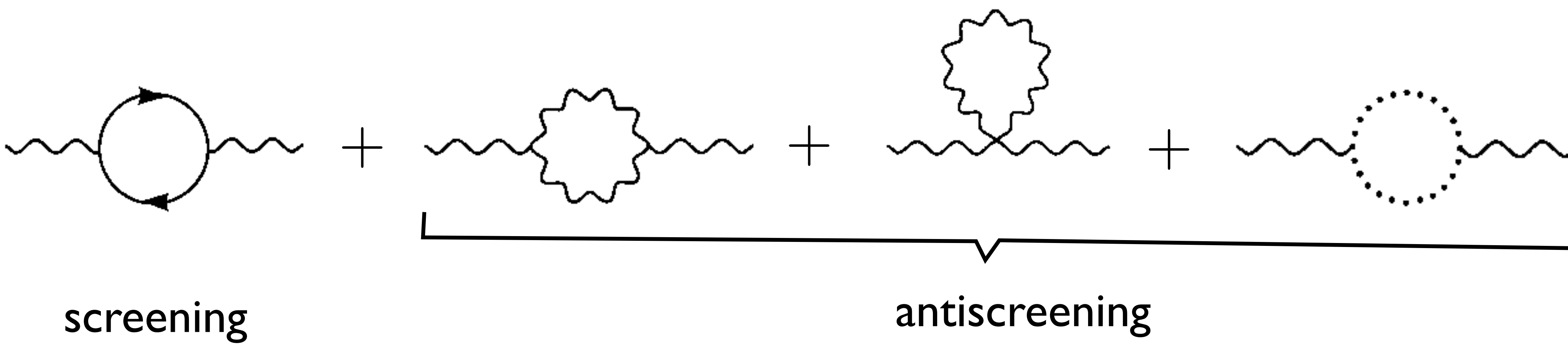
The behavior of the effective coupling is quite different for **non-Abelian gauge theories**

$$\mathscr{L} = -\frac{1}{4}F^A_{\mu\nu}F^{A\mu\nu} + i\sum_{k,\ell=1}^{N_c}\overline{\psi}_k\gamma^\mu(\delta_{k\ell}\partial_\mu - igA^B_\mu T^B_{k\ell})\psi_\ell$$

$$F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + gf^{ABC}A^B_\mu A^C_\nu$$

Now, both **fermions** and **gauge bosons** contribute to the gauge boson polarization tensor



For a $\mathbf{SU}(N_c)$ gauge theory, the **beta function** can be **negative**

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right)$$

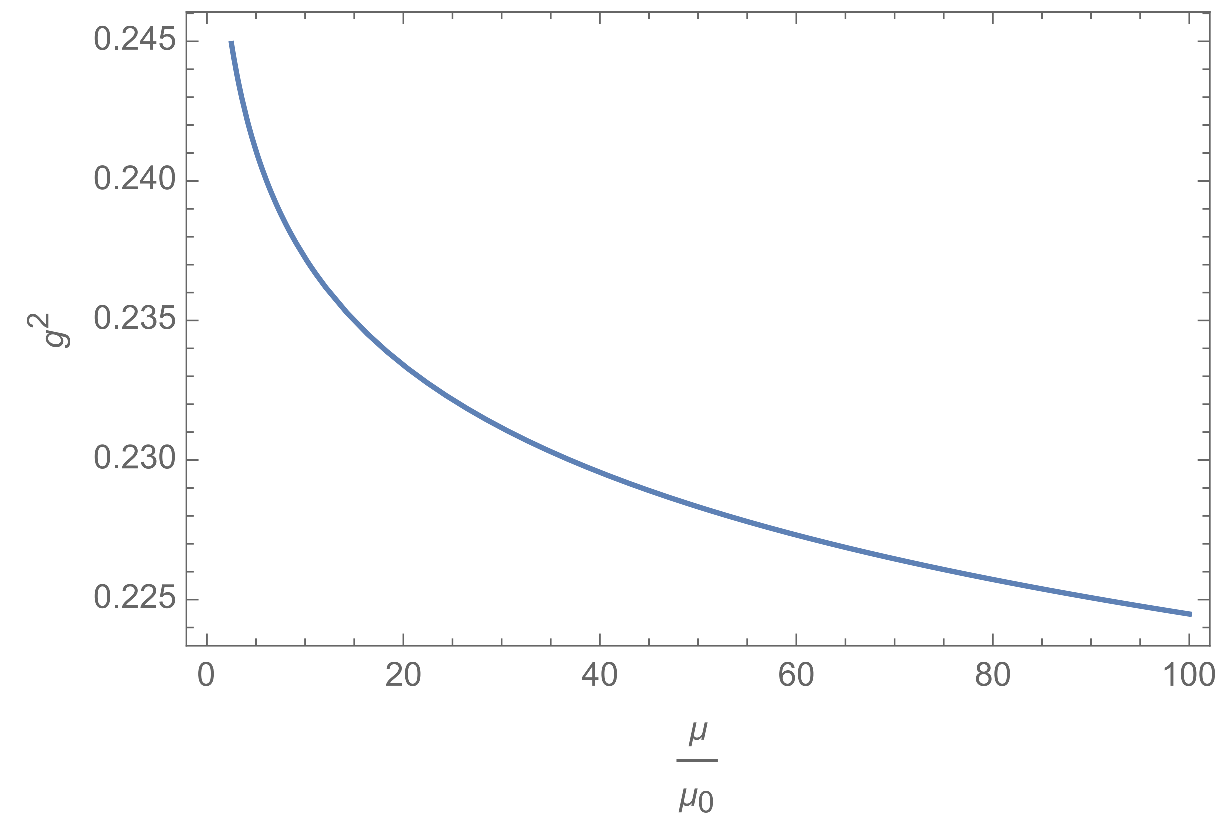
$N_c \equiv$ # of colors

$N_f \equiv$ # of flavors

For **QCD**, the beta function is **negative**

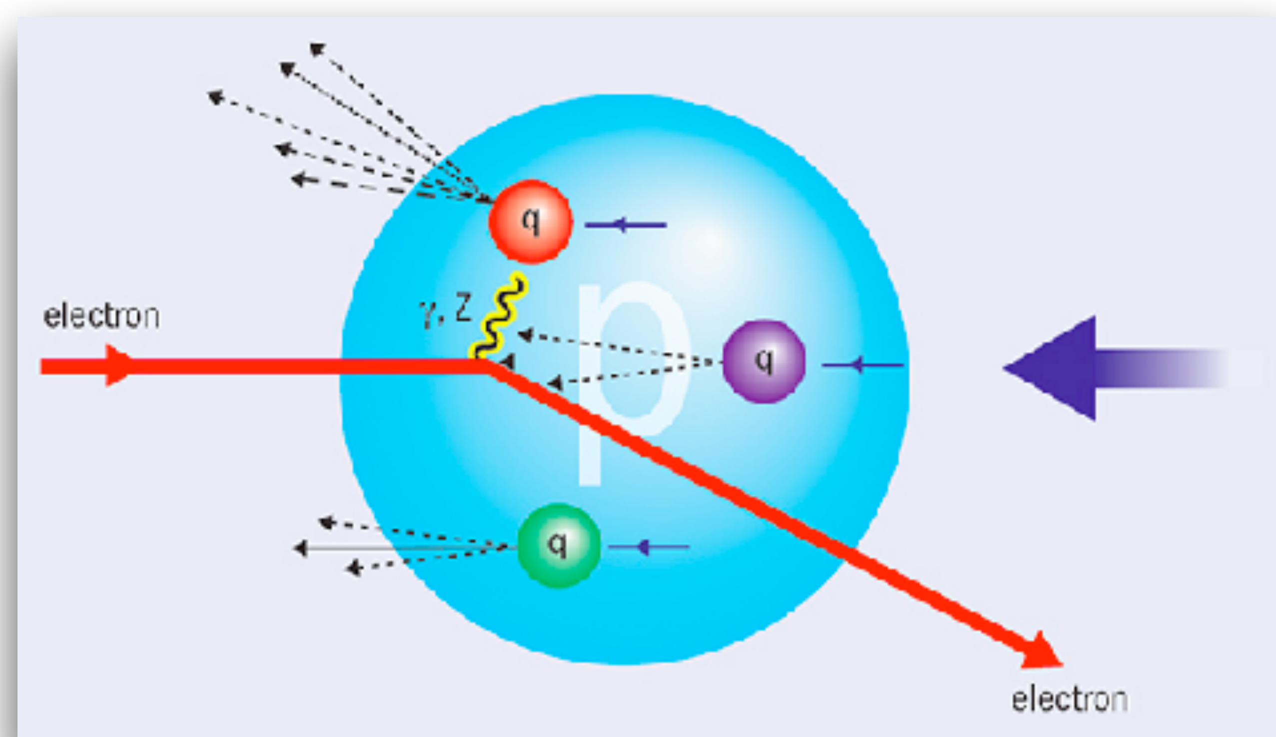
$$\left.\begin{array}{l} N_c = 3 \\ N_f = 6 \end{array}\right\} \Longrightarrow \beta(g)_{\mathrm{QCD}} = -\frac{7g^3}{16\pi^2} < 0$$

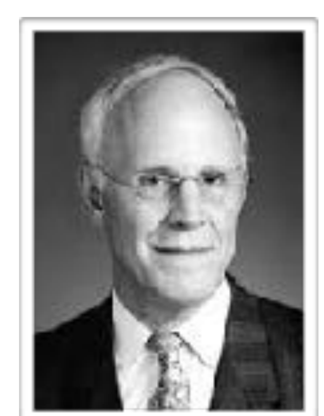
$$g(\mu)^2 = g(\mu_0)^2\left[1 - \frac{7g(\mu_0)^2}{16\pi^2}\log\left(\frac{\mu^2}{\mu_0^2}\right)\right]$$



and the theory is **asymptotically free** at high energies.

This result explains the **quasifree behavior** of **partons** exhibited in **deep inelastic scattering**





David J. Gross
(b. 1941)

H. David Politzer
(b. 1949)

Frank Wilcek
(b. 1951)

What do ϕ^4 and **QED**, and **QCD** have in common?

Infinities are taken care of by renormalizing a **finite number** of **quantites**.

The **renormalized Lagrangian** contains a **finite** number of **operators**, e.g.

$$\mathscr{L}_{\rm ren} = Z_\psi(\Lambda)\overline{\psi}[i\gamma^\mu\partial_\mu - m_0(\Lambda)]\psi - \frac{1}{4}Z_A(\Lambda)F_{\mu\nu}F^{\mu\nu} - e_0(\Lambda)Z_\psi(\Lambda)\sqrt{Z_A(\Lambda)}A_\mu\overline{\psi}\gamma^\mu\psi$$

The theory is **renormalizable**

Rule of thumb: a theory is **renormalizable** if its bare Lagrangian **does not contain** higher-dimensional (**> 4**) operators.

All **coupling** constants have **non-negative energy dimensions**.

What do ϕ^4 and **QED**, and **QCD** have in common?

Infinities are taken care of by renormalizing a **finite number** of **quantites**.

The **renormalized Lagrangian** contains a **finite** number of **operators**, e.g.

Sometimes, we need to **add operators** to renormalize the theory. In the **Yukawa** theory

$$\mathscr{L}_{\text{Yukawa}} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 + \overline{\psi}(i\partial\!\!\!/ - m)\psi + g\phi\overline{\psi}\psi$$

we have

and the renormalized Lagrangian contains **cubic** and **quartic couplings** for the scalar field

$$\mathscr{L}_{\text{ren}} = \frac{1}{2}Z_\phi(\Lambda)\partial_\mu\phi\partial^\mu\phi - Z_\phi(\Lambda)\frac{m_0(\Lambda)^2}{2}\phi^2 + Z_\psi(\Lambda)\overline{\psi}[i\partial\!\!\!/ - m_0(\Lambda)]\psi$$
$$+\sqrt{Z_\phi(\Lambda)}Z_\psi(\Lambda)g_0(\Lambda)\phi\overline{\psi}\psi - Z_\phi(\Lambda)^{\frac{3}{2}}\frac{\alpha_0(\Lambda)}{3!}\phi^3 - Z_\phi(\lambda)^2\frac{\lambda_0(\Lambda)}{4!}\phi^4$$

Take the case of a **scalar ϕ^n theory**.

For a diagram with E **external lines**, I **internal lines** and V **vertices**

total # of legs coming out of vertices → $nV = E + 2I$ ← internal legs join two vertices

E: external legs are attached to a single vertex

$$nV = E + 2I$$

$$L = I - (V - 1) = I - V + 1$$

I: # of integrations; $(V-1)$: # of independent delta functions; -1: global conservation delta function

On the other hand, the **superficial degree of divergence** of a diagram with E **external lines** is

$$D = 4L - 2I$$

This can be expressed in terms of E and V as

Using $L = I - V + 1$ and $2I = nV - E$:

$$D = 4L - 2I = 2I - 4V + 4 = (n-4)V - E + 4$$

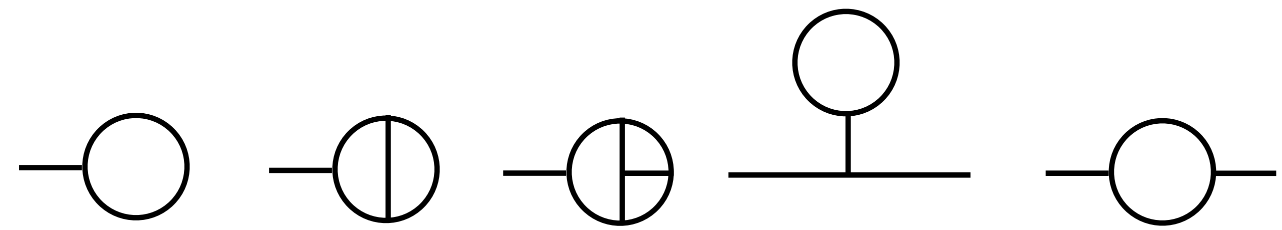
$$\boxed{D = (n-4)V - E + 4}$$

$$D = (n-4)V - E + 4$$

* $n = 3$

$$D = 4 - E - V$$

There is only a **finite** number of **superficially divergent diagrams**.



ϕ^3 theory is **superrenormalizable**

* $n = 4$

$$D = 4 - E$$

There are **infinitely many superficially divergent diagrams**. but **only** with **2** and **4** external legs

ϕ^4 theory is **renormalizable**

$$D = (n-4)V - E + 4$$

* $n = 6$

$$D = 2V - E + 4$$

→ There are **infinitely many divergent diagrams** with an **arbitrary number of external legs**

Thus, we need to add an **infinite number** of **counterterms** with **arbitrary** number of **external legs**.

The **renormalized Lagrangian** contains an **infinite number** of **operators**

$$\mathscr{L}_{\rm ren} = \frac{1}{2}Z_\phi(\Lambda)\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}Z_\phi(\Lambda)m_0(\Lambda)^2\phi^2 - \sum_{n=2}^{\infty}\frac{1}{(2n)!}Z_\phi(\Lambda)^n\lambda_{2n,0}(\Lambda)\phi^{2n}$$

In principle, to compute amplitudes we need to specify **infinitely many renormalizations conditions**!

ϕ^6 **theory** is **not renormalizable** (as well as ϕ^n for $n > 4$)

$$D = (n-4)V - E + 4$$

* $n = 6$

$$D = 2V - E + 4$$ → There are **infinitely many divergent diagrams** with an **arbitrary number of external legs**

Let us look at the **energy dimensions** of the **coupling constants**:

$$[\lambda_n] = 4 - n$$

* ϕ^3 **theory**: $[\lambda_3] = 1$ → superrenormalizable

* ϕ^4 **theory**: $[\lambda_4] = 0$ → renormalizable

* ϕ^n **theory** ($n > 4$): $[\lambda_n] = 4 - n < 0$ → non renormalizable

In

conditions!

ϕ^6 **theory** is **not renormalizable** (as well as ϕ^n for $n > 4$)

A **physical** (i.e., **Wilsonian**) view of **renormalization**.

Kenneth G. Wilson
(1936-2013)

Let us **take** the **cutoff seriously** and start with our quantum field theory defined at the **scale** $\boldsymbol{\Lambda}$

$E = \Lambda$

bare couplings

$$S[\phi_a,\Lambda] = \int d^4x \left\{ \mathscr{L}_0[\phi_a] + \sum_i g_i(\Lambda)\mathscr{O}_i[\phi_a] \right\}$$

bare fields

$$e^{iS[\phi'_a,\mu]} = \int\limits_{\mu<p<\Lambda} \prod_a \mathscr{D}\phi_a \, e^{iS[\phi_a,\Lambda]}$$

$E = \mu$

renormalized couplings

$$S[\phi'_a,\mu] = \int d^4x \left\{ \mathscr{L}_0[\phi'_a] + \sum_i g_i(\mu)\mathscr{O}_i[\phi'_a] \right\}$$

renormalized fields

Part III

The Good QFT:
Locality, Causality & Unitarity

There are a number of **features** to be demanded from a **healthy quantum field theory** (e.g., the **standard model**)

- **Lorentz invariance**: the action should be Lorentz invariant.

- **Locality**: local measurements at a point are determined by what is going on in an **arbitrarily small neighborhood** around that point.

The action **only** contain terms in which the **fields** and their **derivatives** are **evaluated at the same spacetime point**:

Thus, all interactions are **propagated** from point to point (there are **no action at a distance!**)

At the level of the **observables**, locality is identified with **cluster decomposition**

$$\langle \mathscr{O}_1(x)\mathscr{O}_2(y)\rangle = \langle \mathscr{O}_1(x)\rangle \langle \mathscr{O}_2(y)\rangle \qquad \text{if} \qquad (x-y)^2 < 0$$

- **Unitarity**: probability should be conserved

$$\frac{d}{dt}\|\psi(t)\|^2 = 0 \quad \longrightarrow \quad H^\dagger = H \quad \longrightarrow \quad S^\dagger S = SS^\dagger = \mathbf{1}$$

Unitarity of the **S-matrix** have important **implications**:

$$S = \mathbf{1} + iT$$
$$S^\dagger = \mathbf{1} - iT^\dagger$$
$$\longrightarrow \quad \mathbf{1} = (\mathbf{1} - iT^\dagger)(\mathbf{1} + iT) = \mathbf{1} + i(T - T^\dagger) + T^\dagger T$$

Thus, the T-matrix **satisfies**

$$i(T^\dagger - T) = T^\dagger T$$

In a **scattering** experiment $|i\rangle \longrightarrow |f\rangle$

$$i\langle f|T^\dagger|i\rangle - i\langle f|T|i\rangle \equiv i\langle i|T|f\rangle^* - i\langle f|T|i\rangle = \langle f|T^\dagger T|i\rangle$$

$$i\langle i|T|f\rangle^* - i\langle f|T|i\rangle = \langle f|T^\dagger T|i\rangle$$

Now, let us remember that $\langle f|T|i\rangle \equiv (2\pi)^4\delta^{(4)}(P_f - P_i)\mathscr{M}_{i\to f}$

$$i(2\pi)^4\delta^{(4)}(P_f - P_i)\Big[\mathscr{M}^*_{f\to i} - \mathscr{M}_{i\to f}\Big] = \langle f|T^\dagger T|i\rangle$$

Invariant momentum measure (phase space factor)

Next, let us use the **closure relation** $\mathbb{1} = \sum_X \int d\Pi_X |X\rangle\langle X|$ on the right-hand side

$$\langle f|T^\dagger T|i\rangle = \sum_X \int d\Pi_X \langle f|T^\dagger|X\rangle\langle X|T|i\rangle = \sum_X \int d\Pi_X \langle X|T|f\rangle^* \langle X|T|i\rangle$$

$$= \sum_X \int d\Pi_X (2\pi)^4\delta^{(4)}(P_X - P_f)(2\pi)^4\delta^{(4)}(P_X - P_i)\mathscr{M}^*_{f\to X}\mathscr{M}_{i\to X}$$

and we arrive at the **generalized optical theorem**

$$\mathscr{M}_{i\to f} - \mathscr{M}^*_{f\to i} = i\sum_X \int d\Pi_X (2\pi)^4\delta^{(4)}(P_X - P_i)\mathscr{M}^*_{f\to X}\mathscr{M}_{i\to X}$$

$$\mathscr{M}_{i\to f} - \mathscr{M}^*_{f\to i} = i\sum_X \int d\Pi_X (2\pi)^4 \delta^{(4)}(P_X - P_i)\mathscr{M}^*_{f\to X}\mathscr{M}_{i\to X}$$

There are two interesting **particular cases**:

✦ $|i\rangle = |f\rangle = |A\rangle$ is a **single particle state.** Then $\mathscr{M}_{i\to f} \equiv \mathscr{M}_{f\to i}$ and

$$2\mathrm{Im}\,\mathscr{M}_{A\to A} = \sum_X \int d\Pi_X (2\pi)^4 \delta^{(4)}(p_A - P_X)|\mathscr{M}_{A\to X}|^2 \equiv 2\omega_{\mathbf{p}_A}\Gamma_{\text{total}}$$

Thus, the **imaginary part** of the **propagator** gives the **total decay width**.

✦ $|i\rangle = |f\rangle = |A\rangle$ is a **two-particle state** (forward elastic amplitude, $t = 0$)

$4E_1E_2|\mathbf{v}_1 - \mathbf{v}_2|$ in CM frame

$$2\mathrm{Im}\,\mathscr{M}_{A\to A}(s,0) = \sum_X \int d\Pi_X (2\pi)^4 \delta^{(4)}(p_A - P_X)|\mathscr{M}_{A\to X}|^2 \equiv 2E_{\text{CM}}|\mathbf{p}_i|\sigma_{\text{total}}(A \to X)$$

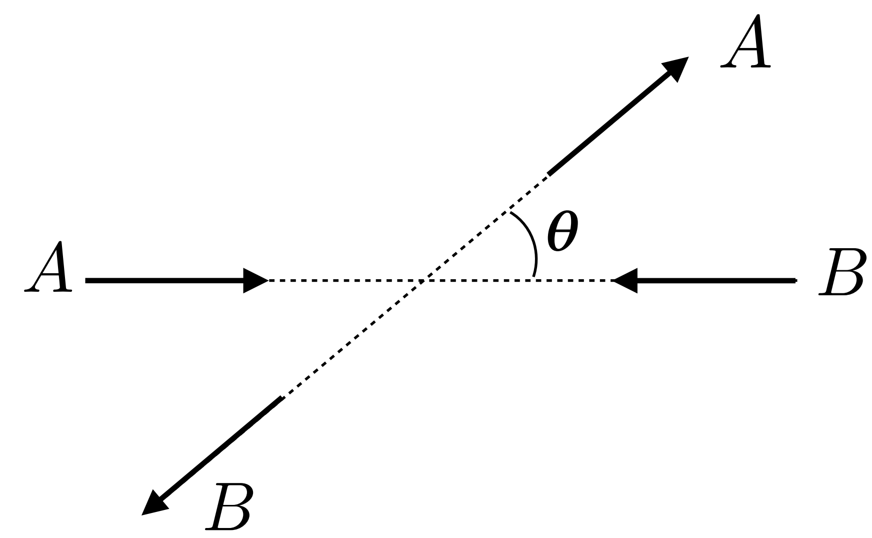
$$2\mathrm{Im}\,\mathscr{M}_{A\to A}(s,0) = 2E_{\text{CM}}|\mathbf{p}_i|\sum_X \sigma_{\text{total}}(A \to X)$$

Optical theorem

Unitary also imposes **strong constraints** on the **growth** of scattering amplitudes with **energy**:

Let us stick to the case of **elastic two-particle scattering** in the **center-of-mass frame**

$$A + B \longrightarrow A + B$$



The amplitude is a function of the **angle of scattering** $\boldsymbol{\theta}$

$$\sigma_{\text{total}}(AB \to AB) = \frac{1}{32\pi E_{\text{CM}}^2} \int_{-1}^{1} d\cos\theta\, |\mathscr{M}_{AB\to AB}(\theta)|^2$$

and we can **expand** it into **partial waves** using the basis of **Legendre polynomials**

partial wave amplitude

$$\mathscr{M}_{AB\to AB}(\theta) = 16\pi \sum_{\ell=0}^{\infty} a_\ell (2\ell+1) P_\ell(\cos\theta)$$

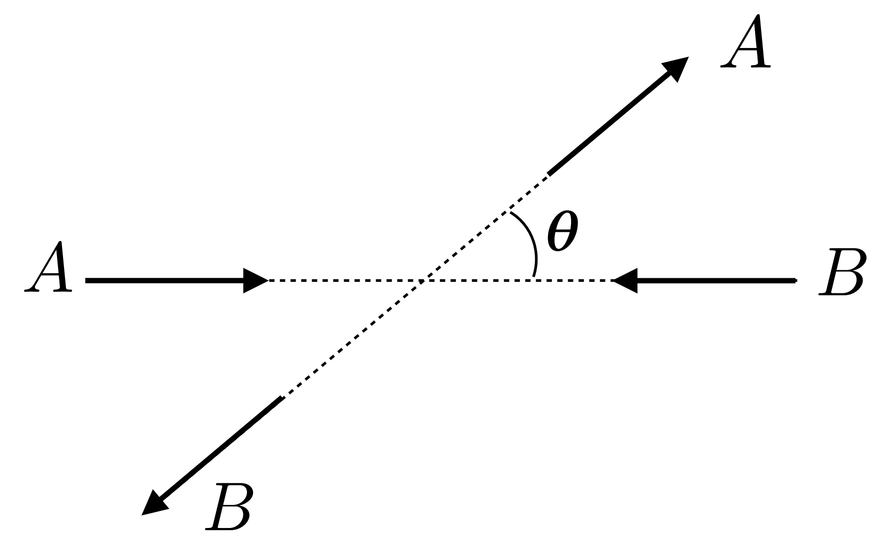
$$\int_{-1}^{1} d\cos\theta\, |\mathscr{M}_{AB\to AB}(\theta)|^2 = 2(16\pi)^2 \sum_{\ell=0}^{\infty} (2\ell+1)|a_\ell|^2$$

$$\sigma_{\text{total}}(AB \to AB) = \frac{16\pi}{E_{\text{CM}}^2} \sum_{\ell=1}^{\infty} (2\ell+1)|a_\ell|^2$$

Unitary also imposes **strong constraints** on the **growth** of scattering amplitudes with **energy**:

Let u ... **scattering** in the **center-of-mass fram**

$$\int_{-1}^{1} d\cos\theta\, P_\ell(\cos\theta)P_{\ell'}(\cos\theta') = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$



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$$\sigma_{\rm total}(AB\to AB) = \frac{16\pi}{E_{\rm CM}^2}\sum_{\ell=1}^{\infty}(2\ell+1)|a_\ell|^2$$

$$\sigma_{\text{total}}(AB \to AB) = \frac{16\pi}{E_{\text{CM}}^2} \sum_{\ell=1}^{\infty} (2\ell+1)|a_\ell|^2$$

$$2\text{Im}\,\mathscr{M}_{AB\to AB}(0) = 2E_{\text{CM}}|\mathbf{p}_i| \sum_X \sigma_{\text{total}}(AB \to X)$$

$$\mathscr{M}_{AB\to AB}(\theta) = 16\pi \sum_{\ell=0}^{\infty} a_\ell (2\ell+1) P_\ell(\cos\theta)$$

Combining these results

$$2\text{Im}\,\mathscr{M}_{AB\to AB}(0) = 2E_{\text{CM}}|\mathbf{p}_i| \sum_X \sigma_{\text{total}}(AB \to X) \geq 2E_{\text{CM}}|\mathbf{p}_i| \sigma_{\text{total}}(AB \to AB)$$

$P_\ell(1) = 1$

$$\sum_{\ell=0}^{\infty} (2\ell+1)\text{Im}\,a_\ell \geq \frac{2|\mathbf{p}_i|}{E_{\text{CM}}} \sum_{\ell=0}^{\infty} (2\ell+1)|a_\ell|^2$$

Expanding the amplitude using the angular momentum basis

$$\text{Im}\,a_\ell \geq \frac{2|\mathbf{p}_i|}{E_{\text{CM}}} |a_\ell|^2$$

Partial wave unitarity bound

$$\text{Im}\, a_\ell \geq \frac{2|\mathbf{p}_i|}{E_{\rm CM}}|a_\ell|^2$$

At **very high energies**, masses can be neglected ($|p_i^0| = |\mathbf{p}_i|$) and

$$E_{\rm CM} = \sqrt{s} = 2|p_i^0| = 2|\mathbf{p}_i|$$

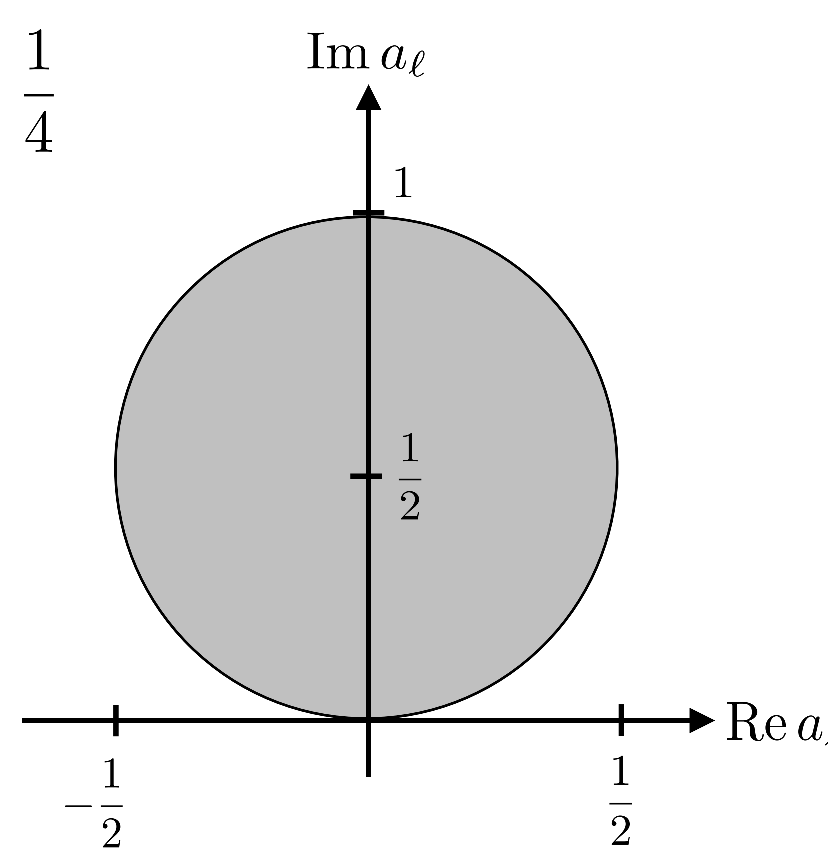
$$\text{Im}\, a_\ell \geq |a_\ell|^2$$

This can be **rewritten** as

$$(\text{Re}\, a_\ell)^2 + (\text{Im}\, a_\ell)^2 \leq \text{Im}\, a_\ell \quad \longrightarrow \quad (\text{Re}\, a_\ell)^2 + \left(\text{Im}\, a_\ell - \frac{1}{2}\right)^2 \leq \frac{1}{4}$$

$$|\text{Re}\, a_\ell| \leq \frac{1}{2}$$

$$\left|\text{Im}\, a_\ell - \frac{1}{2}\right| \leq \frac{1}{2} \quad \longrightarrow \quad 0 \leq \text{Im}\, a_\ell \leq 1 \quad \longrightarrow \quad |a_\ell| \leq 1$$



Example: the **limits** of **Fermi** four-fermion **theory**

Fermi's theory of weak interactions is based on a four-fermion **contact** interaction

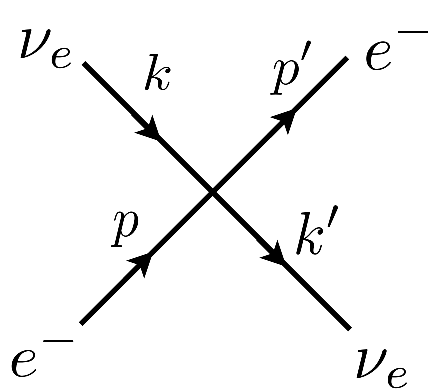
$$\mathscr{L}_{\rm int} = -\frac{G_F}{\sqrt{2}} J^\mu J^\dagger_\mu$$

$$G_F = \left(\frac{1}{292.9\,{\rm GeV}}\right)^2$$

where

$$\begin{aligned} J^\alpha &= J^\alpha_{\rm hadron} + J^\alpha_{\rm lepton} \\ &= \overline{u}\gamma^\alpha(1-\gamma_5)(d\cos\theta_C + s\sin\theta_C) + \overline{c}\gamma^\alpha(1-\gamma_5)(-d\cos\theta_C + s\cos\theta_C) \\ &+\overline{\nu}_e\gamma^\alpha(1-\gamma_5)e + \overline{\nu}_\mu\gamma^\alpha(1-\gamma_5)\mu \qquad (\theta_C = \text{Cabibbo angle} \approx 13^\circ) \end{aligned}$$

Let us study electron-neutrino scattering, $e^-\nu_e \longrightarrow e^-\nu_e$



$$\equiv i\mathscr{M}_{e\nu_e\to e\nu_e} = -i\frac{G_F}{\sqrt{2}}\Big[\overline{u}(p')\gamma^\mu(1-\gamma_5)u(k)\Big]\Big[\overline{u}(k')\gamma_\mu(1-\gamma_5)u(p)\Big]$$

and **averaging** over **polarizations**

$$\overline{|i\mathscr{M}_{e\nu_e\to e\nu_e}|^2} = 32G_F^2(s-m_e^2)^2$$

$$\overline{|i\mathscr{M}_{e\nu_e\to e\nu_e}|^2} = 32G_F^2(s-m_e^2)^2$$

The **differential** and **total cross sections** for **unpolarized scattering** in the center-of mass frame are:

$$\frac{d\sigma}{d\cos\theta} = \frac{G_F^2}{2\pi}\frac{(s-m_e^2)^2}{s} \qquad \sigma_{\rm total} = \frac{G_F^2}{\pi}\frac{(s-m_e^2)^2}{s}$$

This averaged amplitude is **isotropic**, so we only have the **s-wave contribution**

$$\sigma_{\rm total} = \frac{16\pi}{E_{\rm CM}^2}\sum_{\ell=1}^{\infty}(2\ell+1)|a_\ell|^2 \quad \Longrightarrow \quad |a_0|^2 = \frac{G_F^2}{16\pi^2}(s-m_e^2)^2 \approx \frac{G_F^2 s^2}{16\pi^2}$$

$s \gg m_e^2$

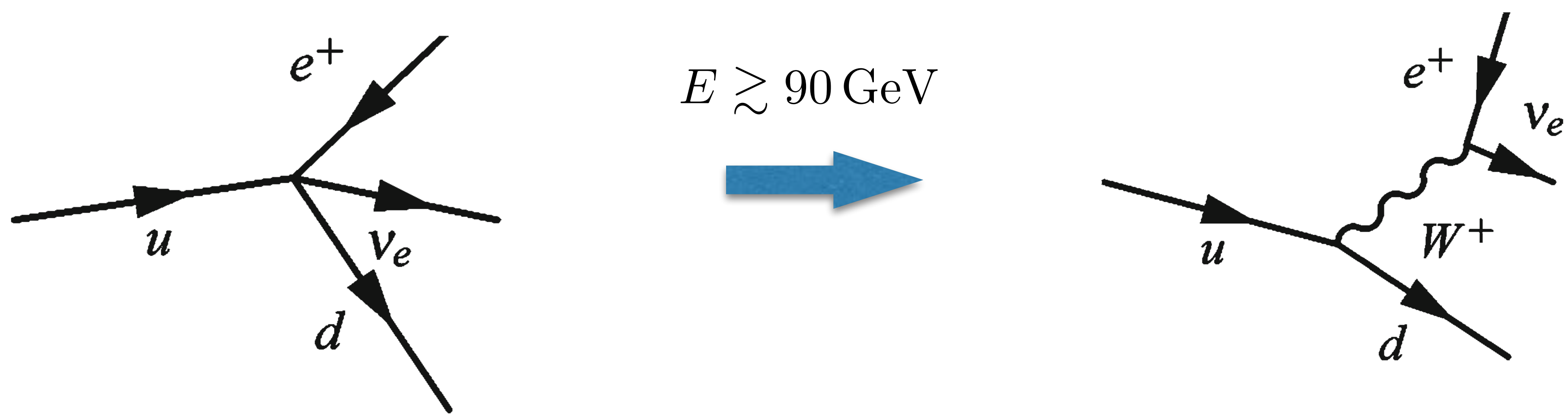
The **strong growth** of the total cross section with energy means that the **unitarity bound** will eventually be **violated**

$$|a_0| < 1 \quad \Longrightarrow \quad \frac{G_F s}{4\pi} < 1$$

Fermi´s theory **breaks down** at the energy

$$E = \sqrt{\frac{4\pi}{G_F}} \approx 1000\,\text{GeV}$$

But we know that Fermi's theory is **replaced** by the standard model well **below this energy**:



The interchange of the gauge boson **tame** the **growth** of the total cross section at **high energies**

$$\sigma_{\text{total}} = \frac{G_F^2}{\pi}\frac{m_W^2 s}{s+m_W^2} \sim \frac{G_F^2 m_W^2}{\pi}$$

$s \gg m_W^2$

Once the Fermi theory is **embedded** in the **full** standard model, **unitarity** is **restored**.